An extensive extension of exponentiated exponential distribution using alpha power transformation – statistical properties and applications in engineering science

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Abstract

In this article, an extension of exponentiated exponential distribution is familiarized by adding an extra parameter to the parent distribution using alpha power technique. The new distribution obtained is referred to as Alpha Power Exponentiated Exponential Distribution. Various statistical properties of the proposed distribution like mean, variance, central and non-central moments, reliability functions and entropies have been derived. Two real life data sets have been applied to check the flexibility of the proposed model. The new density model introduced provides the better fit when compared with other related statistical models.

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1. INTRODUCTION

The advancement of conventional distributions has become a usual exercise in statistical theory, since last few decades. Designing a new distribution from a classical one by adding an additional parameter using different methods has got wide scope in recent years. The main purpose of such developments is to make the classical distributions more flexible for complex data sets. Recently, Mahdavi A, Kundu D, introduced a new technique called Alpha Power Transformation (APT), where an additional parameter is introduced in continuous probability distributions. The Alpha Power Transformation is defined as:

Let $f(x)$ be the probability density function (pdf) of any continuous random variable 'X', then the pdf of Alpha Power Transformation is given by

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} \alpha^{F(x)} f(x), & \text{if } \alpha > 0, \alpha \neq 1 \\ f(x) & , \text{if } \alpha = 1 \end{cases} \quad (1.1)$$

And the corresponding CDF of APT is given as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{\alpha - 1} & , \text{if } \alpha > 0, \alpha \neq 1 \\ F(x) & , \text{if } \alpha = 1 \end{cases} \quad (1.2)$$

The APT of survival function $S_{APT}(x)$ is given by

$$S_{APT}(x) = \begin{cases} \frac{\alpha}{\alpha - 1} (1 - \alpha^{F(x)-1}) & , \text{if } \alpha \neq 1 \\ 1 - F(x) & , \text{if } \alpha = 1 \end{cases} \quad (1.3)$$

The APT of Hazard rate function $H_{APT}(x)$ is given by

$$H_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{1 - \alpha^{F(x)-1}} \log \alpha & , \text{if } \alpha \neq 1 \\ \frac{f(x)}{s(x)} & , \text{if } \alpha = 1 \end{cases} \quad (1.4)$$

The exponential distribution is the probability distribution of the time between events in a Poisson point process, which is described in probability theory and statistics as a process in which events occur continuously and independently at a constant average rate. It's a good example of how the gamma distribution works. It is the geometric distribution's continuous equivalent, and it has the essential virtue of being memory less. It is employed in a variety of additional applications in addition to the analysis of Poisson point processes. The exponential distribution is not the same as the exponential families of distributions, which is a vast class of probability distributions that contains the exponential distribution as one of its members, as well as the normal, binomial, gamma, Poisson, and many more.

The exponentiated exponential distribution formulated by Debasis Kundu is the two parameter right skewed unimodal distribution. It is the particular case of Gompertz-Verhulst distribution function. Therefore, a random variable 'X' is said to follow exponentiated exponential distribution if it follows the pdf

$$f(x; \beta, \lambda) = \beta \lambda (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \quad (1.5)$$

And the corresponding distribution function is given by

$$F(x; \beta, \lambda) = (1 - e^{-\lambda x})^\beta; x > 0 \quad (1.6)$$

2. ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

In this section, we introduce an extension of exponentiated exponential distribution by adding an extra parameter to the original distribution using the technique of alpha power transformation. Then the resultant distribution is known as Alpha Power Exponentiated Exponential Distribution (APEED).

Suppose 'X' be a random variable following Alpha Power Exponentiated Exponential distribution, then the PDF of 'X' is given as

$$f_{APEED}(x; \alpha, \lambda, \beta) = \frac{\beta\lambda}{\alpha-1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta}, \quad \alpha, \lambda, \beta > 0 \quad (2.1)$$

And the corresponding CDF is given by

$$F_{APEED}(x) = \frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1}, \quad \alpha, \lambda, \beta > 0 \quad (2.2)$$

ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

Fig 1. Shows the graph of pdf of APEE distribution for various values of α

Fig.2 Shows the graph of cdf of APEE Distribution

3. RELIABILITY ANALYSIS

In this section, we will obtain the survival function, hazard rate, reverse hazard rate, cumulative hazard rate and mills ratio of APEE distribution.

3.1. Survival function of APEE distribution

The survival function of APEED is defined as under

$$S_{APEED}(x) = 1 - F_{APEED}(x)$$

Substituting the value of $F_{APEED}(x)$ from equation (1.6) we get

$$S_{APEED}(x) = 1 - \frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1}$$

$$S_{APEED}(x) = \frac{\alpha - \alpha^{(1-e^{-\lambda x})^\beta}}{\alpha - 1} \quad (3.1)$$

3.2. Hazard Rate Function of APEE distribution

The hazard rate function of APEED is defined as under

$$H_{APEED}(x) = \frac{f(x)}{S(x)} = \frac{\lambda \beta \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta}}{\alpha - \alpha^{(1-e^{-\lambda x})^\beta}} \quad (3.2)$$

3.3. Reverse Hazard Rate Function of APEE distribution

The reverse hazard rate function of APEED is defined as under

$$RHR_{APEED}(x) = \frac{f(x)}{F(x)} = \frac{\beta \lambda \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta}}{\alpha^{(1-e^{-\lambda x})^\beta} - 1} \quad (3.3)$$

3.4. Cumulative Hazard Rate of APEE distribution

The cumulative hazard rate of APEED is defined as under

$$\begin{aligned} CHR_{APEED}(x) &= -\log[S(x)] \\ &= -\log \left[\frac{\alpha - \alpha^{(1-e^{-\lambda x})^\beta}}{\alpha - 1} \right] \\ &= (1 - e^{-\lambda x})^\beta \log \alpha \end{aligned} \quad (3.4)$$

3.5. Mills Ratio of APEE distribution

The mills ratio of APEED is defined as under

$$MR_{APEED}(x) = \frac{F(x)}{S(x)} = \frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - \alpha^{(1-e^{-\lambda x})^\beta}} \quad (3.5)$$

Plots below show the graphs of the survival function, hazard rate, reverse hazard rate, cumulative hazard rate of APEED

Fig 3. Shows SF of APEED

Fig 4. Shows HR of APEED

Fig 5. Shows CHR of APEED

Fig 6. Shows RHR of APEED

4. STRUCTURAL PROPERTIES OF APEED

In this section, structural properties of APEE distribution are derived. These structural properties include moments about mean, moments about origin, variance, standard deviation, coefficient of variation and index of dispersion.

THEOREM: Let 'x' be a random variable following APEED, then r^{th} moment denoted by μ_r' is given by

$$\begin{aligned}\mu_r' &= \int_0^\infty x^r f(x) dx \\ &= \int_0^\infty x^r f_{APEED}(x) dx \\ &= \int_0^\infty x^r \frac{\beta\lambda}{\alpha-1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx\end{aligned}$$

PROOF:

Take $(1 - e^{-\lambda x})^\beta = t$, then solving the above integral we get

$$\begin{aligned}\mu_r' &= \int_0^1 \left[-\frac{1}{\lambda} \log(1 - t^{\frac{1}{\beta}}) \right]^r \frac{\log \alpha}{\alpha-1} \alpha^t dt \\ &= \left(\frac{\log \alpha}{\alpha-1} \right) \frac{(-1)^r}{\lambda^r} \int_0^1 \alpha^t \log(1 - t^{\frac{1}{\beta}})^r dt \mu_r' = \frac{(-1)^r}{\lambda^r} \left[r \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \quad (4.1)\end{aligned}$$

Putting $r = 1, 2, 3$ and 4, we get the first four moments about origin as under

$$\mu_1' = \frac{-1}{\lambda} \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] = \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \quad (4.2)$$

$$\mu_2' = \frac{1}{\lambda^2} \left[2 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \quad (4.3)$$

$$\mu_3' = \frac{(-1)^3}{\lambda^3} \left[3 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] = \frac{1}{\lambda^3} \left[1 - 3 \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \quad (4.4)$$

$$\mu_4' = \frac{1}{\lambda^4} \left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \quad (4.5)$$

Therefore, Mean $\mu_1' = \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right]$

The moments about mean are derived by using the relationship between moments about origin and moments about mean

$$\begin{aligned}
\mu_2 &= \mu_2' - (\mu_1')^2 \\
&= \frac{1}{\lambda^2} \left[2 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\}^2 \\
&= \frac{1}{\lambda^2} \left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} \right]^2 - 2 \right] \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^2 \\
&= \frac{(-1)^3}{\lambda^3} \left[3 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] - 3 \left\{ \frac{1}{\lambda^2} \left[2 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \right\} \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\} \\
&\quad + 2 \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\}^2 \\
&= \frac{(-1)}{\lambda^3} \left[\left(3 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right) + 6 \left(\log \frac{1}{(1-e^{-\lambda x})^\beta} \right)^2 - 9 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 3 \right] + \\
&\quad \frac{2}{\lambda^2} \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} \right]^2 \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 \\
&= \frac{1}{\lambda^4} \left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \\
&\quad - 4 \left\{ \frac{(-1)^3}{\lambda^3} \left[3 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \right\} \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\} \\
&\quad - \log \frac{1}{(1-e^{-\lambda x})^\beta} \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\} \\
&\quad + 6 \left\{ \frac{1}{\lambda^2} \left[2 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 1 \right] \right\} \left\{ \frac{1}{\lambda} \left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right] \right\} \\
&= \frac{1}{\lambda^4} \left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 12 \left(\log \frac{1}{(1-e^{-\lambda x})^\beta} \right)^2 + 16 \log \frac{1}{(1-e^{-\lambda x})^\beta} - 5 \right] - \\
&\quad \frac{6}{\lambda^3} \left[2 \left(\log \frac{1}{(1-e^{-\lambda x})^\beta} \right)^2 - 3 \log \frac{1}{(1-e^{-\lambda x})^\beta} + 1 \right] \tag{4.8}
\end{aligned}$$

The standard deviation, coefficient of variation and index of dispersion are obtained as under

Standard deviation

$$\sigma = \sqrt{\text{variance}} = \frac{1}{\lambda} \sqrt{\left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} \right]^2 - 2 \right]} \quad (4.9)$$

$$\text{C.V} = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} \right]^2 - 2 \right]}}{\left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right]} \quad (4.10)$$

$$\text{Index of dispersion } \gamma = \frac{\sigma^2}{\mu_1'} = \frac{\frac{1}{\lambda} \left[4 \log \frac{1}{(1-e^{-\lambda x})^\beta} - \left[\log \frac{1}{(1-e^{-\lambda x})^\beta} \right]^2 - 2 \right]}{\left[1 - \log \frac{1}{(1-e^{-\lambda x})^\beta} \right]} \quad (4.11)$$

5. MOMENT GENERATING FUNCTION AND CHARACTERISTIC FUNCTION OF ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

In this section, moment generating function and characteristic function of APEED are derived. Moment Generating Function is generally defined as

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \int_0^\infty e^{tx} f(x) dx \\ &= \int_0^\infty e^{tx} \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx \end{aligned}$$

Using Taylors Expansion, we get

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{t^r x^r}{r!} \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r \frac{\beta\lambda}{\alpha-1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{(-1)^r}{\lambda^r} \left[r \log \frac{1}{(1-e^{-\lambda x})^{\beta}} - 1 \right] \right] \quad (5.1)$$

The characteristics function of APEED is generally defined as

$$\phi_x(t) = \int_0^{\infty} e^{itx} f(x) dx$$

Using Taylor expansion, we get

$$\begin{aligned} \phi_x(t) &= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(it)^r x^r}{r!} f(x) dx \\ &= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x) dx \\ &= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r \frac{\beta\lambda}{\alpha-1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} dx = \\ &\sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\left[\frac{(-1)^r}{\lambda^r} \left[r \log \frac{1}{(1-e^{-\lambda x})^{\beta}} - 1 \right] \right] \right] \end{aligned} \quad (5.2)$$

6. ENTROPIES

Shanon Entropy:

THEOREM: Let X be a random variable following APEED then Shanon entropy is defined as

$$H(x; \alpha, \lambda, \beta) = E(-\log f(x; \alpha, \lambda, \beta))$$

PROOF:

$$\begin{aligned} H(x; \alpha, \lambda, \beta) &= E \left[-\log \left(\frac{\beta\lambda}{\alpha-1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} \right) \right] \\ &= -E \left(\log \left(\frac{\beta\lambda}{\alpha-1} \right) \right) + (\beta - 1) E \left(\log (1 - e^{-\lambda x}) \right) + \log \alpha E (1 - e^{-\lambda x})^{\beta} - \\ &E [\log (\log \alpha)] - \lambda E(x) \end{aligned} \quad (6.1)$$

Now

$$\begin{aligned} E(1 - e^{-\lambda x})^\beta &= \int_0^\infty (1 - e^{-\lambda x})^\beta f(x) dx \\ &= \int_0^\infty (1 - e^{-\lambda x})^\beta \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx \end{aligned}$$

Taking $(1 - e^{-\lambda x})^\beta = t$, then solving the above integral we get

$$E(1 - e^{-\lambda x})^\beta = \left[\frac{\alpha}{\alpha-1} - \frac{\alpha}{(\alpha-1)\log \alpha} + \frac{1}{(\alpha-1)\log \alpha} \right] \quad (6.2)$$

and

$$\begin{aligned} E(1 - e^{-\lambda x}) &= \int_0^\infty (1 - e^{-\lambda x}) f(x) dx \\ &= \int_0^\infty (1 - e^{-\lambda x}) \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx \end{aligned}$$

Making the substitution of $(1 - e^{-\lambda x})^\beta = t$, then solving the above integral we get

$$E(1 - e^{-\lambda x}) = -\frac{1}{\beta} \quad (6.3)$$

Substituting the values of equation (6.2) and (6.3) in equation (6.1) we have

$$\begin{aligned} H(x; \alpha, \lambda, \beta) &= -\left(\log \left(\frac{\beta \lambda}{\alpha - 1} \right) \right) + (\beta - 1) \left(-\frac{1}{\beta} \right) + \log \alpha \left[\frac{\alpha}{\alpha - 1} - \frac{\alpha}{(\alpha - 1)\log \alpha} + \right. \\ &\quad \left. \frac{1}{(\alpha - 1)\log \alpha} \right] - \log(\log \alpha) - \lambda \left\{ \frac{-1}{\lambda} \left[\log \frac{1}{(1 - e^{-\lambda x})^\beta} - 1 \right] \right\} \\ &= \log \frac{1}{(1 - e^{-\lambda x})^\beta} + \frac{1}{\alpha - 1} \{ \alpha(\log \alpha - 1) + 1 \} - \log \left(\frac{\beta \lambda}{\alpha - 1} \right) - \log(\log \alpha) + \frac{1}{\beta} - 2 \end{aligned} \quad (6.4)$$

RENYI ENTROPY

$$I_{\delta}(x) = (1 - \delta)^{-1} \log \int_{-\infty}^{\infty} \{f_{APEED}(x; \alpha, \lambda, \beta)\}^{\delta} dx \quad (6.5)$$

Substituting the value of $f_{APEED}(x; \alpha, \lambda, \beta)$ from equation (2.1) in equation (6.5) we have

$$I_{\delta}(x) = (1 - \delta)^{-1} \log \int_0^{\infty} \left\{ \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta - 1} e^{-\lambda x} \alpha^{(1 - e^{-\lambda x})^{\beta}} \right\}^{\delta} dx$$

Putting $(1 - e^{-\lambda x})^{\beta} = t$ in the above equation, we get

$$I_{\delta}(x) = (1 - \delta)^{-1} \log \left[\frac{\log \alpha}{\alpha - 1} \right]^{\delta} \int_0^1 \alpha^{\delta t} dt$$

Solving the above integral we get

$$I_{\delta}(x) = (1 - \delta)^{-1} \left\{ \log \left[\frac{\log \alpha}{\alpha - 1} \right]^{\delta} - \log \left[\frac{\alpha^{\delta} - 1}{\delta \log \alpha} \right] \right\} \quad (6.6)$$

7. ORDER STATISTICS

Suppose $X_1, X_2, \dots, X_n$ be n random samples of size n from APEE distribution with pdf (2.1) and cdf (2.2), then pdf of k^{th} order statistics is given by

$$f_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x) \quad (7.1)$$

Substituting the values from equation (2.1) and (2.2) we get the pdf of k^{th} order statistics as

$$f_{X(k)}(x) = \left[\frac{n!}{(k-1)!(n-k)!} \left[\frac{\alpha^{(1 - e^{-\lambda x})^{\beta} - 1}}{\alpha - 1} \right]^{k-1} \left[1 - \frac{\alpha^{(1 - e^{-\lambda x})^{\beta} - 1}}{\alpha - 1} \right]^{n-k} \frac{\beta \lambda}{\alpha - 1} \log \alpha (1 - e^{-\lambda x})^{\beta - 1} e^{-\lambda x} \alpha^{(1 - e^{-\lambda x})^{\beta}} \right] \quad (7.2)$$

The pdf of 1st order statistics is derived by putting $k=1$ in the above equation

$$f_{x(1)}x = \frac{n!}{(1-1)!(n-1)!} \left[\frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1} \right]^{1-1} \left[1 - \frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1} \right]^{n-1} \frac{\beta\lambda}{\alpha - 1} \log\alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta}$$

Solving the above equation we get

$$f_{x(1)}x = \frac{n\lambda\beta\log\alpha}{(\alpha-1)^n} \left[\alpha - \alpha^{(1-e^{-\lambda x})^\beta} \right] \left[\alpha^{(1-e^{-\lambda x})^\beta} e^{-\lambda x} (1 - e^{-\lambda x})^{\beta-1} \right] \quad (7.3)$$

And the pdf of nth order statistics is derived by putting $k = n$ in equation (7.2), we get

$$f_{x(n)}x = \left[\frac{n!}{(n-1)!(n-n)!} \left[\frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1} \right]^{n-1} \left[1 - \frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1} \right]^{n-n} \frac{\beta\lambda}{\alpha - 1} \log\alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} \right]$$

Solving above equation we get

$$f_{x(n)}x = n \left[\frac{\alpha^{(1-e^{-\lambda x})^\beta} - 1}{\alpha - 1} \right]^{n-1} \frac{\beta\lambda}{\alpha - 1} \log\alpha (1 - e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} \quad (7.4)$$

Application in Real Data Analysis:

In this section, we'll look at how to apply the proposed distribution to a real-world situation. As a result, two genuine data sets are employed, and analyses are carried out using R software (1.4.1717).

The first set of data concerns the breaking stress of carbon fibres with a length of 50 mm (GPa). Nichols and Padgett (2006), Cordeiro and Lemonte (2011), and Al-Aqtashet al. had already used the data (2014). The following is the information:

0.39 0.85 1.08 1.25 1.47 1.57 1.61 1.61 1.69 1.80 1.84 1.87 1.89 2.03 2.03
2.05 2.12 2.87 2.88 2.93 2.95 2.96 2.97 3.09 3.11 3.11 3.15 3.15 3.19 3.22
3.22 3.27 3.28 3.31 2.35 2.41 2.43 2.48 2.50 2.53 2.55 2.55 2.56 2.59 2.67
2.73 2.74 2.79 2.81 2.82 2.85 3.31 3.33 3.39 3.39 3.56 3.60 3.65 3.68 3.70
3.75 4.20 4.38 4.42 4.70 4.90

Table 1 presents a summary of the data.

DATA	Min	Mean	Median	variance	1 st Qu.	3 rd Qu.	max
1	0.390	2.760	2.835	0.7947	2.178	3.277	4.900

Table 2. Performance of the distributions:

Distribution	AIC	BIC	CAIC	HQIC
IE	274.057	276.247	274.247	277.787
GIE	203.240	207.619	203.431	204.971
EE	194.745	199.124	194.935	196.475
L	246.768	248.958	246.959	250.499
APEE_{pro}	194.320	198.699	194.510	196.050

Table 3. The estimation of parameters for the first data set:

Distribution	Estimated Parameters
IE	2.2992439
GIE	13.2850451 7.6013295
EE	9.199203 1.007550
L	0.59025384
APEE_{pro}	11.3002074 1.3687862

The strength of 1.5 cm glass fibres is the subject of the second data set. Smith and Naylor (1987) and Bourguignon et al. (1987) utilised data originally gathered by employees at the UK National Physical Laboratory (2014). The following is the information:

1.42 1.48 1.48 1.49 1.49 1.50 1.50 1.51 1.52 1.53 1.54 1.55 1.55 1.58 1.59
 1.60 1.61
 0.55 0.74 0.77 0.81 0.84 0.93 1.04 1.11 1.13 1.24 1.25 1.27 1.28 1.29 1.30
 1.36 1.39

1.76 1.76 1.77 1.78 1.81 1.82 1.84 1.84 1.89 2.00 2.01 2.24 1.61 1.61 1.61
1.62 1.62 1.63 1.64 1.66 1.66 1.66 1.67 1.68 1.68 1.69 1.70 1.70 1.73

Table 4 presents a summary of the data.

DATA	Min	Mean	Median	variance	1 st Qu.	3 rd Qu.	max
2	0.550	1.507	1.590	0.1050	1.375	1.685	2.240

Table 5. Performance of the distributions:

Distribution	AIC	BIC	CAIC	HQIC
IE	180.878	183.021	181.078	184.564
GIE	64.243	68.529	64.443	65.929
EE	66.767	71.053	66.967	68.453
L	164.557	166.700	164.757	168.243
APEE _{pro}	55.323	59.609	55.523	57.009

Table 6. The estimation of parameters for the second data set:

05	Estimated Parameters
IE	1.4083964
GIE	163.206558 8.1508240
EE	31.348914 2.611571
L	0.99611636
APEE _{pro}	39.0846906 3.5083968

CONCLUSION

In this article, a modified distribution namely Alpha Power Exponentiated Exponential Distribution is derived. Various structural properties including mgf, cf, moments about mean, moments about origin, variance, standard deviation, and index of dispersion are obtained. Apart from the above properties some reliability measures of the said distribution are drawn. These reliability measures contain survival

function, hazard rate function, reverse hazard rate, cumulative hazard rate and mills ratio. Moreover Shannon's entropy and Renyi entropy of the new distribution are derived. Finally the newly developed distribution was applied to the two real life data sets which show APEE distribution resulted better than the models compared to it.

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