Integral transforms for three-dimensional pumping in confined, leaky, and unconfined aquifers

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Abstract: Analytical or hybrid numerical-analytical solutions based on the Generalized Integral Transform Technique (GITT) are obtained for the transient three-dimensional pumping problem of aquifers with a fully penetrating vertical well between two parallel streams. The problem formulation for confined and leaky aquifers allows for achieving exact analytical solutions through integral transforms, while the unconfined aquifer case introduces a fourth kind boundary condition which leads to a coupled transformed head ordinary differential system, that can be solved either analytically or numerically. A convergence analysis is performed to illustrate the consistency of the numerical results achieved for the head distribution, as well as for the related pumping rates. Results are obtained for selected cases and comparisons with literature results are performed. A solution verification confirms the agreement of the integral transform solutions with available simulations and provides additional confidence for the analysis of a few physical parameters that influence the hydrological behavior.

Keywords: Integral transforms; Hybrid methods; Aquifer pumping problem; Head distribution; Stream depletion rate.

INTRODUCTION

The water collected in wells near streams can reduce the flow in a stream through a process known as streamflow depletion. This problem has become particularly important in recent years, especially around densely populated regions, due to the significant increase in water resources demands for supplying large cities, large agricultural areas, and other uses, such as processing industries or mining activities (Knowling and Werner, 2016; Loizeau et al., 2017; Pastore et al., 2017). For this reason, regulatory agencies seek to establish the maximum groundwater pumping for wells located near streams. The pumping near streams causes depletion of the river to the aquifer, characterized by a stream depletion rate (SDR). Thus, more realistic solutions are required, leading to a better understanding of the SDR transient behavior under the pumping effect. According to Huang et al. (2014), the SDR only begins after the drawdown cone reaches the streambed once pumping has started. In the steady state, the water pumped out of the aquifer is solely from the infiltration of rivers through their semi-permeable beds.

The first studies about SDR were developed by Theis (1941). Glover and Balmer (1954) found a solution to the same problem, written in terms of the complementary error function, and the Theis solution was then renamed the Glover-Balmer solution. Hantush (1965) attempted to make the Theis problem slightly more realistic by aligning the edge flow with a semi-permeable aquitard of semi-infinite length. Hunt (1999) obtained a solution to the two-dimensional problem of a partially penetrating stream with infinite length in a homogeneous Theis' aquifer, and later on, Hunt (2001) compared his solution to experimental results. Butler et al. (2001) proposed a solution to the depletion aquifer with semi-infinite geometry, caused by a

full penetration vertical well adjacent to a limited lateral extension stream, i.e., the stream and aquifer both have finite width. The contributions reviewed above divide the domain into two different regions that contain a stream between them, while the governing equation for each region was determined by coupling the head and flow continuity at the interfaces between the intermediate zone and the lateral zones.

Other models of aquifers admit horizontal wells and have been developed for draining, which have been increasingly common, such as in the works of Hantush and Papadopoulos (1962), Zhan and Zlotnik (2002), Chen et al. (2003), Zhan and Park (2003), Sun and Zhan (2006), Huang et al. (2011), Huang et al. (2012a), and Huang et al. (2016).

Most of the analytical work has been developed in the semi-infinite domain, and the solutions are commonly given in the time or Laplace-domains. Analytical solutions for multilayer configurations or more general heterogeneous aquifer problems remain scarce in the literature. However, some researchers include the assumption of soil heterogeneity as layers of different physical properties, such as in Butler et al. (2007), Malama et al. (2007), Hunt (2009), and Ward and Lough (2011).

Strack (1999, 2003), Bakker and Strack (2003), Bakker (2004), and Bakker et al. (2005) used the analytic elements method for the mathematical modeling of groundwater flow. The technique resembles the classical boundary element method but differs through the analytical nature of the chosen solutions. Sun and Zhan (2007) modelled the two-dimensional problem in a fully penetrating vertical well located between two parallel streams, simulating the streambed effect as a low-permeability contour, analyzing how the streambed affects the stream depletion rate, and obtaining the solution by Laplace transform. Similarly, Intaraprasong and Zhan (2007) applied

the Fourier transform to obtain the solution of the two-dimensional steady-state problem. Huang et al. (2012b) solved the problem of a two-dimensional leaky aquifer, while Huang et al. (2014) presented a model of the three-dimensional problem to describe the groundwater flow induced by a fully penetrating vertical well on aquifers between two parallel streams. A general formulation was considered to represent the upper contour boundary condition directly applicable to the confined, leaky, or unconfined aquifer cases. The solution proposed by Huang et al. (2014) is given by Laplace transform, and inversions for the time domain are obtained analytically for the confined case. However, for the leaky and unconfined cases, the Padé approximant is used to obtain the solutions, which are then compared to a numerical solution obtained by Crump's method.

There is no questioning about the advantages of obtaining analytical solutions for partial differential equations, once the specific problem formulation allows for, due to the inherent full precision control and the resulting minimal computational cost. The availability of an analytical expression directly offers information at any specific location of the domain and mathematical operations such as differentiation and integration are readily implemented. Besides, analytical solutions permit the inspection of asymptotic trends and parametric dependences with very little or no numerical calculation effort. Although numerical methods are, in principle, more flexible and straightforward, numerical error propagation and instabilities may eventually impair (or at least bring up costly mesh refinement) the universal use of the traditional computational approaches, such as in problems with asymptotic behaviors and/or abrupt variations. For this reason, hybrid numerical-analytical approaches find a niche of application, providing a bridge between classical analytical methods and modern numerical analysis. The most successful hybrid methods lead to computational-analytical algorithms that still offer error control within mild computational effort. The range of problems that can provide sets of benchmark independent results for verification of numerical codes is then naturally and widely extended. In addition, once enough flexibility is incorporated into the hybrid approach algorithm, it may serve itself as an alternative computational path for different classes of problems. Here, the formulation proposed by Huang et al. (2014) is revisited but a different solution path is considered through application of the Generalized Integral Transform Technique (GITT) (Cotta, 1990, 1993, 1994, 1998; Cotta and Mikhailov, 1997, 2006; Cotta et al., 2013, 2014, 2018a, 2018b; Santos et al., 2001). The GITT methodology consists of reducing partial differential equations (PDE) to an infinite system of coupled ordinary differential equations (ODE) for the transformed potentials through integral transformation, which is then either solved analytically for certain linear problems, thus yielding an exact solution, or numerically for both coupled linear or nonlinear problems, then providing a hybrid numerical-analytical solution. Besides the general reference books and review papers above mentioned, a few previous works are here pointed out that employed the GITT on streams and groundwater flow and contaminant dispersion problems, such as Almeida and Cotta (1995, 1996a), Liu et al. (2000), Cotta et al. (2003), Barros et al. (2006), Barros and Cotta (2007), Naveira-Cotta et al. (2013), Cotta et al. (2018c). Also, the nature of the source terms here to be analyzed and the three-dimensional geometry under consideration require some sort of convergence acceleration techniques for the computational implementation of the proposed eigenfunction expansions. Therefore, filtering solutions are employed (Almeida and Cotta,

1996b; Cotta, 1993; Cotta and Mikhailov, 1997; Cotta et al., 2013, 2014), together with reordering strategies for multiple expansions (Correa et al., 1997; Cotta et al., 2013, 2014; Mikhailov and Cotta, 1996), and the resulting convergence behavior is critically analyzed.

MATHEMATICAL FORMULATION

The mathematical model that describes the pumping problem using a fully penetrating vertical well is the same as proposed by Huang et al. (2014). The governing equation for a three-dimensional transient hydraulic head distribution, $h(x, y, z, t)$, in a homogeneous and anisotropic aquifer, induced by a fully penetrating vertical well positioned at point (x_0, y_0) , is expressed in dimensionless form as:

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + k_v \frac{\partial^2 h}{\partial z^2} - \delta(x-1)\delta(y-y_0) \quad (1a)$$

$$0 < x < w_x, 0 < y < w_y, 0 < z < 1, \text{ and } t > 0$$

$$h(x, y, z, 0) = 0 \quad (1b)$$

$$\frac{\partial h(0, y, z, t)}{\partial x} - k_1 h(0, y, z, t) = 0; \quad (1c, d)$$

$$\frac{\partial h(w_x, y, z, t)}{\partial x} + k_2 h(w_x, y, z, t) = 0$$

$$\frac{\partial h(x, 0, z, t)}{\partial y} = 0; \quad \frac{\partial h(x, w_y, z, t)}{\partial y} = 0 \quad (1e, f)$$

$$\frac{\partial h(x, y, 0, t)}{\partial z} = 0; \quad (1g, h)$$

$$\frac{\partial h(x, y, 1, t)}{\partial z} + \frac{\sigma}{k_v} \frac{\partial h(x, y, 1, t)}{\partial t} + k' h(x, y, 1, t) = 0$$

where the following dimensionless groups were adopted:

$$\begin{aligned} h &= \frac{K_h H}{Q} h^*; & x &= \frac{x^*}{x_0^*}; & y &= \frac{y^*}{x_0^*}; & z &= \frac{z^*}{H}; \\ y_0 &= \frac{y_0^*}{x_0^*}; & t &= \frac{K_h}{S_s x_0^{*2}} t^*; & \sigma &= \frac{S_y}{S_s H}; \\ k_v &= \frac{K_v x_0^{*2}}{H^2 K_h}; & k' &= \frac{K' H}{B' K_v}; & k_1 &= \frac{K_1 x_0^*}{B_1 K_h}; \\ k_2 &= \frac{K_2 x_0^*}{B_2 K_h}; & w_x &= \frac{W_x}{x_0^*}; & w_y &= \frac{W_y}{x_0^*} \end{aligned} \quad (2a-m)$$

The boundary condition, Eq. (1h), offers three different aquifer models:

- Confined aquifer, if $\sigma = 0$ and $k' = 0$;
- Leaky aquifer, if $\sigma = 0$ and $k' > 0$;
- Unconfined aquifer, if $\sigma > 0$ and $k' = 0$.

Equation (1) governs the three-dimensional transient head behavior in an aquifer with thickness H and specific storage S_s that contains a fully penetrating vertical well to a drainage rate Q between two parallel streams of negligible width. The streams near the well are partially isolated by the streambeds, with the conductivities and thicknesses denoted by (K_1, B_1) and (K_2, B_2) , respectively, as proposed by Sun and Zhan (2007) and

Intaraprasong and Zhan (2007). The aquifer has a finite width in the x and y directions, denoted by W_x and W_y , respectively. The proposed model also allows for the existence of an aquitard of thickness B' with conductivity K' between the aquifer and an overlying adjacent aquifer. The well is located at (x_0^*, y_0^*) , as shown schematically in Fig. 1(a), indicating the well horizontal position. The z -direction of the aquifer depth has a top boundary condition capable of defining the three types of aquifers, as mentioned above: confined, Fig. 1(b), leaking, Fig. 1(c), and unconfined, Fig. 1(d), depending on the conductivity of the aquitard K' between aquifers and their specific yield S_y . When the conductivity K' is exceptionally low, the aquitard becomes an aquiclude and the aquifer is then confined. The model describes an anisotropic medium. Therefore, the hydraulic conductivity on the aquifer is equal on the x and y directions, K_h , but different in the z -direction, K_v .

Equation (1h) generalizes the boundary condition at the top that models the three different types of aquifers: confined, leaky and unconfined for the problem here analyzed, where S_y is the specific yield of unconfined aquifers, K' is the vertical hydraulic conductivity of the connected aquitard of thickness B . The general boundary condition (1h) simulates a leaky aquifer if the specific yield $S_y = 0$, containing an aquitard of width B with conductivity $K' > 0$ between the aquifer and an overlying aquifer. It also models a confined aquifer if the aquitard conductivity is too low, i.e., $K' = 0$. When K' is low, the aquitard becomes an aquiclude or an impermeable layer, passing to a non-flow condition and if no have specific yield, $S_y = 0$, models a confined aquifer. Finally, if $S_y > 0$ and $K' = 0$, the equation models an unconfined aquifer, disregarding the existence of an aquitard among the adjacent aquifers, making a single aquifer.

SOLUTION METHODOLOGY

In this work, the Generalized Integral Transform Technique (GITT), Cotta (1990, 1993, 1994, 1998), Cotta and Mikhailov (1997, 2006), Santos et al. (2001), Cotta et al. (2013, 2014, 2018a, 2018b), is applied to the proposed model. The GITT automatically reduces to the classical integral transform method (Özsisik, 1980; Mikhailov and Özsisik, 1984) once the transformed problem becomes a decoupled linear ODE system. Otherwise, when an exact analytical solution cannot be recovered, the generalized approach still provides hybrid numerical-analytical solutions based on the numerical solution of a truncated transformed ODE system.

For convergence rates improvement, the first step in the GITT formalism is the proposition of a filtering solution to reduce the importance of source terms in the eigenfunction expansions convergence. It is well known that optimal convergence rates in eigenfunction expansions are achieved in homogeneous problems, such as in the solutions achieved through separation of variables, characterized by summation terms with an exponential decay with the squared eigenvalue as argument. For nonhomogeneous problems, this exponential behavior does not prevail, and a typical rational pattern in terms of the inverse of the squared eigenvalues can markedly slow down the overall expansion convergence. Thus, it pays back to filter the original problem with known analytical solutions of simpler formulations that account for the source terms in the equations and boundary conditions, so as to reduce the relative importance of these nonhomogeneous terms in the final expression for the expansion terms. For this purpose, a filter is here proposed that represents the steady state solution of the three-dimensional head distribution, in the form:

Fig. 1. Schematic representation of hydraulic head distribution in an aquifer between two rivers being drained by a fully penetrating vertical well: (a) well position; (b) confined aquifer; (c) leaky aquifer; (d) unconfined aquifer.

$$h(x, y, z, t) = h_H(x, y, z, t) + h_f(x, y, z) \quad (3)$$

Writing the filter formulation and substituting Eq. (3) into Eqs. (1), the two distinct problems are obtained as follows:

– Proposed filter problem:

$$\frac{\partial^2 h_f}{\partial x^2} + \frac{\partial^2 h_f}{\partial y^2} + k_v \frac{\partial^2 h_f}{\partial z^2} = \delta(x-1)\delta(y-y_0) \quad (4a)$$

$$0 < x < w_x, \quad 0 < y < w_y, \quad \text{and} \quad 0 < z < 1$$

$$\frac{\partial h_f(0, y, z)}{\partial x} - k_1 h_f(0, y, z) = 0; \quad (4b, c)$$

$$\frac{\partial h_f(w_x, y, z)}{\partial x} + k_2 h_f(w_x, y, z) = 0$$

$$\frac{\partial h_f(x, 0, z)}{\partial y} = 0; \quad \frac{\partial h_f(x, w_y, z)}{\partial y} = 0 \quad (4d, e)$$

$$\frac{\partial h_f(x, y, 0)}{\partial z} = 0; \quad \frac{\partial h_f(x, y, 1)}{\partial z} + k' h_f(x, y, 1) = 0 \quad (4f, g)$$

– Filtered homogeneous problem:

$$\frac{\partial h_H}{\partial t} = \frac{\partial^2 h_H}{\partial x^2} + \frac{\partial^2 h_H}{\partial y^2} + k_v \frac{\partial^2 h_H}{\partial z^2} \quad (5a)$$

$$0 < x < w_x, \quad 0 < y < w_y, \quad 0 < z < 1, \quad \text{and} \quad t > 0,$$

$$h_H(x, y, z, 0) = -h_f(x, y, z) \quad (5b)$$

$$\frac{\partial h_H(0, y, z, t)}{\partial x} - k_1 h_H(0, y, z, t) = 0; \quad (5c, d)$$

$$\frac{\partial h_H(w_x, y, z, t)}{\partial x} + k_2 h_H(w_x, y, z, t) = 0$$

$$\frac{\partial h_H(x, 0, z, t)}{\partial y} = 0; \quad \frac{\partial h_H(x, w_y, z, t)}{\partial y} = 0 \quad (5e, f)$$

$$\frac{\partial h_H(x, y, 0, t)}{\partial z} = 0; \quad (5g, h)$$

$$\frac{\partial h_H(x, y, 1, t)}{\partial z} + \frac{\sigma}{k_v} \frac{\partial h_H(x, y, 1, t)}{\partial t} + k' h_H(x, y, 1, t) = 0$$

Homogeneous problem solution

The solution for the homogeneous problem is obtained through an extension of the GITT approach (Cotta, 1993). In principle, an exact integral transform solution could be derived from the general theory for fourth and fifth kind boundary conditions as first proposed in (Mikhailov and Özisik, 1985), once the corresponding three-dimensional eigenvalue problem is obtained by Separation of Variables from Eqs. (5) and then solved by the GITT itself. Here choosing a simpler three-dimensional eigenvalue problem that is uncoupled in the three coordinates, which allows for the straightforward definition of the three-dimensional eigenfunction expansion, given by:

$$\frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} + k_v \frac{\partial^2 \psi_i}{\partial z^2} + \mu_i^2 \psi_i(x, y, z) = 0 \quad (6a)$$

$$\frac{\partial \psi_i(0, y, z)}{\partial x} - k_1 \psi_i(0, y, z) = 0; \quad (6b, c)$$

$$\frac{\partial \psi_i(w_x, y, z)}{\partial x} + k_2 \psi_i(w_x, y, z) = 0$$

$$\frac{\partial \psi_i(x, 0, z)}{\partial y} = 0; \quad \frac{\partial \psi_i(x, w_y, z)}{\partial y} = 0 \quad (6d, e)$$

$$\frac{\partial \psi_i(x, y, 0)}{\partial z} = 0; \quad \frac{\partial \psi_i(x, y, 1)}{\partial z} + (k' - \frac{\sigma}{k_v} \gamma_r^2) \psi_i(x, y, 1) = 0 \quad (6f, g)$$

The problem described above is called the Sturm-Liouville problem and its solution is obtained through expansions of eigenvalues that equally satisfy the boundary conditions. Its solution is later obtained through the superposition of the linearly independent eigenfunctions in a series that converges when the number of used eigenvalues is sufficiently large. The three-dimensional eigenfunction $\psi_i(x, y, z) = X_m(x)Y_p(y)Z_r(z)$ corresponds to the product of the one-dimensional eigenfunctions in directions x , y and z , and the three indices combination provides the single index i ($m, p, r \rightarrow i$), and the corresponding eigenvalue of the three-dimensional eigenfunction becomes the sum of the squared eigenvalues corresponding to each one-dimensional eigenfunction, $\alpha_m^2 + \beta_p^2 + \gamma_r^2 = \mu_i^2$. The separated one-dimensional eigenvalue problems and their corresponding solutions are shown in Table 1. It should be noticed that the eigenvalue problem separation was possible due to the choice of writing the boundary condition, Eq. (6g), only with the eigenvalue corresponding to the z coordinate auxiliary eigenvalue problem, instead of the global eigenvalue μ .

To reduce computational costs, a reordering scheme of the eigenvalues was implemented according to Mikhailov and Cotta (1996) and Cotta and Mikhailov (1997), reducing the 3D combination of eigenvalues $\alpha_m^2 + \beta_p^2 + \gamma_r^2$ to the single squared eigenvalue μ_i^2 . The reordering scheme reduces the indices, selecting the smallest values obtained from the sums of the squared one-dimensional eigenvalues. This procedure is based on the principle that the convergence of such series depends mainly on the lower positive arguments that govern the exponential behavior of the time-dependent term (Cotta and Mikhailov, 1997), thus, the appropriate order of choosing these combinations is from lowest to highest values of the t -variable exponential arguments. The major contribution of this procedure is to reduce the multiple summations resulting from the transformation to a single summation dependent on a single index. An algorithm for reordering eigenvalues was developed by Mikhailov and Cotta (1996) for solving multi-dimensional diffusion problems. Cotta and Mikhailov (1997) also described this process applied to heat transfer problems using symbolic computation to obtain these sorted eigenvalues.

The presence of the squared eigenvalue in the boundary condition, Eq. (6g), makes problem (6) a non-classical eigenvalue problem, which requires the definition of a modified orthogonality condition, as proposed in Mikhailov and Özisik (1985). Then, problem (6) requires the definition of the following integral transform pair:

$$\bar{h}_i(t) = \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i(x, y, z) h_H(x, y, z, t) dz dy dx + \sigma \int_0^{w_x} \int_0^{w_y} \psi_i(x, y, 1) h_H(x, y, 1, t) dy dx, \quad \text{transform} \quad (7a)$$

$$h_H(x, y, z, t) = \sum_{i=1}^{\infty} \tilde{\psi}_i(x, y, z) \bar{h}_i(t), \text{ inverse} \quad (7b)$$

where

$$N_i = Nx_m \cdot Ny_p \cdot Nz_r, \quad \tilde{\psi}_i(x, y, z) = \frac{\psi_i(x, y, z)}{N_i} \quad (7c, d)$$

The integral transformation is accomplished through operation over Eq. (5a) and the initial condition (5b) employing both components of the transformation formula, Eq. (7a), through

the transformation kernel, $\int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i(x, y, z) dz dy dx$, which is the

operator used for equation transform as

$$\begin{aligned} \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i \frac{\partial h_H}{\partial t} dz dy dx &= \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i \frac{\partial^2 h_H}{\partial x^2} dz dy dx + \\ &+ \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i \frac{\partial^2 h_H}{\partial y^2} dz dy dx + k_v \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i \frac{\partial^2 h_H}{\partial z^2} dz dy dx \\ \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i h_H(x, y, z, 0) dz dy dx &= - \int_0^{w_x} \int_0^{w_y} \int_0^1 \psi_i h_f(x, y, z) dz dy dx \end{aligned}$$

Table 1. Eigenvalue problems in the x, y and z directions.

x-direction	
Eigenvalue problem	$\frac{d^2 X_m}{dx^2} + \alpha_m^2 X_m(x) = 0$ $\frac{dX_m(0)}{dx} - k_1 X_m(0) = 0; \quad \frac{dX_m(w_x)}{dx} + k_2 X_m(w_x) = 0$
Eigenquantities	$X_m(x) = \alpha_m \cos(\alpha_m x) + k_1 \sin(\alpha_m x); \quad \tan(\alpha_m w_x) = \frac{\alpha_m (k_1 + k_2)}{(\alpha_m^2 - k_1 k_2)}, \quad m = 1, 2, 3, \dots$ $\int_0^{w_x} X_m(x) X_n(x) dx = \begin{cases} 0, & \text{if } m \neq n \\ Nx_m, & \text{if } m = n \end{cases}$ $Nx_m = \int_0^{w_x} X_m^2(x) dx = \frac{\alpha_m^2 + k_1}{2} \left(w_x + \frac{k_2}{\alpha_m^2 + k_2} \right) + \frac{k_1}{2}$
y-direction	
Eigenvalue problem	$\frac{d^2 Y_p}{dy^2} + \beta_p^2 Y_p(y) = 0$ $\frac{dY_p(0)}{dy} = 0; \quad \frac{dY_p(w_y)}{dy} = 0$
Eigenquantities	$Y_p(y) = \cos(\beta_p y); \quad \beta_p = \frac{(p-1)\pi}{w_y}, \quad p = 1, 2, 3, \dots$ $\int_0^{w_y} Y_p(y) Y_q(y) dy = \begin{cases} 0, & \text{if } p \neq q \\ Ny_p, & \text{if } p = q \end{cases}$ $Ny_p = \int_0^{w_y} Y_p^2(y) dy = \begin{cases} w_y, & \text{if } p = 1 \\ w_y/2, & \text{if } p > 1 \end{cases}$
z-direction	
Eigenvalue problem	$\frac{d^2 Z_r}{dz^2} + \frac{\gamma_r^2}{k_v} Z_r(z) = 0$ $\frac{dZ_r(0)}{dz} = 0; \quad \frac{dZ_r(1)}{dz} + [k' - \frac{\sigma \gamma_r^2}{k_v}] Z_r(1) = 0$
Eigenquantities	$Z_r(z) = \cos(\gamma_r z / k_v^{1/2}); \quad \tan(\gamma_r / k_v^{1/2}) = \sqrt{k_v} [k' - \frac{\sigma \gamma_r^2}{k_v}] / \gamma_r, \quad r = 1, 2, 3, \dots$ $\int_0^1 Z_r(z) Z_s(z) dz + \sigma Z_r^2(1) = \begin{cases} 0, & \text{if } r \neq s \\ Nz_r, & \text{if } r = s \end{cases}$ $Nz_r = \int_0^1 Z_r^2(z) dz + \sigma Z_r^2(1) = \begin{cases} 1 + \sigma, & \text{if } r = 1 \\ \frac{1}{4} \left[2 + \frac{\sqrt{k_v}}{\gamma_r} \sin(2\gamma_r / k_v^{1/2}) \right] + \sigma \cos^2(\gamma_r / k_v^{1/2}), & \text{if } r > 1 \end{cases}$

After the usual manipulations, the following transformed ordinary differential system is obtained:

$$\frac{d\bar{h}_i(t)}{dt} + \mu_i^2 \bar{h}_i(t) = \sum_{j=1}^{\infty} A_{ij} \bar{h}_j(t) \quad (8a)$$

$$\bar{h}_i(0) = \bar{f}_i \quad (8b)$$

where

$$A_{ij} = -\sigma(\alpha_{m(i)}^2 + \beta_{p(i)}^2) \delta_{m(i)n(j)} \delta_{p(i)q(j)} Z_{r(i)}(1) Z_{s(j)}(1) / N_{z_{s(j)}} \quad (8c)$$

$$\bar{f}_i = -\int_0^1 \tilde{Z}_{m(i)p(i)}(z) Z_{r(i)}(z) dz - \sigma \tilde{Z}_{m(i)p(i)}(1) Z_{r(i)}(1) \quad (8d)$$

$$\mu_i^2 = \alpha_{m(i)}^2 + \beta_{p(i)}^2 + \gamma_{r(i)}^2 \quad (8e)$$

and

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}; \quad \delta_{pq} = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}; \quad \delta_{rs} = \begin{cases} 0, & r \neq s \\ 1, & r = s \end{cases} \quad (9a-c)$$

The derivation of Eq. (8d) is detailed in the Appendix. The function $\tilde{Z}_{m(i)p(i)}(z)$ is obtained from the filter solution developed in the next section.

Filter problem solution

The filter problem is also solved by GITT (Cotta, 1993), by applying a two-dimensional eigenfunction expansion, leaving the spatial variable z untransformed, thus recovering a decoupled ODE system in this coordinate for the transformed filter potential. The two-dimensional eigenvalue problem is obtained from the three-dimensional problem given by Eqs. (6) with solution described in Table 1, retaining only the x and y -directions, and the two-dimensional eigenfunction is then given as $\phi_i(x, y) = X_m(x) Y_p(y)$. The two indices associated with each coordinate are then merged into a single index, as $(m, p) \rightarrow i^*$ or $\alpha_{m(i^*)}^2 + \beta_{p(i^*)}^2 = \lambda_{i^*}^2$, and the appropriate sequence combination is obtained by reordering the eigenvalues. The one-dimensional Sturm-Liouville problems and respective solutions are the same presented in Table 1 for the x and y -directions.

The integral transformation of problem (4) is achieved through the following integral transform pair:

$$\tilde{Z}_{i^*}(z) = \int_0^{w_x} \int_0^{w_y} \phi_{i^*}(x, y) h_f(x, y, z) dy dx, \text{ transform} \quad (10a)$$

$$h_f(x, y, z) = \sum_{i^*=1}^{\infty} \tilde{\phi}_{i^*}(x, y) \tilde{Z}_{i^*}(z), \text{ inverse} \quad (10b)$$

where

$$N_{i^*} = N_{x_m} N_{y_p}, \quad \tilde{\phi}_{i^*}(x, y) = \phi_{i^*}(x, y) / N_{i^*} \quad (10c, d)$$

The integral transformation is accomplished through operation over Eq. (4a) and the boundary condition (4f, g) with the

transformation kernel $\int_0^{w_x} \int_0^{w_y} \phi_{i^*}(x, y) dy dx$. After the usual manipulations, the following transformed ordinary differential system is obtained:

$$k_v \frac{d^2 \tilde{Z}_{i^*}(z)}{dz^2} - \lambda_{i^*}^2 \tilde{Z}_{i^*}(z) = \tilde{G}_{i^*} \quad (11a)$$

$$\frac{d\tilde{Z}_{i^*}(0)}{dz} = 0; \quad \frac{d\tilde{Z}_{i^*}(1)}{dz} + k' \tilde{Z}_{i^*}(1) = 0 \quad (11b, c)$$

where

$$\begin{aligned} \tilde{G}_{i^*} &= \int_0^{w_x} \int_0^{w_y} \phi_{i^*}(x, y) \delta(x-1) \delta(y-y_0) dy dx = \phi_{i^*}(1, y_0) = \\ &= X_{m(i^*)}(1) Y_{p(i^*)}(y_0) \end{aligned} \quad (11d)$$

The decoupled system of ordinary differential equations (Eqs. (11)) has analytical solution in the form:

$$\tilde{Z}_{i^*}(z) = \frac{\tilde{G}_{i^*}}{k_v \lambda_{i^*}^2} \left[\frac{k' \cosh(v_{i^*} z)}{v_{i^*} \sinh(v_{i^*}) + k' \cosh(v_{i^*})} - 1 \right] \quad (12a)$$

where

$$v_{i^*} = \lambda_{i^*} / \sqrt{k_v} \quad (12b)$$

Therefore, invoking Eq. (3) and the inverse formulas (7b) and (10b), the original problem solution is given by:

$$h(x, y, z, t) = \sum_{i=1}^{\infty} \tilde{\Psi}_i(x, y, z) \bar{h}_i(t) + \sum_{i^*=1}^{\infty} \tilde{\phi}_{i^*}(x, y) \tilde{Z}_{i^*}(z) \quad (13)$$

Eqs. (8) consist of a coupled infinite system of linear differential equations for the computation of the transformed potentials, $\bar{h}_i(t)$. It is important to note that for $\sigma = 0$ the solution for the transformed system may be analytically obtained as:

$$\bar{h}_i(t) = \bar{f}_i e^{-\mu_i^2 t} \quad (14)$$

For computational purposes in the general case of $\sigma \neq 0$, the expansion is then truncated to NT terms, with NT sufficiently high to ensure the targeted accuracy in each case. The analytical solution of Eqs. (8) in the general case can still be achieved by the proper algebraic matrix eigenvalue problem analysis, or alternatively, employing the matrix exponential function directly available in modern symbolic computation platforms, such as the *Mathematica* system (Wolfram, 2016). Here, projecting the extension of this theory to nonlinear situations such as in partially saturated porous media, a numerical approach to handle the transformed ODE system has been preferred. Several numerical methods are described in the literature for solving initial value problems such as that given by Eqs. (8). In particular, the numerical subroutine DIVPAG from the IMSL Library (2018). The DIVPAG subroutine solves an initial-value problem for ordinary differential equations using either Adams-Moulton's or Gear's BDF method. It is a well-established method used on the solution of stiff systems of ODEs. It is part of

the IMSL Library written in the Fortran programming language. Such a subroutine is well-tested on the integration of this type of problem. Alternatively, the built-in routine NDSolve of the *Mathematica* system (Wolfram, 2016) can be considered, which was the routine employed for the case of $\sigma \neq 0$ in the present work. In light of the hybrid numerical-analytical nature of the proposed methodology, it is a perfect match to employ a mixed symbolic-numerical software system, such as *Mathematica* (Wolfram, 2016), for implementing the computational algorithm. First, all analytical derivations can be symbolically performed, such as the eigenvalue problem solutions, integrations, coefficients matrix assemblies, and ordinary differential equations solver automatic routines. Second, once the transformed system does not allow for an analytical solution, the numerical solver (NDSolve function) can be readily invoked in place of the symbolic one (DSolve function). It should be added that the NDSolve function automatically provides an interpolating function object that yields a continuous representation of the potentials along with the associated independent variable.

Calculation of pumping rates

The rates involved in pumping from the aquifer can be obtained by integrating Eq. (1a) over the volume $dV_c = dz dy dx$. Based on Darcy's law, the stream depletion rate SDR is the flow in the adjacent streambeds to the aquifer and can be calculated through

$$SDR_1 = - \int_0^{w_y} \int_0^1 \frac{\partial h(x, y, z, t)}{\partial x} \Big|_{x=0} dz dy; \quad (15a, b)$$

$$SDR_2 = \int_0^{w_y} \int_0^1 \frac{\partial h(x, y, z, t)}{\partial x} \Big|_{x=w_x} dz dy$$

The pumping system of a well also depends on the contribution of other sources, namely:

1 – The Storage Release Rate (SRR) is the volume of water released from the aquifer for the total volume pumped.

2 – The Leakage Rate (LR), which is related to when a semi-confined aquifer is overlapped by another (separated by an aquitard with a different conductivity of the two); there is a flow of water between them, which can happen from the overlying aquifer to the underlying aquifer or vice-versa.

3 – The Gravity Drainage Rate (GDR) is caused on the upper surface by decreasing the soil pressure, leading to empty release.

The integrals below define these pumping rates:

$$SRR = - \int_0^{w_x} \int_0^{w_y} \int_0^1 \frac{\partial h(x, y, z, t)}{\partial t} dz dy dx;$$

$$LR = k_v \int_0^{w_x} \int_0^{w_y} \frac{\partial h(x, y, z, t)}{\partial z} \Big|_{z=1} dy dx; \quad (15c-e)$$

$$GDR = -\sigma \int_0^{w_x} \int_0^{w_y} \frac{\partial h(x, y, z, t)}{\partial t} \Big|_{z=1} dy dx$$

In leaky aquifers, this pumping receives the contribution of the adjacent aquifer leakage rate LR , Eq. (15d), whereas in an unconfined aquifer, this contribution is given by the gravity drainage rate GDR , Eq. (15e). Therefore, the pumping rates are

calculated replacing the head distribution, Eq. (13), into Eqs. (15), resulting in:

$$SDR_1 = - \sum_{i=1}^{\infty} \bar{f}_i \frac{B_i C_i}{N x_{m(i)} N y_{p(i)} N z_{r(i)}} \frac{dX_{m(i)}(0)}{dx} \bar{h}_i(t) - \sum_{i^*=1}^{\infty} \frac{B_{i^*}}{N x_{k(i^*)} N y_{l(i^*)}} \frac{\tilde{G}_{i^*}}{\lambda_{i^*}^2} \left[\frac{k' C_{i^*}}{v_{i^*} \sinh(v_{i^*}) + k' \cosh(v_{i^*})} - 1 \right] \frac{dX_{k(i^*)}(0)}{dx} \quad (16a)$$

$$SDR_2 = \sum_{i=1}^{\infty} \bar{f}_i \frac{B_i C_i}{N x_{m(i)} N y_{p(i)} N z_{r(i)}} \frac{dX_{m(i)}(w_x)}{dx} \bar{h}_i(t) + \sum_{i^*=1}^{\infty} \frac{B_{i^*}}{N x_{k(i^*)} N y_{l(i^*)}} \frac{\tilde{G}_{i^*}}{\lambda_{i^*}^2} \left[\frac{k' C_{i^*}}{v_{i^*} \sinh(v_{i^*}) + k' \cosh(v_{i^*})} - 1 \right] \frac{dX_{k(i^*)}(w_x)}{dx} \quad (16b)$$

$$SRR = \sum_{i=1}^{\infty} \bar{f}_i \frac{B_i C_i D_i}{N x_{m(i)} N y_{p(i)} N z_{r(i)}} \frac{d\bar{h}_i(t)}{dt} \quad (16c)$$

$$LR = k_v \sum_{i=1}^{\infty} \bar{f}_i \frac{B_i D_i}{N x_{m(i)} N y_{p(i)} N z_{r(i)}} \frac{dZ_{r(i)}(1)}{dz} \bar{h}_i(t) + k_v \sum_{i^*=1}^{\infty} \frac{B_{i^*} D_{i^*} \tilde{G}_{i^*}}{\lambda_{i^*}^2 N x_{k(i^*)} N y_{l(i^*)}} \frac{k' v_{i^*} \sinh(v_{i^*})}{[v_{i^*} \sinh(v_{i^*}) + k' \cosh(v_{i^*})]} \quad (16d)$$

$$GDR = -\sigma \sum_{i=1}^{\infty} \frac{B_i D_i Z_{r(i)}(1)}{N x_{m(i)} N y_{p(i)} N z_{r(i)}} \frac{d\bar{h}_i(t)}{dt} \quad (16e)$$

where

$$B_i = \int_0^{w_y} Y_{p(i)}(y) dy; \quad C_i = \int_0^1 Z_{r(i)}(z) dz; \quad D_i = \int_0^{w_x} X_{m(i)}(x) dx \quad (16f-k)$$

$$B_{i^*} = \int_0^{w_y} Y_{l(i^*)}(y) dy; \quad C_{i^*} = \int_0^1 \cosh(v_{i^*} z) dz; \quad D_{i^*} = \int_0^{w_x} X_{k(i^*)}(x) dx$$

In Eqs. (16a–e), for $\sigma = 0$, the transformed potential $\bar{h}_i(t)$ is obtained from Eq. (14).

RESULTS AND DISCUSSION

The input parameters that were used in reporting the results are listed in Table 2. The eigenfunction expansions were truncated to a finite number of terms NT in the summations associated with the i index and MT in the summations of the i^* index. Table 3 illustrates the reordering scheme used for reducing double and triple summations into single sums for the first 20 eigenvalues.

The truncation order of MT terms for calculating the steady filter problem was set to a fixed sufficiently large value for all cases studied here, since it has little effect on the total computational cost. The coupled system of ordinary differential equations for the filtered transformed potentials, Eqs. (8), were numerically solved for variable truncation orders NT . Then, a thorough convergence analysis was performed to verify the behavior of the proposed integral transform solution. It was observed that the proposed solution can accurately reproduce the depletion rates in stream 1, including the behavior for short

Table 2. Dimensional parameters data modified from Huang et al. (2014).

Notations	Default value (unit)
K_h, K_v	1 m/day, 0.1 m/day
S_{ys}, S_s	0.1, 10^{-5} m^{-1}
H, W_x, W_y	20 m, 500 m, 500 m
T, S	$20 \text{ m}^2/\text{day}$, 2×10^{-4}
K', B'	0.01 m/day, 1 m
K_1, B_1	1 m/day, 1 m
K_2, B_2	1 m/day, 1 m
x_0^*, y_0^*	50 m, 250 m
Q	none

Table 3. Eigenvalues reordering scheme for the double and triple eigenfunction expansions ($k_2 = 0$ and $w_x = 10$).

Double summation reordering				Triple summation reordering				
m	p	i^*	λ_{i^*}	m	p	r	i	μ_i
1	1	1	0.156766	1	1	1	1	0.156766
1	2	2	0.351101	1	2	1	2	0.351101
2	1	3	0.470298	2	1	1	3	0.470298
2	2	4	0.565576	2	2	1	4	0.565576
1	3	5	0.647580	1	3	1	5	0.647580
3	1	6	0.783831	3	1	1	6	0.783831
2	3	7	0.784834	2	3	1	7	0.784834
3	2	8	0.844444	3	2	1	8	0.844444
1	4	9	0.955427	1	4	1	9	0.955427
3	3	10	1.004577	3	3	1	10	1.004577
2	4	11	1.053302	2	4	1	11	1.053302
4	1	12	1.097363	4	1	1	12	1.097363
4	2	13	1.141447	4	2	1	13	1.141447
3	4	14	1.225828	3	4	1	14	1.225828
4	3	15	1.264512	1	1	2	15	1.252677
1	5	16	1.266378	4	3	1	16	1.264512
2	5	17	1.341759	1	5	1	17	1.266378
5	1	18	1.410896	1	2	2	18	1.291471
5	2	19	1.445449	2	1	2	19	1.328836
4	4	20	1.446537	2	5	1	20	1.341759

periods, as can be seen in Table 4. Convergence analysis of the SDR_1 for $k_2 = 0$ and $w_x = 10$ (unconfined aquifer) is briefly illustrated in Table 4, at different dimensionless times t , varying the number of terms in the expansion, NT. Note that full convergence to three significant digits, even at the shortest time $t = 20$ or $t^* = 0.5$ day, is obtained for NT always less than 1000, with a marked improvement of convergence rate for larger values of t . The complete set of dimensionless parameters employed in this simulation is: $w_x = w_y = 10$, $k_1 = 50$, $k_2 = 0$, $k_v = 0.625$, $k' = 5$, $\sigma = 500$, $x_0 = 1$, $y_0 = 5$.

Figure 2 shows the SDR_1 evolution for the confined, leaky and unconfined aquifer cases obtained through Eq. (16a). Previous approaches obtain analytical solution only confined case, $\sigma = 0$ and $k' = 0$. However, the present work proposed an analytical solution which include the leakage aquifer case, $\sigma = 0$ and $k' \neq 0$. The transformed potential $\bar{h}_i(t)$ is only numerically obtained from Eqs. (8) for unconfined case $\sigma \neq 0$. The transformed potentials are obtained from the numerical solution of the system of ODEs given in Eq. (8a), subject to the transformed initial conditions given in Eq. (8b). The present results show excellent agreement with previously reported approximate analytical solutions (Huang et al., 2014). Figure 3 shows

Fig. 2. SDR_1 distribution for confined, leaky and unconfined aquifers.

Fig. 3. SDR_1 , SRR and GDR distributions for an unconfined aquifer.

the rates SDR_1 , SRR and GDR for $k' = 0$ and $\sigma \neq 0$ as obtained from Eqs. (16a), (16c) and (16e), respectively. Again, it is verified an excellent agreement between the present results shown in Figure 3 with those of reference works. Here, it is essential to mention that for unconfined aquifers, at any time, the sum of the GDR , SRR , and SDR is equal to unity. Therefore, in Fig. 3, we observe SRR beginning to decrease from a value near one until it reaches a zero value approximately in the interval $1 \leq t \leq 100$, and the GDR reaches its maximum. This location is precisely the first mode place for the SDR . As both SRR and GDR decline after this interval, the SDR continues to increase until the second mode occurs when the pumping reaches a steady state.

Figure 4 shows the effect of varying the distance between streams 1 and 2, with $k_2 \neq 0$, on the time variation of SDR_1 and SDR_2 , where it can be noted how the SDR is affected by the

Fig. 4. SDR_1 and SDR_2 distributions for an unconfined aquifer varying w_x and for $k_1 = k_2 = 50$.

distance between the well and the streambed. Miller et al. (2007) discussed the relationship between the depletion rate and the distance between the well and the streams, showing that there is an inverse relationship between them, i.e., SDR increases faster when w_x decreases. It is worth noting that the increase in distance between streams makes the depletion rate tend to resemble the non-flow case in streambed 2 or $SDR_2 \rightarrow 0$ when $w_x > 50$ (or $W_x > 2500$ meters for the employed parameters). As can be seen in Figs. 2–4 the stream depletion rate reaches steady state when $t > 10^5$ and the head distribution remains constant from this moment, since from this point on the well pumping has contribution only from SDR_1 and SDR_2 . In Fig. 4 we may also note that increasing the river distances, keeping the bed position fixed, for $w_x > 10$ ($W_x > 500$ m), the SDR in the streambed two is only started after approximated $t > 10^3$ ($t^* > 25$ days). Increasing w_x to 30 ($W_x > 1500$ m, i.e., the streambed two is 1450 m far from the well), the SDR is started only when $t > 10^4$ ($t^* > 250$ days).

Figure 5 shows the dimensionless head profiles for an unconfined aquifer in the x -direction at $y = y_0$, with $k_2 = 0$ and $w_x = 10$. Fig. 5a shows a head variation profile varying in time for a constant specific yield, $S_y = 0.1$, whereas Fig. 5b shows the head profile for fixed time $t = 10^3$, with S_y ranging from 0.001 to 0.3. We noted that the increase of the time, $t > 10^3$, and decrease in the specific yield, $S_y < 0.1$, implies the confinement of the aquifer. Based on Fig. 3, near $t = 10^3$, the $GDR = SDR$ and after they tend to zero while SDR changes to behave like a confined aquifer. The recovery reduction of the aquifer can

Fig. 5. Dimensionless head profiles for an unconfined aquifer as functions of the x -direction at $y = y_0$ and $z = 1$ with $k_2 = 0$ and $w_x = 10$ for: (a) different times with $S_y = 0.1$; (b) for different specific yields S_y with $t = 10^3$.

explain this behavior due to the action of gravity, which would represent the reduction in its recharging capacity. As shown in Fig. 5, this occurs immediately, decreasing the specific yield or prolonging pumping times.

Table 4. Convergence behavior of the SDR_1 for $k_2 = 0$ and $w_x = 10$ for the case of an unconfined aquifer.

t	t^* (days)	$NT = 300$	$NT = 400$	$NT = 500$	$NT = 600$	$NT = 700$	$NT = 800$	$NT = 900$	$NT = 1000$
20	0.5	0.215	0.247	0.244	0.247	0.250	0.251	0.248	0.248
50	1.25	0.237	0.252	0.259	0.261	0.263	0.263	0.263	0.263
100	2.5	0.272	0.279	0.285	0.287	0.288	0.288	0.289	0.289
500	12.5	0.468	0.468	0.468	0.468	0.468	0.468	0.468	0.468
1000	25	0.592	0.592	0.592	0.592	0.592	0.592	0.592	0.592
5000	125	0.810	0.810	0.810	0.810	0.810	0.810	0.810	0.810

Fig. 6. Dimensionless head profiles at $z = 0$ for $k_1 = k_2 = 50$ and $w_x = 10$: (a) $t = 10^3$; (b) $t = 10^5$.

Fig. 7. Dimensionless head profiles for an unconfined aquifer in the z -direction at $y = y_0$ for $x \in [0, w_x/2]$ for different times: (a) $t = 10^2$; (b) $t = 10^3$; (c) $t = 10^4$.

Since the source term representing the vertical well is the Dirac delta equation, it consists of an infinite pulse located at (x_0, y_0) . Depending on its intensity, it acts with equal distribution throughout the domain in the z -direction. For this reason, the pressures h in the z -direction vary only in the initial pumping times and become constant for long times, changing only in the other directions involved, as shown in Figs. 6 and 7. In Figure 6, we note that the variable horizontal pressure behavior in anisotropic medium, and we can identify the expansion of the drawdown cone varying in space. Fig 7 demonstrates that the pressure in the z -direction ceases to exert influence on the system for $t \geq 10^3$.

CONCLUDING REMARKS

The present work proposes analytical and hybrid (numerical-analytical) solutions through the Generalized Integral Transform Technique (GITT) to obtain the transient three-dimensional hydrological behavior of a limited aquifer between two parallel streams. A semipermeable streambed defined by boundary conditions of the third type and no flow condition in the direction parallel to the streams is considered, that is drained by a vertical, fully penetrating well and with an upper boundary condition that models so much a confined, leaky, or unconfined aquifer. For accelerate the convergence of this eigenfunction expansion solution, a steady-state filter was used to eliminate the source term in the general equation of the head, associated with the well. The filter problem solution was obtained through integral transforms as well, considering a two-dimensional eigenvalue problem as the base of the eigenfunction expansion. The filtered head solution formulated as a homogeneous problem can then be explicitly and analytically obtained for $\sigma = 0$, while a numerical-analytical solution is achieved when $\sigma \neq 0$. Previous works have achieved analytical solutions only for the confined aquifer situation. Although the GITT is a hybrid method, it automatically recovers fully analytical solutions once the appropriate eigenfunction expansion base is employed, such as here proposed for the case of a leaking aquifer, while obtaining an analytical-numerical solution only the more involved case of a free aquifer. To reduce computational costs, a reordering scheme for the eigenvalues involved in the filter and filtered homogeneous problems was employed. In addition to reducing the sums, the reordering speeds up convergence, and the program developed to calculate the eigenvalues orders them from the smallest (most important) to the largest (least important). Since the solution is obtained by a series governed by an exponential, these eigenvalues lose influence when they assume exceptionally large values which resulted in a marked reduction of the number of terms NT necessary for convergence. The computed results have shown that the proposed solution offers fully converged flow rates and head distributions, even close to the well, at modest computation effort. The verification with previously reported simulations provided additional confidence in the GITT solutions, offering the possibility for a brief analysis on the variation of other parameters of physical interest or different mediums.

The results obtained in the present work show that GITT (widely used in heat transfer problems) can be very well applied to fluid flow problems in saturated or even unsaturated porous media. More complex models, including irregular geometries, can be obtained by combining previous advancements in the same methodology.

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APPENDIX: Integral transformation of the initial condition

The integral transformation of the initial condition given by Eq. (5b) is given by:

$$\int_0^{w_x} \int_0^{w_y} \int_0^1 \Psi_i(x, y, z) h_H(x, y, z, 0) dz dy dx = - \int_0^{w_x} \int_0^{w_y} \int_0^1 \Psi_i(x, y, z) h_f(x, y, z) dz dy dx \quad (A1)$$

From Eq. (7a), it follows that:

$$\bar{h}_{mpr}(0) - \sigma \int_0^{w_x} \int_0^{w_y} \Psi_i(x, y, 1) h_H(x, y, 1, 0) dy dx = - \int_0^{w_x} \int_0^{w_y} \int_0^1 \Psi_i(x, y, z) h_f(x, y, z) dz dy dx \quad (A2)$$

However, from Eq. (5b), $h_H(x, y, 1, 0) = -h_f(x, y, 1)$, then:

$$\bar{h}_{mpr}(0) = - \int_0^{w_x} \int_0^{w_y} \int_0^1 \Psi_i(x, y, z) h_f(x, y, z) dz dy dx - \sigma \int_0^{w_x} \int_0^{w_y} \Psi_i(x, y, 1) h_f(x, y, 1) dy dx \quad (A3)$$

Replacing in Eq. (A3) the head filter $h_f(x, y, z)$, given by Eq. (12), into Eq. (7b) and the eigenfunctions expression, $\Psi_i(x, y, z) = X_m(x)Y_p(y)Z_r(z)$, it results

$$\bar{h}_{mpr}(0) = - \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \int_0^{w_x} \int_0^{w_y} \frac{X_m(x)X_n(x)Y_p(y)Y_q(y)}{Nx_n Ny_q} dy dx \int_0^1 Z_r(z) \tilde{Z}_{nq}(z) dz - \sigma \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \int_0^{w_x} \int_0^{w_y} \frac{X_m(x)X_n(x)Y_p(y)Y_q(y)}{Nx_n Ny_q} Z_r(1) \tilde{Z}_{nq}(1) dy dx \quad (A4)$$

In light of the orthogonality properties for the eigenfunctions $X_m(x)$ and $Y_p(y)$ given in Table 1, the summations in Eq. (A4) contain only one term, as $m = n$ and as $p = q$, respectively. Therefore, Eq. (A4) reduces to:

$$\bar{h}_{mpr}(0) = \bar{f}_{mpr} = - \int_0^1 \tilde{Z}_{mp}(z) Z_r(z) dz - \sigma \tilde{Z}_{mp}(1) Z_r(1) \quad (A5)$$

or in terms of the reordering index i

$$\bar{h}_i(0) = \bar{f}_i = - \int_0^1 \tilde{Z}_{m(i)p(i)}(z) Z_{r(i)}(z) dz - \sigma \tilde{Z}_{m(i)p(i)}(1) Z_{r(i)}(1) \quad (A6)$$

NOMENCLATURE

B'	Aquitard thickness
B_1	Thickness of streambed 1
B_2	Thickness of streambed 2
$\bar{f}_i$	Initial transformed condition
h	Dimensionless hydraulic head
h_f	Filter potential
h_H	Homogeneous filtered potential

$\bar{h}_i$	Transformed potential
h^*	Hydraulic head
H	Aquifer height
k_v	Dimensionless vertical hydraulic conductivity
k'	Dimensionless aquitard conductivity
k_1	Dimensionless hydraulic conductivity of streambed 1
k_2	Dimensionless hydraulic conductivity of streambed 2
K_h	Aquifer horizontal hydraulic conductivity
K_v	Aquifer vertical hydraulic conductivity
K'	Aquitard conductivity
K_1	Hydraulic conductivity of streambed 1
K_2	Hydraulic conductivity of streambed 2
N_i	Norm
N_{i^*}	Norm for the filter solution
Nx_m	Norm of the eigenvalue problem in the x-direction
Ny_p	Norm of the eigenvalue problem in the y-direction
Nz_r	Norm of the eigenvalue problem in the z-direction
Q	Pumping rate of the vertical well
S	Aquifer storage coefficient
S_s	Aquifer specific storage
S_y	Aquifer specific yield
t	Dimensionless time
t^*	Time
T	Aquifer transmissivity coefficient
w_x	Dimensionless aquifer length
w_y	Dimensionless aquifer width
W_x	Aquifer length
W_y	Aquifer width
x	Dimensionless longitudinal coordinate
x^*	Longitudinal coordinate
x_0^*	Location of the vertical well in the x-direction
X_m	Eigenfunctions in the x-direction defined in Table 1
y	Dimensionless transversal coordinate
y_0	Dimensionless location of the vertical well in the y-direction
y^*	Transversal coordinate
y_0^*	Location of the vertical well in the y-direction
Y_p	Eigenfunctions in the y-direction defined in Table 1
z	Dimensionless vertical coordinate
z^*	Vertical coordinate
Z_r	Eigenfunctions in the z-direction defined in Table 1
$\tilde{Z}_i$	Transformed potential for the filter solution

Greek symbols

α_m	Eigenvalues in the x-direction defined in Table 1
β_p	Eigenvalues in the y-direction defined in Table 1
γ_r	Eigenvalues in the z-direction defined in Table 1
μ_i	Eigenvalues defined by Eqs. (6)
σ	Ratio of aquifer specific yield to specific storage
ϕ_{i^*}	Eigenfunctions for the filter problem
Ψ_i	Eigenfunctions defined by Eqs. (6)

Subscripts and superscripts

i, i^*, j	Original orders from eigenvalue problems
k, l, m, p, r	Derived orders from eigenvalue problems
$\rightarrow, \simeq$	Integral transformed quantities

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