Fractional calculus pertaining to multivariable I -function defined by Prathima

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Abstract

In this paper, we study a pair of unified and extended fractional integral operator involving the multivariable I -functions and general class of multivariable polynomials. Here, we use Mellin transforms to obtain our main results. Certain properties of these operators concerning to their Mellin-transforms have been investigated. On account of the general nature of the functions involved herein, a large number of known (may be new also) fractional integral operators involved simpler functions can be obtained. We will also quote the particular case of the multivariable H -function.

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1. INTRODUCTION AND PRELIMINARIES

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences can be described very successfully by models using mathematical tools by models using mathematical tools from fractional calculus. Chaurasia and Srivastava [4], Choi et al. [5], Daiya et al. [6], Kumar and Daiya [10], Kumar et al. [12], and others have studied the fractional calculus pertaining to multivariable H -function defined by Srivastava and Panda [24].

The $\bar{I}$ -function, introduced by Rathie [16], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral:

$$\bar{I}(z) = \bar{I}_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{n,n+1}, (a_j, \alpha_j; A_j)_p \\ (b_j, \beta_j; B_j)_{m,m+1}, (b_j, \beta_j; B_j)_q \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \Omega_{p,q}^{m,n}(s) z^{-s} ds, \quad (1)$$

for all $z \neq 0$, and

$$\Omega_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^m \Gamma^{B_j}(b_j + \beta_j s) \prod_{j=1}^n \Gamma^{A_j}(1 - a_j - \alpha_j s)}{\prod_{j=n+1}^p \Gamma^{A_j}(a_j + \alpha_j s) \prod_{j=m+1}^q \Gamma^{B_j}(1 - b_j - \beta_j s)}. \quad (2)$$

If $\Delta > 0$, the integral (1) converges when $|\arg z| < \frac{1}{2}\Delta$; where

$$\Delta = \sum_{j=1}^m B_j \beta_j - \sum_{j=m+1}^q B_j \beta_j + \sum_{j=1}^n A_j \alpha_j - \sum_{j=n+1}^p A_j \alpha_j. \quad (3)$$

The generalized polynomials defined by Srivastava [23], is given in the following manner:

$$S_{N_1, \dots, N_s}^{M_1, \dots, M_s} [y_1, \dots, y_s] = \sum_{K_1=0}^{[N_1/M_1]} \dots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1 K_1}}{K_1!} \dots \frac{(-N_s)_{M_s K_s}}{K_s!} \\ \times A [N_1, K_1; \dots; N_s, K_s] y_1^{K_1} \dots y_s^{K_s}, \quad (4)$$

where $M_1, \dots, M_s$ are arbitrary positive integers; and the coefficients $A[N_1, K_1; \dots; N_s, K_s]$ are arbitrary constants (real or complex).

The multivariable I -function defined by Prathima et al. [14] (see also, [8]) is a extension of the multivariable H -function [5; 6; 22; 24]. It is defined in term of multiple Mellin-Barnes type integral, given by

$$I(z_1, \dots, z_r) = \int_{L_1}^{z_1} \dots \int_{L_r}^{z_r} \left[\begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j)_{1,p} : \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j)_{1,q} : \\ (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \end{matrix} \right] \quad (5)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \left\{ \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} \right\} ds_1 \dots ds_r, \quad (6)$$

where $\omega = \sqrt{-1}$; $\phi(s_1, \dots, s_r)$ and $\theta_i(s_i)$ for $i = 1, \dots, r$ are given as follows

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma^{A_j} (1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} s_i)}{\prod_{j=n+1}^p \Gamma^{A_j} (a_j - \sum_{i=1}^r \alpha_j^{(i)} s_i) \prod_{j=1}^q \Gamma^{B_j} (1 - b_j + \sum_{i=1}^r \beta_j^{(i)} s_i)}, \quad (7)$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} (1 - c_j^{(i)} + \gamma_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}} (d_j^{(i)} - \delta_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} (1 - d_j^{(i)} + \delta_j^{(i)} s_i)}, \quad (8)$$

for more details, reader can refer to recent work given by Prathima et al. [14].

Following the result of Braaksma [3], the I -function of r variables is analytic if

$$U_i = \sum_{j=1}^p A_j \alpha_j^{(i)} - \sum_{j=1}^q B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0 \quad (i = 1, \dots, r). \quad (9)$$

The integral (6) converges absolutely if

$$|\arg(z_k)| < \frac{1}{2} \Delta_k \pi \quad (k = 1, \dots, r), \text{ where}$$

$$\begin{aligned} \Delta_k = & - \sum_{j=n+1}^p A_j \alpha_j^{(k)} - \sum_{j=1}^q B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} \\ & - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0. \end{aligned} \quad (10)$$

The complex numbers z_i are not zero. Throughout this paper, we assume the existence and absolute convergence conditions of the multivariable I -function.

We will note

$$A = \left(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}; A_j \right)_{1,p}. \quad (11)$$

$$B = \left(b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}; B_j \right)_{1,q}. \quad (12)$$

$$C = \left(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)} \right)_{1,p_1}; \dots; \left(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)} \right)_{1,p_r}. \quad (13)$$

$$D = \left(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)} \right)_{1,q_1}; \dots; \left(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)} \right)_{1,q_r}. \quad (14)$$

The Mellin transform of $f(x)$ will be denoted by $M[f(x)]$ or $F(s)$. If p and y are real, we write $s = p^{-1} + iy$. If $p \geq 1$, $f(x) \in L_p(0, \infty)$, then for $p = 1$ we have

$$M[f(x)] = F(s) = \int_0^\infty x^{s-1} f(x) dx \quad \text{and} \quad f(x) = \frac{1}{2i\pi} \int_L F(s) x^{-s} ds. \quad (15)$$

$$\text{For } p > 0, \quad M[f(x)] = F(s) = l.i.m. \int_{1/x}^x x^{s-1} f(x) dx, \quad (16)$$

where, $l.i.m.$ denotes the usual limit in the mean for L_p -spaces.

2. DEFINITIONS

The pair of new extended fractional integral operators are defined by the following equations:

$$\begin{aligned}
 & \mathcal{Q}_{\gamma_n}^{\alpha, \beta} [f(x)] \\
 &= t x^{-\alpha-t\beta-1} \int_0^x y^\alpha (x-y)^{\beta} I \left(\begin{matrix} \gamma_1 v_1 \\ \vdots \\ \gamma_n v_n \end{matrix} \middle| \begin{matrix} A : C \\ \vdots \\ B : D \end{matrix} \right) \prod_{j=1}^k I_{M_j', N_j''}^{M_j', N_j''} \left[z_j \left(\frac{y^t}{x^t} \right)^{a_j} \left(1 - \frac{y^t}{x^t} \right)^{b_j} \right] \\
 & \quad \times \prod_{j=1}^r S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{y^t}{x^t} \right)^{g_1^{(j)}} \left(1 - \frac{y^t}{x^t} \right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{y^t}{x^t} \right)^{g_s^{(j)}} \left(1 - \frac{y^t}{x^t} \right)^{h_s^{(j)}} \end{matrix} \right) \psi \left(\frac{y^t}{x^t} \right) f(y) dy. \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 & R_{\gamma_n}^{\rho, \beta} [f(x)] \\
 &= t x^{\rho} \int_x^{\infty} y^{-\rho-t\beta-1} (y^t - x^t)^{\beta} I \left(\begin{matrix} \gamma_1 \mu_1 \\ \vdots \\ \gamma_n \mu_n \end{matrix} \middle| \begin{matrix} A : C \\ \vdots \\ B : D \end{matrix} \right) \prod_{j=1}^k I_{M_j', N_j''}^{M_j', N_j''} \left[z_j \left(\frac{x^t}{y^t} \right)^{a_j} \left(1 - \frac{x^t}{y^t} \right)^{b_j} \right] \\
 & \quad \times \prod_{j=1}^r S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{x^t}{y^t} \right)^{g_1^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{x^t}{y^t} \right)^{g_s^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_s^{(j)}} \end{matrix} \right) \psi \left(\frac{x^t}{y^t} \right) f(y) dy, \quad (18)
 \end{aligned}$$

where, $v_i = \left(\frac{y^t}{x^t} \right)^{u_i} \left(1 - \frac{y^t}{x^t} \right)^{v_i}$, $\mu_i = \left(\frac{x^t}{y^t} \right)^{u_i} \left(1 - \frac{x^t}{y^t} \right)^{v_i}$ and $t, u_i, v_i, g_i^{(j)}, h_i^{(j)}, a_j, b_j$ are positive numbers.

The kernels $\psi \left(\frac{y^t}{x^t} \right)$ and $\psi \left(\frac{x^t}{y^t} \right)$ appearing in (17) and (18) respectively, are assumed to be continuous functions such the integrals make sense for wide classes of function

$f(x)$.

The conditions for existence of these operators are as follows:

(a) $f(x) \in L_p(0, \infty)$, (b) $1 \leq p, q < \infty$, $p^{-1} + q^{-1} = 1$.

$$(c) \Re \left(\alpha + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -q^{-1}.$$

$$(d) \Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -q^{-1}.$$

$$(e) \Re \left(\rho + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -p^{-1} \text{ where } j = 1 \dots, k; j' = 1, \dots, M'_j.$$

Condition (a) ensures that both operators defined in (17) and (18) exist and belong to. These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [7], Love [13], Saigo et al. [17], Saxena and Kiryakova [18], Saxena and Kumbhat [20; 21], and etc.

3. MAIN RESULTS

THEOREM 1. If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also following conditions satisfied:

$$p^{-1} + q^{-1} = 1,$$

$$\Re \left(\alpha + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -q^{-1},$$

and the integrals present are absolutely convergent, then

$$M \{ Q_n^{\alpha, \beta} [f(x)] \} = M \{ f(x) \} R_n^{\alpha-s+1, \beta} [1], \quad (19)$$

where $M_p(0, \infty)$ stands for the class of all functions $f(x)$ of $L_p(0, \infty)$ with $p > 2$, which are inverse Mellin-transforms of the function of $L_p(-\infty, \infty)$.

PROOF. By making Mellin transform of (17), we get

$$M\{Q_{\eta}^{\alpha,\beta}[f(x)]\} = \int_0^\infty x^{s-1} \left\{ t x^{-\alpha-t\beta-1} \int_0^x y^\alpha (x'-y')^\beta I \left(\begin{matrix} \gamma_1 v_1 & A:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \gamma_n v_n & B:D \end{matrix} \right) \right. \\ \left. \prod_{j=1}^k \bar{I}_{M_j', N_j''}^{M_j', N_j'} \left[z_j \left(\frac{y'}{x'} \right)^{a_j} \left(1 - \frac{y'}{x'} \right)^{b_j} \right] S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{y'}{x'} \right)^{g_1^{(j)}} \left(1 - \frac{y'}{x'} \right)^{h_1^{(j)}} \\ \cdot \\ \cdot \\ z_s^{(j)} \left(\frac{y'}{x'} \right)^{g_s^{(j)}} \left(1 - \frac{y'}{x'} \right)^{h_s^{(j)}} \end{matrix} \right) f(y) dy \right\} dx. \quad (20)$$

On interchanging the order of integration, which is permissible under the conditions, the result (19) follows in view of (18).

THEOREM 2. If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also satisfied following conditions:

$$p^{-1} + q^{-1} = 1,$$

$$\Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\rho + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > -p^{-1},$$

and the integrals present are absolutely convergent, then

$$M\{R_{\eta}^{\rho,\beta}[f(x)]\} = M\{f(x)\} Q_{\eta}^{\rho+s-1,\beta}[1], \quad (21)$$

where $M_p(0, \infty)$ stands for the class of all functions $f(x)$ of $L_p(0, \infty)$ with $p > 2$, which are inverse Mellin-transforms of the function of $L_p(-\infty, \infty)$.

PROOF. By making Mellin transform of (18), we get

$$\begin{aligned}
 M\{R_{\gamma_n}^{\alpha,\beta}[f(x)]\} &= \int_0^\infty x^{s-1} \left\{ t x^p \int_x^\infty y^{-p-t\beta-1} (y^t - x^t)^\beta I \left(\begin{matrix} \gamma_1 \mu_1 \\ \vdots \\ \gamma_n \mu_n \end{matrix} \middle| \begin{matrix} A:C \\ \vdots \\ B:D \end{matrix} \right) \right. \\
 &\times \prod_{j=1}^k \tilde{I}_{M_j'', N_j''}^{M_j', N_j'} \left[z_j \left(\frac{x^t}{y^t} \right)^{a_j} \left(1 - \frac{x^t}{y^t} \right)^{b_j} \right] \\
 &\times \prod_{j=1}^r S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{x^t}{y^t} \right)^{g_1^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{x^t}{y^t} \right)^{g_s^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_s^{(j)}} \end{matrix} \right) \psi \left(\frac{x^t}{y^t} \right) f(y) dy \Bigg\} dx. \quad (22)
 \end{aligned}$$

THEOREM 3. If $f(x) \in L_p(0, \infty)$, $v(x) \in L_p(0, \infty)$, also satisfied following conditions:

$$p^{-1} + q^{-1} = 1,$$

$$\Re \left(\alpha + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > \max\{p^{-1}, q^{-1}\},$$

$$\Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\bar{\delta}_j^{(i)}} \right) > 0,$$

and the integrals present are absolutely convergent, then

$$\int_0^\infty v(x) \mathcal{Q}_{\gamma_n}^{\alpha,\beta}[f(x)] dx = \int_0^\infty f(x) R_{\gamma_n}^{\alpha,\beta}[v(x)] dx. \quad (23)$$

PROOF. The result of (23) can be obtained in view of (17) and (18).

4. INVERSION FORMULAS

THEOREM 4. If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also following conditions satisfied:

$$p^{-1} + q^{-1} = 1,$$

$$\Re \left(\alpha + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

and the integrals present are absolutely convergent, also we have

$$Q_{\gamma_n}^{\alpha, \beta} [f(x)] = v(x), \quad (24)$$

then

$$f(x) = \int_0^\infty y^{-1} [v(y)] \left[h \left(\frac{x}{y} \right) \right] dy, \quad (25)$$

where

$$h(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} y^{-1} \frac{x^{-s}}{R(s)} ds, \quad (26)$$

$$R(s) = R_{\gamma_n}^{\alpha-s+1, \beta} [1]. \quad (27)$$

PROOF. On taking Mellin transform of (24) and then applying Theorem 1, we get

$$M\{f(x)\} = \frac{M\{v(x)\}}{R(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{M\{v(x)\}}{R(s)} ds = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{R(s)} \left\{ \int_0^\infty [v(y)] dy \right\} ds.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{v(y)}{y} \left\{ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{y} \right)^s \frac{1}{R(s)} ds \right\} dy.$$

Now in view of (26), we obtain the desired result (25).

THEOREM 5. If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also satisfied following conditions:

$$p^{-1} + q^{-1} = 1,$$

$$\Re \left(\beta + t b_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\rho + t a_j \frac{b_{j'j}}{\beta_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -p^{-1},$$

and the integrals present are absolutely convergent, also we have

$$R_{\gamma_n}^{\rho, \beta} [f(x)] = w(x), \quad (28)$$

then

$$f(x) = \int_0^\infty y^{-1} [w(y)] \left[g \left(\frac{x}{y} \right) \right] dy, \quad (29)$$

where

$$g(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} y^{-1} \frac{x^{-s}}{T(s)} ds, \quad (30)$$

$$T(s) = Q_{\gamma_n}^{\rho+s-1, \beta} [1]. \quad (31)$$

PROOF. On taking Mellin transform of (28) and then applying Theorem 2, we get

$$M\{f(x)\} = \frac{M\{w(x)\}}{T(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{M\{w(x)\}}{T(s)} ds = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{T(s)} \left\{ \int_0^\infty [w(y)] dy \right\} ds.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{w(y)}{y} \left\{ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{y} \right)^s \frac{1}{T(s)} ds \right\} dy.$$

Now in view of (30), we obtain the desired result (29).

5. GENERAL PROPERTIES

The properties given below are consequences of the definitions (17) and (18).

$$x^{-1} Q_{\gamma_n}^{\alpha,\beta} \left[\frac{1}{x} f \left(\frac{1}{x} \right) \right] = R_{\gamma_n}^{\alpha,\beta} [f(x)], \quad (32)$$

$$x^{-1} R_{\gamma_n}^{\rho,\beta} \left[\frac{1}{x} f \left(\frac{1}{x} \right) \right] = Q_{\gamma_n}^{\rho,\beta} [f(x)], \quad (33)$$

$$x^\mu Q_{\gamma_n}^{\alpha,\beta} [f(x)] = Q_{\gamma_n}^{\alpha-\mu,\beta} [x^\mu f(x)], \quad (34)$$

$$x^\mu R_{\gamma_n}^{\rho,\beta} [f(x)] = R_{\gamma_n}^{\rho+\mu,\beta} [x^\mu f(x)]. \quad (35)$$

The properties given below express the homogeneity of operator Q and R respectively.

$$\text{If } Q_{\gamma_n}^{\alpha,\beta} [f(x)] = v(x) \text{ then } Q_{\gamma_n}^{\alpha,\beta} [f(cx)] = v(cx), \quad (36)$$

$$\text{If } R_{\gamma_n}^{\rho,\beta} [f(x)] = w(x) \text{ then } R_{\gamma_n}^{\rho,\beta} [f(cx)] = w(cx). \quad (37)$$

6. MULTIVARIABLE H -FUNCTION

If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I -function defined by Prathima et al. [14] reduces to multivariable H -function. We obtain the two following operators:

$$\begin{aligned} & Q_{\gamma_n}^{\alpha,\beta} [f(x)] \\ &= t x^{-\alpha-t\beta-1} \int_0^x y^\alpha (x' - y')^\beta H \left(\begin{matrix} \gamma_1 v_1 \\ \vdots \\ \gamma_n v_n \end{matrix} \middle| \begin{matrix} A : C \\ \vdots \\ B : D \end{matrix} \right) \prod_{j=1}^k I_{M_j', N_j'}^{M_j, N_j'} \left[z_j \left(\frac{y'}{x'} \right)^{a_j} \left(1 - \frac{y'}{x'} \right)^{b_j} \right] \\ & \quad \times \prod_{j=1}^r S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{y'}{x'} \right)^{g_1^{(j)}} \left(1 - \frac{y'}{x'} \right)^{h_1^{(j)}} \\ \vdots \\ z_s^{(j)} \left(\frac{y'}{x'} \right)^{g_s^{(j)}} \left(1 - \frac{y'}{x'} \right)^{h_s^{(j)}} \end{matrix} \right) \psi \left(\frac{y'}{x'} \right) f(y) dy, \quad (38) \end{aligned}$$

under the same notations and conditions that (17) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$.

$$\begin{aligned}
& R_{\gamma_n}^{\rho, \beta} [f(x)] \\
&= t x^\rho \int_x^\infty y^{-\rho-t\beta-1} (y^t - x^t)^\beta H \left(\begin{matrix} \gamma_1 \mu_1 & A : C \\ \cdot & \cdot \\ \cdot & \cdot \\ \gamma_n \mu_n & B : D \end{matrix} \right) \prod_{j=1}^k \tilde{T}_{M_j'', N_j''}^{M_j', N_j'} \left[z_j \left(\frac{x^t}{y^t} \right)^{a_j} \left(1 - \frac{x^t}{y^t} \right)^{b_j} \right] \\
&\quad \times \prod_{j=1}^r S_{N_1^{(j)}, \dots, N_s^{(j)}}^{M_1^{(j)}, \dots, M_s^{(j)}} \left(\begin{matrix} z_1^{(j)} \left(\frac{x^t}{y^t} \right)^{g_1^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_1^{(j)}} \\ \cdot \\ \cdot \\ z_s^{(j)} \left(\frac{x^t}{y^t} \right)^{g_s^{(j)}} \left(1 - \frac{x^t}{y^t} \right)^{h_s^{(j)}} \end{matrix} \right) \psi \left(\frac{x^t}{y^t} \right) f(y) dy, \quad (39)
\end{aligned}$$

under the same notations and conditions that (18) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$.

We obtain the same theorems and properties concerning these operators.

7. CONCLUSION

The functions involved in the results established in this paper are unified and general nature, hence a large number of known results lying in the literature follows as special cases. Further, on suitable specifications of the parameters involved, numerous new results involving simpler functions may also be obtained.

8. DISCLOSURE STATEMENT

No potential conflict of interest was reported by the authors.

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