Some fractional derivatives of *A*-function of multivariable

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Abstract

In the present paper, we study and develop Fractional derivatives of multivariable *A* – function. We derive two theorems which will act as the key formulas from which can obtain their special cases.

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1. INTRODUCTION

A number of earlier works on the subject of fractional calculus give interesting account of the theory and application of fractional calculus operators in many different areas of mathematical analysis. In this paper, we define the Fractional Derivatives involving A – function of multivariable and derive two main theorems involving Fractional Derivative of the product of A – function of multivariable and the Horn's function. Some new and known results are also established as special cases of our main results. The Fractional Derivative of the product of the multivariable A – function and Horn's function has not been established so far, and some new Fractional Derivative formulae for the product of the multivariable A – function and Horn's function are derived by making use of generalized Leibnitz rule. Recently, Berndt and Bowman [1], Chaurasia and Godika [2], Saxena [3], Tripathi et al [4] gives some integrals and series.

Gautam and Asgar [5, 6], Ram and Kumar [7], Srivastava and Panda [8] and several other authors have evaluated some definite and indefinite integrals involving the *A* – function of one, two and multivariables.

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2. DEFINITION OF FRACTIONAL DERIVATIVE

Following Oldham and Spainer [9], we define the (Riemann Liouville) fractional derivatives of a function f(x) of complex order ϑ or alternatively $(a-\vartheta)^{th}$ by the following

$$\alpha D_x^{\vartheta}\{f(x)\} = \begin{cases} \frac{1}{\Gamma(-\vartheta)}\int_{\alpha}^{x}(x-t)^{-\vartheta-1}f(y)dy\,, Re\,(\vartheta)<0, \\ \frac{d^n}{dx^n}\alpha D_x^{\vartheta-k}\{f(x)\}, 0\le Re\,(\vartheta)<n, \end{cases} \tag{2.1}$$

where n is a positive integer .

For simplicity, the special case of the Fractional Derivative Operator αD_x^{ϑ} when $\alpha = 0$ will be written as αD_x^{ϑ}. Thus, we have

$$0D_x^{\vartheta} = D_x^{\vartheta}. \tag{2.2}$$

3. MAIN RESULTS

THEOREM 1. If $\min\{\rho_{r,}\sigma_r\} > 0, |\arg(x/\xi)| < \pi$, Re (m) $+\rho_r$ min {Re $(b_{j,}\beta_j)$}$> -1\ (j=1,\ldots,r)$,$|z_r x^{\rho_r}| < r_1, |(x+\xi)^{\sigma_r} z_r| < r_2, r_1+r_2=1$; then

$$D_x^{\vartheta}\{x^m(x+\xi)^{\lambda} A_{p_r,q_{r:Y}}^{0.n_r;X}\left[\begin{matrix} z_1 x^{\rho_1}(x+\xi)^{\sigma_1} \\ \vdots \\ z_r x^{\rho_r}(x+\xi)^{\sigma_r}\end{matrix}\ \middle|\ \begin{matrix} ,\ldots,\ldots \\ ,\ldots,\ldots, \end{matrix}\right] G_1(\gamma,\delta,\delta' : Z_2\, x^{\rho_2}, (x+\xi)^{\sigma_2} z_3,\ldots,\ldots,z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} (z_2 x^{\rho_2})^r (z_3 \xi^{\sigma_2})^s z_r \xi^{\lambda} x^{m-\vartheta} \sum_{R=0}^{\infty} \frac{(x/\xi)^R}{(R)!}$$

$$A_{p_{r+2},q_{r+2:Y}}^{0.n_r+2;X}\left[\begin{matrix} z_1 \xi^{\sigma_1} x^{\rho_1} \\ \vdots \\ z_r \xi^{\sigma_r} x^{\rho_r}\end{matrix}\ \middle|\ \begin{matrix} (-\lambda-\sigma_2 s,\sigma_1,\ldots,\sigma_r),(-R-m-\rho_2 r,\rho_1,\ldots,\rho_r),\ldots,\ldots \\ \ldots,\ldots(R-\lambda-\sigma_2 s,\sigma_1,\ldots,\sigma_r)(\vartheta-m-R-\sigma_2 s,\rho_1,\ldots,\ldots,\rho_r) \end{matrix}\right]. \tag{3.1}$$

PROOF. We first replace the A – function of multivariable occurring on the left –hand – side by its Mellin –Barnes type contour integral and Horn's function G_1, and changing the order of integration and differentiation, which is readily justified in view of conditions stated above and collecting the powers of x and (x+ξ), we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} z_2{}^r z_3{}^s \frac{1}{(2\pi\omega)^r} \int_{L_1},\ldots,\int_{L_r} \emptyset(s_1,\ldots,s_r)\prod_{i=1}^{r}\emptyset_i\,(s_i) z_i{}^{s_i}$$

$$\{D_x^{v} x^{m+\rho_1 s+\rho_2 r}\alpha(x+\xi)^{\lambda+\sigma_1 s+\sigma_2 r})\}\, ds_1,\ldots,ds_r \tag{3.2}$$

Now, applying well known binomial expansion , we have

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!}\, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1},\ldots,\int_{L_r} \emptyset(s_1,\ldots,s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} \xi^{\lambda+\sigma_1 s+\sigma_2 r}$$

$$D_x^{v} x^{m+\rho_1 s+\rho_2 r} \sum_{R=0}^{\infty} \binom{\lambda+\sigma_1 s+\sigma_2 r}{R} \left(\frac{x}{\xi}\right)^R \mathrm{d}s_1,\ldots,ds_r\,. \tag{3.3}$$

Making use of the formula [the result Oldham and Spanier [9]], we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \frac{1}{(2\pi\omega)^r} \int_{L_1},\ldots,\int_{L_r} \emptyset(s_1,\ldots,s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} \xi^{\lambda+\sigma_1 s+\sigma_2 r-R}$$

$$\times \frac{\Gamma[1-(-\lambda-\sigma_2 s)+\sigma_1 s,\ldots,\sigma_r s]\,\Gamma[1-(-m-R-\rho_2 r)+\rho_1 s,\ldots,\rho_r s]}{(R)!\;\Gamma[1-(R-\lambda-\sigma_2 S)+\sigma_1 s,\ldots,\sigma_r s]\,\Gamma[1-(\vartheta-m-R-\rho_2 r)+\rho_1 s,\ldots,\rho_r s]}$$

$$\times (z_2 x^{\rho_2})^r (z_3 \xi^{\sigma_2})^s x^{m+\rho_1 s+\rho_2 r+R-\vartheta} {z_1}^s,\ldots,{z_r}^s \mathrm{d}s_1,\ldots,ds_r \tag{3.4}$$

If we interpret the resulting Mellin–Barnes contour integral as an A- function of multivariable, we shall arrive (3.1).

THEOREM 2. If $\min\{\rho_{r,}\sigma_r\} > 0, |\arg(-x/\xi)\ | < \pi$, Re (m) $+\rho_r$ min {Re $(\delta_{j,}\gamma_j)\} > -1\ (j=1,2,\ldots,r)$, $|z_2(x-\xi)^{\rho_r}| < r_1, |(\eta-x)^{\sigma_r} z_r| < r_2, r_1+r_2+r_n=1$; then

$$D_x^{\vartheta}\{(x-\xi)^{\lambda}(\eta-x)^{\lambda} A_{p_r,q_{r:Y}}^{0.n_r;X} \left[\begin{matrix} z_1(x-\xi)^{\rho_1}(\eta-\xi)^{\sigma_1} \\ \vdots \\ z_r x-\xi)^{\rho_r}(\eta-\xi)^{\sigma_r} \end{matrix} \left| \begin{matrix} \ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots \end{matrix} \right.\right]$$

$$G_1(\gamma,\delta,\delta' : Z_2\, x^{\rho_2}, (\text{x-}\xi)^{\sigma_2}, (\eta-x)^{\sigma_r} z_3, \ldots, \ldots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} \, [(z_2(-\xi)^{\rho_2}]^r\ [z_3\eta^{\sigma_2}]^s\ (\text{-}\xi)^m(\eta)^{\lambda}$$

$$\sum_{R_1=0}^{\infty}\sum_{R_2=0}^{\infty} \frac{x^{-\vartheta}\,(x/\xi)^{R_1}(x/\eta)^{R_2}}{(R_1)!(R_2)!} \frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)}$$

$$A_{p_{r+2,}q_{r+2:Y}}^{0.n_r+2;X} \left[\begin{matrix} z_1(-\xi)^{\rho_1}\eta^{\sigma_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\eta^{\sigma_r} \end{matrix} \left| \begin{matrix} (-\lambda-\rho_2 s,\rho_1,\ldots\ldots,\rho_r),(-\rho-\sigma_2 s,\sigma_1,\ldots,\ldots\sigma_r),\ldots,\ldots\ldots \\ \ldots,\ldots(R_1-m-\rho_2 r,\rho_1,\ldots,\ldots\rho_r)(R_2-\sigma_2 s,\sigma_1,\ldots,\ldots,\sigma_r) \end{matrix} \right.\right]. \tag{3.5}$$

PROOF. we first replace the A-function of several variable occurring on the left – hand side by its Mellin –Barnes type contour integral and Horn's function G_1 by its definition and changing the order of integration and differentiation, which is readily justified in view of conditions stated above and collecting the powers of (x-ξ) and (η-x), we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i}$$
$$\{D_x^{v}(x-\xi)^{m+\rho_1 s+\rho_2 r} \times (\eta - x)^{\lambda+\sigma_1 s+\sigma_2 r})\}\, \mathrm{d}s_1, \dots, ds_r \qquad (3.6)$$

Now, applying well known Binomial expansion, we have

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} (-\xi)^{m+\rho_1 s+\rho_2 r}$$
$$\times (\eta)^{\lambda+\sigma_1 s+\sigma_2 r} D_x^{v} \, \sum_{R_1=0}^{\infty} \binom{m+\rho_1 s+\rho_2 r}{R_1} \left(\frac{-x}{\xi}\right)^{R_1} \sum_{R_2=0}^{\infty} \binom{\lambda+\sigma_1 s+\sigma_2 r}{R_2} \left(\frac{-x}{\eta}\right)^{R_2} \}$$
$${z_1}^s, \dots, {z_r}^s \, \mathrm{d}s_1, \dots, ds_r \qquad (3.7)$$

Making the use of the formula [the result Oldham and Spanier [9]], we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \, {z_2}^r {z_3}^s (-\xi)^{m+\rho_1 s} (\eta)^{\lambda+\sigma_1 s} \sum_{R_1=0}^{\infty} \sum_{R_2=0}^{\infty} \frac{(-1)^{R_1+R_2} \left(\frac{x}{\xi}\right)^{R_1} \left(\frac{x}{\eta}\right)^{R_2} x^{-\vartheta}}{(R_1)!\,(R_2)!}$$
$$\frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)} \, \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_r) {z_i}^{s_r} (-\xi)^{\rho_1 s} (-\eta)^{\sigma_1 s}$$
$$\times \frac{\Gamma[1-(-m-\rho_2 r)+\rho_1 s, \dots, \rho_r s]\, \Gamma[1-(-\lambda-\sigma_2 s)+\sigma_1 s, \dots, \sigma_r s]}{\Gamma[1-(R_1-m-\rho_2 r)+\rho_1 s, \dots, \rho_r s]\, \Gamma[1-(R_2-\sigma_2 S)+\sigma_1 s, \dots, \sigma_r s]}$$
$${z_1}^s, \dots, {z_r}^s \, \mathrm{d}s_1, \dots, ds_r \;. \qquad (3.8)$$

If we interpret the resulting Mellin –Barnes contour integral as an A- function of multivariable, we shall arrive (3.5).

4. SPECIAL CASES OF (3.1) AND (3.5)

(1) Putting $\sigma_r \to 0$ another four Fractional Derivative formulae corresponding to (3.1) and (3.5):

$$D_x^{\vartheta} x^m (x\xi)^{\lambda} A^{0.n_r;X}_{p_r,q_{r:Y}} \left[\begin{matrix} z_1(-\xi)^{\rho_1}\eta^{\sigma_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\eta^{\sigma_r} \end{matrix} \middle| \begin{matrix} \dots,\dots \\ \dots,\dots \end{matrix} \right] G_1(\gamma, \delta, \delta' : Z_2\, x^{\rho_2}, (\mathrm{x}+\xi)^{\sigma_2} z_3, \dots, \dots, z_r)\}$$
$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} (z_2 x^{\rho_2})^r (z_3 \xi^{\sigma_2})^s z_r \xi^{\lambda} x^{m-\vartheta} \sum_{R=0}^{\infty} \frac{(x/\xi)^R}{(R)!}$$
$$\times \frac{\Gamma(1+\lambda+\sigma_2 s)}{\Gamma(1+\lambda+\sigma_2 s-R)} A^{0.n_r+1;X}_{p_{r+1},q_{r+1:Y}} \left[\begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{matrix} \middle| \begin{matrix} (-R-m-\rho_2 r, \rho_1, \dots, \rho_r), \dots, \dots \\ \dots, \dots (\vartheta-m-R-\sigma_2 s, \rho_1, \dots, \dots, \rho_r) \end{matrix} \right], \qquad (4.1)$$

$\min\{\rho_{r,}\} > 0, |\arg(x/\xi)| < \pi,$

Re (m) $+\rho_r$ min {Re $(\delta_{j,}\gamma_j)\} > -1$ $(j = 1, \ldots, r)$,

$|z_r x^{\rho_r}| < r_1, |(x+\xi)^{\sigma_r} z_r| < r_2, r_1 + r_2 +, \ldots, r_n = 1$;

$$D_x^{\vartheta}\{(x-\xi)^{\lambda}(\eta-x)^{\lambda} A^{0.n_r;X}_{p_r,q_{r:Y}}\left[\begin{matrix} z_1(x-\xi)^{\rho_1} \\ \vdots \\ z_r(x-\xi)^{\rho_r}\end{matrix}\left|\begin{matrix}\ldots,\ldots \\ \ldots,\ldots\end{matrix}\right.\right]$$

$$\times\, G_1(\gamma,\delta,\delta' : Z_2, (x-\xi)^{\rho_r}, (x-\eta)^{\sigma_r} z_3, \ldots, \ldots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} [(z_2(-\xi)^{\rho_2}]^r\, [z_3\eta^{\sigma_2}]^s\, (-\xi)^m(\eta)^{\lambda}$$

$$\sum_{R_1=0}^{\infty}\sum_{R_2=0}^{\infty} \frac{x^{-\vartheta}\,(x/\xi)^{R_1}(x/\eta)^{R_2}}{(R_1)!(R_2)!} \frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)} \times \frac{\Gamma(1+\lambda+\sigma_r s)}{\Gamma(1+\sigma_r s-R_2)}$$

$$A^{0.n_r+1;X}_{p_{r+1},q_{r+1:Y}}\left[\begin{matrix} z_1(-\xi)^{\rho_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\end{matrix}\left|\begin{matrix}(-m-\rho_2 r,\rho_1,\ldots,\ldots\rho_r),\ldots,\ldots\ldots \\ \ldots,\ldots(R_1-m-\rho_2 r_1,\rho_1,\ldots,\ldots\rho_r)(R_2-\sigma_2 s,\sigma_1,\ldots,\ldots,\sigma_r)\end{matrix}\right.\right], \tag{4.2}$$

$\min\{\rho_{r,}\} > 0, |\arg(-x/\xi)| < \pi,$

Re (m) $+\rho_r$ min {Re $(\delta_{j,}\gamma_j)\} > -1$ $(j = 1,2, \ldots, r)$,

$|z_2(x-\xi)^{\rho_r}| < r_1, |(\eta-x)^{\sigma_r} z_3| < r_2, r_1 + r_2 + r_n = 1$;

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