On Accuracy Estimation Using Parametric Bootstrap in small Area Prediction Problems

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We consider longitudinal data and the problem of prediction of subpopulation (domain) characteristics that can be written as a linear combination of the variable of interest, including cases of small or zero sample sizes in the domain and time period of interest. We consider the empirical version of the predictor proposed by Royall (1976) showing that it is a generalization of the empirical version of the predictor presented by Henderson (1950). We propose a parametric bootstrap MSE estimator of the predictor. We prove its asymptotic unbiasedness and derive the order of its bias. Considerations are supported by Monte Carlo simulation analyses to compare its accuracy (not only the bias) with other MSE estimators, including jackknife and weighted jackknife MSE estimators that we adapt for the considered predictor.

Key words: Empirical best linear unbiased predictor; model approach in survey sampling; parametric bootstrap; properties of MSE estimators; small area estimation.

1. Introduction

To estimate or to predict subpopulation characteristics with small or even zero sample sizes, small area estimation methods are used. In the model approach, empirical (estimated) versions of the best linear unbiased predictor (EBLUP) proposed by Henderson (1950) are widely studied. The first approximation (based on the Taylor's expansion) of the MSE of the predictor is proposed by Kackar and Harville (1984), but they do not study the order of the approximation and the order of the bias of their MSE estimator. These problems are discussed by Prasad and Rao (1990), although in cases where estimators of model parameters are unbiased. Datta and Lahiri (2000) generalize the idea for biased estimators of model parameters including maximum likelihood and restricted maximum likelihood estimators as special cases. More general model and different assumptions leading to different asymptotic results are studied by Das et al. (2004). In all of the above mentioned papers, the additional term of the MSE, resulting from the estimation of model parameters, is derived based on the Taylor's expansion. However, alternatively the jackknife method, proposed by Jiang et al. (2002), or the weighted jackknife method, see Chen and Lahiri (2002, 2003), can be used as well. A similar solution, but using the parametric bootstrap method, is presented in Butar and Lahiri (2003). The parametric bootstrap MSE estimator can also be defined in a different

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way and under more general models as shown by [González-Manteiga et al. \(2007\)](#), and [González-Manteiga et al. \(2007, 2008\)](#).

However, the empirical version of the best linear unbiased predictor proposed by [Henderson \(1950\)](#) is not the predictor of the linear combination of the variable of interest (that we would like to predict). This predictor can also be used to make predictions for the unsampled part of the population, as described for example, in [Rao and Molina \(2015, 7.1.3\)](#). Even in this case, they are not generally the best linear unbiased predictors of the linear combination of the variable of interest. The additional condition is required, as we show in Section 3.

In this case, as well as under some additional assumptions presented in Section 3, the resulting predictor becomes the empirical best linear unbiased predictor (EBLUP) of the linear combination of the variable of interest.

A more general predictor is the empirical version of the best linear unbiased predictor proposed by [Royall \(1976\)](#), but it is very rarely considered in the small area estimation literature, for example, [Rao and Molina \(2015, 178\)](#) only mentioned it once. We compare it analytically with the empirical version of the best linear unbiased predictor proposed by [Henderson \(1950\)](#). The aim of the article is to propose an asymptotically unbiased parametric bootstrap MSE estimator of the predictor and derive the order of its bias, generalizing the results of [Butar and Lahiri \(2003\)](#) for:

- a more general predictor (see Section 3 and Remark 1 in Section 4),
- and a more general model, taking into account the problem of changes of domain affiliations of population elements and covering many models considered in small area estimation.

Furthermore, our proposal of the MSE estimator will have the advantage that the additional component of the MSE resulting from the estimation of model parameters will not have to be derived, unlike in the case of Taylor's expansion, considered, for example, by [Żadło \(2009\)](#).

In Section 2, we will present our proposal of the superpopulation model, which is a longitudinal linear mixed model with the block-diagonal covariance matrix. It covers many models known from small area estimation literature and allows taking into account possible changes of population and subpopulations in time. In Section 3, we will show that the BLUP proposed by [Royall \(1976\)](#) is a generalization of the BLUP proposed by [Henderson \(1950\)](#) together with a condition when they are equivalent. The dependence between their MSE components will be also presented. In Section 4, we will study the empirical version of the BLUP proposed by [Royall \(1976\)](#) and propose an estimator of its MSE based on the parametric bootstrap method. We will also prove that the order of the bias of the proposed bootstrap MSE estimator is $o(K^{-1})$, where K is the number of blocks in the covariance matrix of the variable of interest. Considerations are supported by simulation studies based on real data.

2. Longitudinal Superpopulation Model

We will introduce a superpopulation model which belongs to the class of linear mixed models with a block-diagonal variance-covariance matrix of the variable of interest.

For further considerations the diagonal structure will be crucial, but we will also show that our model is (in our opinion) very flexible in the sense that it covers different types of longitudinal data even in cases of population and subpopulation changes in time. We will also present many longitudinal models used for small area estimation purposes as special cases of our proposal. In the next two sections, different predictors and their MSEs will be studied under the model.

Longitudinal data for periods $t = 1, \dots, M$ are considered, where the number M may include future periods. In the period t the population of size N_t is denoted by Ω_t . The population in the period t is divided into D disjoint subpopulations (domains) Ω_{dt} of size N_{dt} , where $d = 1, \dots, D$. Let the set of population elements for which observations are available in the period t be denoted by s_t and its size by n_t . The set of subpopulation elements for which observations are available in the period t is denoted by s_{dt} and its size by n_{dt} . Let $\Omega_{rdt} = \Omega_{dt} \setminus s_{dt}$, $N_{rdt} = N_{dt} - n_{dt}$, $\bigcup_{t=1}^M \Omega_t = \Omega$, $\bar{\Omega} = N$, $\bigcup_{t=1}^M \Omega_{dt} = \Omega_d$, $\bar{\Omega}_d = N_d$, $\bigcup_{t=1}^M \Omega_{rdt} = \Omega_{rd}$, $\bar{\Omega}_{rd} = N_{rd}$, $\bigcup_{t=1}^M s_t = s$, $\bar{s} = n$, $\bigcup_{t=1}^M s_{dt} = s_d$, $\bar{s}_d = n_d$. If the period t is the future period, then $s_t = s_{dt} = \emptyset$, $n_t = n_{dt} = 0$, $\Omega_{rdt} = \Omega_{dt}$ and $N_{rdt} = N_{dt}$.

Let the vector of random variables of interest in M periods, denoted by $\mathbf{Y}$, be divided into K subvectors denoted by $\mathbf{Y}_k$, where $k = 1, 2, \dots, K$. The division can be made according to any rule, including but not limited to the division based on subsets Ω_t ($t = 1, \dots, M$ and hence $K = M$), Ω_d ($d = 1, \dots, D$ and hence $K = D$) or Ω_{dt} ($t = 1, \dots, M$, $d = 1, 2, \dots, D$ and hence $K = D \times M$). We assume that population data obey the following model:

$$\left\{ \begin{array}{l} \mathbf{Y}_k = \mathbf{X}_k \boldsymbol{\beta} + \mathbf{Z}_k \mathbf{v}_k + \mathbf{e}_k \\ E(\mathbf{e}_k) = 0 \\ E(\mathbf{v}_k) = 0 \\ D^2 \begin{bmatrix} \mathbf{v}_k \\ \mathbf{e}_k \end{bmatrix} = \begin{bmatrix} \mathbf{G}_k(\boldsymbol{\delta}) & 0 \\ 0 & \mathbf{R}_k(\boldsymbol{\delta}) \end{bmatrix} \end{array} \right., \quad (1)$$

where $\mathbf{Y}_k$ of size $N_k \times 1$ (where $k = 1, 2, \dots, K$) are assumed to be independent, $\mathbf{X}_k$ and $\mathbf{Z}_k$ are known matrices of auxiliary variables of sizes $N_k \times p$ and $N_k \times r_k$, respectively, and of full ranks, $\boldsymbol{\delta}$ is a vector of q unknown parameters, $\boldsymbol{\beta}$ is a vector of p unknown parameters, random vectors $\mathbf{e}_k$ and $\mathbf{v}_k$ of sizes $N_k \times 1$ and $r_k \times 1$, respectively, are independent and the symbol $D^2(\cdot)$ denotes the variance-covariance matrix. Hence, the covariance matrix of $\mathbf{Y}_k$ denoted by $D^2(\mathbf{Y}_k) = \mathbf{V}_k(\boldsymbol{\delta})$ is given by:

$$\mathbf{V}_k(\boldsymbol{\delta}) = \mathbf{Z}_k \mathbf{G}_k(\boldsymbol{\delta}) \mathbf{Z}_k^T + \mathbf{R}_k(\boldsymbol{\delta}). \quad (2)$$

If we assume, without loss of generality, that first n_k elements of $\mathbf{Y}_k$ are observed then matrices in Equation (1) can be decomposed as follows: where $\mathbf{Y}_k = \begin{bmatrix} \mathbf{Y}_{sk}^T & \mathbf{Y}_{rk}^T \end{bmatrix}^T$, $\mathbf{X}_k = \begin{bmatrix} \mathbf{X}_{sk}^T & \mathbf{X}_{rk}^T \end{bmatrix}^T$, $\mathbf{Z}_k = \begin{bmatrix} \mathbf{Z}_{sk}^T & \mathbf{Z}_{rk}^T \end{bmatrix}^T$, $\mathbf{e}_k = \begin{bmatrix} \mathbf{e}_{sk}^T & \mathbf{e}_{rk}^T \end{bmatrix}^T$, $\mathbf{Y}_{sk}$, $\mathbf{Y}_{rk}$, $\mathbf{X}_{sk}$, $\mathbf{X}_{rk}$, $\mathbf{Z}_{sk}$, $\mathbf{Z}_{rk}$, $\mathbf{e}_{sk}$, $\mathbf{e}_{rk}$ are of sizes $n_k \times 1$, $(N_k - n_k) \times 1$, $n_k \times p$, $(N_k - n_k) \times p$, $n_k \times r_k$, $(N_k - n_k) \times r_k$, $n_k \times 1$, $(N_k - n_k) \times 1$, respectively. In this case, covariance matrices in Equation (2) are given by: $\mathbf{V}_k = \begin{bmatrix} \mathbf{V}_{kss} & \mathbf{V}_{ksr} \\ \mathbf{V}_{krs} & \mathbf{V}_{krr} \end{bmatrix}$ and $\mathbf{R}_k = \begin{bmatrix} \mathbf{R}_{kss} & \mathbf{R}_{ksr} \\ \mathbf{R}_{krs} & \mathbf{R}_{krr} \end{bmatrix}$, where matrices $\mathbf{V}_{kss}$ and $\mathbf{R}_{kss}$ are $n_k \times n_k$, $\mathbf{V}_{krr}$ and $\mathbf{R}_{krr}$ are $(N_k - n_k) \times (N_k - n_k)$, $\mathbf{V}_{ksr}$ and $\mathbf{R}_{ksr}$ are $n_k \times (N_k - n_k)$, $\mathbf{V}_{krs} = \mathbf{V}_{ksr}^T$ and $\mathbf{R}_{krs} = \mathbf{R}_{ksr}^T$.

Model (1) can be written as follows:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{v} + \mathbf{e}, \quad (3)$$

where $\mathbf{Y} = \text{col}_{1 \leq k \leq K}(\mathbf{Y}_k)$, $\mathbf{X} = \text{col}_{1 \leq k \leq K}(\mathbf{X}_k)$, $\mathbf{Z} = \text{diag}_{1 \leq k \leq K}(\mathbf{Z}_k)$, $\mathbf{v} = \text{col}_{1 \leq k \leq K}(\mathbf{v}_k)$, $\mathbf{e} = \text{col}_{1 \leq k \leq K}(\mathbf{e}_k)$. Hence, the covariance matrix of $\mathbf{Y}$ denoted by $D^2(\mathbf{Y}) = \mathbf{V}(\delta)$ is given by:

$$\mathbf{V}(\delta) = \mathbf{ZG}(\delta)\mathbf{Z}^T + \mathbf{R}(\delta), \quad (4)$$

where $\mathbf{G} = \text{diag}_{1 \leq k \leq K}(\mathbf{G}_k)$ and $\mathbf{R} = \text{diag}_{1 \leq k \leq K}(\mathbf{R}_k)$.

Model (1) is a very general model covering many special cases, including cases presented below, in which changes of the population and subpopulations are taken into account. The assumption that one population element may change its domain (or group) affiliation in time is very important in longitudinal surveys. For example, let us consider the population of households and the division of the population into domains made according to household size. In this case, we should assume that some households can change their sizes in time and hence their domain affiliation. If some human population is under the study, one may be interested in its characteristics for subpopulations defined according to some social or economic criteria (e.g., the job position). In the case of business surveys, the population of firms may be divided into subpopulations according to some economic or financial criteria, which may imply even stronger changes in the division of the population in time than in the case of human populations.

Let M_{id} denotes the number of periods when the i th population element belongs to the d th domain (it may include future periods). Values of the variable of interest are realizations of random variables Y_{idj} for the i th population element that belongs to the d th domain in the period t_{ij} , where $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M_{id}$, $d = 1, 2, \dots, D$. The vector $\mathbf{Y}_{id} = [Y_{idj}]_{M_{id} \times 1}$ is called the profile. We have defined $\Omega = \cup_{t=1}^M \Omega_t$, where Ω_t is the population in the period t . Now, let us assume that profiles in Ω are divided into K groups Ω_k for which where the number of random variables within each group equals N_k , where $k = 1, 2, \dots, K$. Grouping profiles means that groups always have a longitudinal character in this special case of Model (1). We consider the following four cases:

- C1 - the additional division of the profiles is not taken into account and hence, the number of groups of profiles equals the number of profiles ($K = N$),
- C2 - domains Ω_d ($d = 1, 2, \dots, D$) are unions of groups of profiles Ω_k ($K > D$ and $K \neq N$),
- C3 - groups of profiles are domains Ω_d ($d = 1, 2, \dots, D$) and hence $K = D$,
- C4 - sets Ω_k ($k = 1, 2, \dots, K$) are unions of domains Ω_d ($K < D$).

Now we can define the profile more precisely, including the information on the division of the population into K groups of profiles. The vector of random variables for the i th population element in different periods will be denoted by $\mathbf{Y}_{idk} = [Y_{idjk}]_{M_{idk} \times 1}$, where M_{idk} is the number of periods when the i th element belongs to the d th domain and the k th group of profiles. Let the number of profiles within the k th group be denoted by M_k . Hence, the number of random variables within Ω_k equals $N_k = \sum_{i=1}^{M_k} M_{idk}$. In the cases C1 and C3, the additional subscript k is not necessary because i and d explicitly define k . Let m_{idk} be the number of periods when the i th population element (which belongs to the d th domain

and the k th group of profiles) is observed. The vector $\mathbf{Y}_{sidk} = [Y_{idkj}]_{m_{idk} \times 1}$, will be called the sample profile. Let the number of profiles observed in the sample in the k th group be denoted by m_k . Hence, the number of sample observations from Ω_k equals $n_k = \sum_{i=1}^{m_k} m_{idk}$. Let the vector $\mathbf{Y}_{ridk} = [Y_{idkj}]_{M_{ridk} \times 1}$, where $M_{ridk} = M_{idk} - m_{idk}$, be the profile of random variables with non-observed realizations (including out of sample and future values). This notation allows the inclusion of possibilities of changes of the population in time, changes of subpopulations in time and even changes of domains or group affiliations of population elements in time in the considerations.

Let us consider the cases in which random variables for one population element observed in different periods form more than one profile. It is possible in the following cases:

- for C1: a change of the domain affiliation,
- for C2: a change of the group affiliation within a domain or a change of the domain affiliation,
- for C3: a change of the domain affiliation equivalent to a change of the group affiliation,
- for C4: the change of the domain affiliation within the group or the change of the group and the domain affiliation at the same time.

It must be pointed out that the period of time can be denoted by one out of two indexes: t (where $t = 1, 2, \dots, M$) - to distinguish between different periods of the longitudinal data, and j (where $j = 1, 2, \dots, M_{id}$) - to distinguish between different periods within profiles.

Model (1) covers many unit-level longitudinal models with block-diagonal covariance matrix considered in the literature. Case C1 of the model includes, for example, the model with independent profile specific random effects considered by Verbeke and Molenbergh (2000, 20). Random regression coefficient models considered by Hobza and Morales (2013) and the multilevel model studied by Moura and Holt (1999), both assumed for one period, and also heteroscedastic models with domain specific and domain and domain-and-time specific random effects considered by Morales and Santamaría (2019) are special cases of case C3. The model with two random effects (domain specific and profile specific) considered by Stukel and Rao (1999) and Nissinen (2009, 22) and the model with two random effects (domain specific and domain-and-time specific) considered by Molina et al. (2010, 143–180) are covered by the case C3 of our model as well. Case C3 also covers models considered by Żądło (2014, 2015a) with profile-specific random effects spatially correlated within domains. Case C4 of the model includes the model with profile-specific random effects spatially correlated within groups of domains proposed by Żądło (2015b). What is more, longitudinal area-level models considered by Rao and Yu (1994) and Marhuenda et al. (2013) can also be written as a Case C3 of our model. However, not all special cases of Model (1) are covered by Cases C1-C4 due to assumed non-longitudinal character of groups including the longitudinal model with domain-and-time specific random effects (autocorrelated in time) considered by Saei and Chambers (2003,13).

It must be pointed out that Model (1) does not cover linear mixed models without block-diagonal covariance matrix, see models studied by, for example Fabrizi et al. (2007, 189), Pagliarelli and Salvatore (2016, 232–235), D'Aló et al. (2017), and nonlinear mixed

models studied by, for example [Hobza et al. \(2018\)](#). The overview of different small area models can be found in [Jiang and Lahiri \(2006\)](#).

3. Best Linear Unbiased Predictors

Under the linear mixed model presented in the previous section, we will compare analytically different predictors and their MSEs, including a predictor that is widely discussed in the literature, proposed by [Henderson \(1950\)](#) and a more general predictor presented by [Royall \(1976\)](#).

We study the problem of prediction of a linear combination $\theta = \gamma^T \mathbf{Y}$. For example, if we are interested in the prediction of the d th subpopulation (domain) total in the period t then the k th element of the vector γ equals 1 for $k \in \Omega_{dt}$ and 0 otherwise. Under Model (3):

$$\theta = \gamma^T \mathbf{Y} = \gamma^T \mathbf{X}\beta + \gamma^T \mathbf{Z}\mathbf{v} + \gamma^T \mathbf{e}. \quad (5a)$$

Let us present the results obtained by [Henderson \(1950\)](#) for surveys conducted in one period by changing the sizes of matrices in his model to cover the case of the longitudinal Model (1). He considers the problem of prediction of:

$$\theta^s = \mathbf{l}^T \beta + \mathbf{m}^T \mathbf{v}. \quad (5b)$$

For $\mathbf{l}^T = \gamma^T \mathbf{X}$ and $\mathbf{m}^T = \gamma^T \mathbf{Z}$ from Equations (5a) and (5b) we obtain:

$$\theta = \theta^s + \gamma^T \mathbf{e}. \quad (5c)$$

Theorem 1. (see [Henderson 1950](#)). Assume that sample data obey the following assumptions:

$$\left\{ \begin{array}{l} \mathbf{Y}_s = \mathbf{X}_s \beta + \mathbf{Z} \mathbf{v} + \mathbf{e}_s \\ E(\mathbf{e}_s) = 0 \\ E(\mathbf{v}) = 0 \\ D^2 \begin{bmatrix} \mathbf{v} \\ \mathbf{e}_s \end{bmatrix} = \begin{bmatrix} \mathbf{G}(\delta) & 0 \\ 0 & \mathbf{R}_{ss}(\delta) \end{bmatrix} \end{array} \right. \quad (6)$$

Among linear, model-unbiased predictors $\hat{\theta}^s = \mathbf{a}^T \mathbf{Y}_s + \mathbf{b}$ of linear combination of β and the realization of $\mathbf{v}$ given by $\theta^s = \mathbf{l}^T \beta + \mathbf{m}^T \mathbf{v}$ (for specified vectors, $\mathbf{l}$ and $\mathbf{m}$, of constants) the MSE is minimized by:

$$\hat{\theta}_{BLUP}^s = \mathbf{1}^T \tilde{\beta}(\delta) + \mathbf{m}^T \tilde{\mathbf{v}}(\delta), \quad (7a)$$

where

$$\tilde{\beta}(\delta) = (\mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\delta) \mathbf{X}_s)^{-1} \mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\delta) \mathbf{Y}_s, \quad (7b)$$

$$\tilde{\mathbf{v}}(\delta) = \mathbf{G}(\delta) \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}(\delta) (\mathbf{Y}_s - \mathbf{X}_s \tilde{\beta}(\delta)). \quad (7c)$$

The MSE of $\hat{\theta}_{BLUP}^s$ is given by

$$MSE\left(\hat{\theta}_{BLUP}^s\right) = Var\left(\hat{\theta}_{BLUP}^s - \theta^s\right) = g_1^s(\boldsymbol{\delta}) + g_2^s(\boldsymbol{\delta}), \quad (7d)$$

where

$$g_1^s(\boldsymbol{\delta}) = \mathbf{m}^T \left(\mathbf{G}(\boldsymbol{\delta}) - \mathbf{G}(\boldsymbol{\delta}) \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{Z}_s \mathbf{G}(\boldsymbol{\delta}) \right) \mathbf{m}, \quad (7e)$$

$$g_2^s(\boldsymbol{\delta}) = \left(\mathbf{1}^T - \mathbf{m}^T \mathbf{G}(\boldsymbol{\delta}) \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{X}_s \right) \left(\mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{X}_s \right)^{-1} \times \\ \times \left(\mathbf{1}^T - \mathbf{m}^T \mathbf{G}(\boldsymbol{\delta}) \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{X}_s \right)^T. \quad (7f)$$

The proof of the theorem (for surveys conducted in one period) is presented in detail in, for example, [Rao and Molina \(2015, 119–120\)](#).

Because in Theorem 1 the problem of prediction of Equation (5b) instead of Equation (5c) is considered, in small area estimation literature, see, for example, [Rao and Molina \(2015, 178–179\)](#), the problem of prediction of Equation (5c) using predictor (7a) is studied. We assume longitudinal Model (1) and study the problem of prediction of Equation (5c), but in a more general framework than in [Rao and Molina \(2015, 178–179\)](#), where the crucial difference is the lack of the independence assumption of random components.

Let us consider the problem of prediction of

$$\theta = \gamma^T \mathbf{Y} = \gamma_s^T \mathbf{Y}_s + \gamma_r^T \mathbf{Y}_r = \gamma_s^T \mathbf{Y}_s + \theta_r, \quad (8)$$

where $\gamma = [\gamma_s^T \gamma_r^T]^T$, $\theta_r = \gamma_r^T \mathbf{Y}_r$ and the realization of $\gamma_s^T \mathbf{Y}_s$ is known. Because realization of $\gamma_s^T \mathbf{Y}_s$ is known, the problem of prediction of Equation (8) is reduced to the problem of prediction of:

$$\theta_r = \gamma_r^T \mathbf{Y}_r = \gamma_r^T \mathbf{X}_r \beta + \gamma_r^T \mathbf{Z}_r \mathbf{v} + \gamma_r^T \mathbf{e}_r = \theta_r^s + \gamma_r^T \mathbf{e}_r, \quad (9)$$

where

$$\theta_r^s = \gamma_r^T \mathbf{X}_r \beta + \gamma_r^T \mathbf{Z}_r \mathbf{v}. \quad (10)$$

Based on Theorem 1 the BLUP of Equation (10) is given by:

$$\hat{\theta}_{BLUPr}^s = \mathbf{1}^T \tilde{\beta}(\boldsymbol{\delta}) + \mathbf{m}^T \tilde{\mathbf{v}}(\boldsymbol{\delta}), \quad (11)$$

where $\mathbf{1}^T = \gamma_r^T \mathbf{X}_r$, $\mathbf{m}^T = \gamma_r^T \mathbf{Z}_r$, $\tilde{\beta}(\boldsymbol{\delta})$ and $\tilde{\mathbf{v}}(\boldsymbol{\delta})$ are given by Equations (7b) and (7c), respectively.

Finally, we obtain the following predictor:

$$\hat{\theta} = \gamma_s^T \mathbf{Y}_s + \hat{\theta}_{BLUPr}^s = \gamma_s^T \mathbf{Y}_s + \\ + \gamma_r^T [\mathbf{X}_r \tilde{\beta}(\boldsymbol{\delta}) + (\mathbf{Z}_r \mathbf{G}(\boldsymbol{\delta}) \mathbf{Z}_s^T) \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) (\mathbf{Y}_s - \mathbf{X}_s \tilde{\beta}(\boldsymbol{\delta}))] \quad (12a)$$

of Equation (8), which generally is not the BLUP of Equation (8). Later in this section, we will show that if $\mathbf{R}_{sr} = 0$, then the predictor given by Equation (12a) becomes the BLUP of Equation (8).

The MSE of the predictor (12a) of Equation (8) is given by

$$\begin{aligned}
 \text{MSE}(\hat{\theta}) &= E\left(\gamma_s^T \mathbf{Y}_s + \hat{\theta}_{BLUPr}^s - \gamma_r^T \mathbf{Y}\right)^2 = \\
 &= E\left(\hat{\theta}_{BLUPr}^s - \gamma_r^T \mathbf{Y}_r\right)^2 = \\
 &= E\left(\hat{\theta}_{BLUPr}^s - \theta_r^s - \gamma_r^T \mathbf{e}_r\right)^2 = \\
 &= \text{MSE}\left(\hat{\theta}_{BLUPr}^s\right) + \\
 &\quad + g_4^s(\boldsymbol{\delta}) - 2\text{Cov}\left(\left(\hat{\theta}_{BLUPr}^s - \theta_r^s\right), \gamma_r^T \mathbf{e}_r\right),
 \end{aligned} \tag{12b}$$

where $\text{MSE}\left(\hat{\theta}_{BLUPr}^s\right)$ is given by Equation (7d) (where $\mathbf{1}^T = \gamma_r^T \mathbf{X}_r$ and $\mathbf{m}^T = \gamma_r^T \mathbf{Z}_r$), $g_4^s(\boldsymbol{\delta}) = \gamma_r^T \mathbf{R}_{rr}(\boldsymbol{\delta}) \gamma_r$ and

$$\begin{aligned}
 \text{Cov}\left(\left(\hat{\theta}_{BLUPr}^s - \theta_r^s\right), \gamma_r^T \mathbf{e}_r\right) &= \\
 &= \gamma_r^T \mathbf{X}_r \left(\mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{X}_s\right)^{-1} \mathbf{X}_s^T \mathbf{V}_{ss}^{-1} \mathbf{R}_{sr}(\boldsymbol{\delta}) \gamma_r + \\
 &\quad + \gamma_r^T \mathbf{Z}_r \mathbf{G}(\boldsymbol{\delta}) \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \times \\
 &\quad \times \left(\mathbf{I} - \mathbf{X}_s \left(\mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta}) \mathbf{X}_s\right)^{-1} \mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\boldsymbol{\delta})\right) \mathbf{R}_{sr}(\boldsymbol{\delta}) \gamma_r.
 \end{aligned} \tag{12c}$$

Firstly, we assume that $\mathbf{R}_{sr} = 0$. Then, $\text{Cov}\left(\left(\hat{\theta}_{BLUPr}^s - \theta_r^s\right), \gamma_r^T \mathbf{e}_r\right) = 0$ and Equation (12b) simplifies to

$$\begin{aligned}
 \text{MSE}(\hat{\theta}) &= \text{MSE}\left(\hat{\theta}_{BLUPr}^s\right) + g_4^s(\boldsymbol{\delta}) = \\
 &= g_1^s(\boldsymbol{\delta}) + g_2^s(\boldsymbol{\delta}) + g_4^s(\boldsymbol{\delta})
 \end{aligned} \tag{12d}$$

and the predictor (12a) is the BLUP of Equation (8) (which will be shown later in this section).

Royall (1976) for surveys conducted in one period derived a more general predictor than predictors (7a) and (12a). In the following consideration we change the sizes of matrices in his theorem to cover the following longitudinal model:

$$\begin{cases} E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta} \\ D^2(\mathbf{Y}) = \mathbf{V} \end{cases}, \tag{13}$$

where matrices in Equation (13) were defined in Section 1.

Theorem 2. (compare Royall 1976). Assume that the population data obey the longitudinal general linear model (see Equation (13)). Among linear, model-unbiased predictors $\hat{\theta} = g^T \mathbf{Y}_s$ of linear combination of random variables $\theta = \gamma^T \mathbf{Y}$, the MSE is

minimized by:

$$\hat{\theta}_{BLUP} = \gamma_s^T \mathbf{Y}_s + \gamma_r^T [\mathbf{X}_r \tilde{\beta}(\delta) + \mathbf{V}_{rs}(\delta) \mathbf{V}_{ss}^{-1}(\delta) (\mathbf{Y}_s - \mathbf{X}_s \tilde{\beta}(\delta))], \quad (14a)$$

where $\tilde{\beta}(\delta)$ is given by (7b).

The MSE of $\hat{\theta}_{BLUP}$ is given by

$$MSE(\hat{\theta}_{BLUP}) = g_1(\delta) + g_2(\delta), \quad (14b)$$

where

$$g_1(\delta) = \gamma_r^T (\mathbf{V}_{rr}(\delta) - \mathbf{V}_{rs}(\delta) \mathbf{V}_{ss}^{-1}(\delta) \mathbf{V}_{sr}(\delta)) \gamma_r, \quad (14c)$$

$$g_2(\delta) = \gamma_r^T (\mathbf{X}_r - \mathbf{V}_{rs}(\delta) \mathbf{V}_{ss}^{-1}(\delta) \mathbf{X}_s) \times (\mathbf{X}_s^T \mathbf{V}_{ss}^{-1}(\delta) \mathbf{X}_s)^{-1} \times (\mathbf{X}_r - \mathbf{V}_{rs}(\delta) \mathbf{V}_{ss}^{-1}(\delta) \mathbf{X}_s)^T \gamma_r. \quad (14d)$$

The proof of the theorem is presented in details by, for example, [Valliant et al. \(2000, 29–30\)](#).

Firstly, let us compare predictor (14a) proposed by [Royall \(1976\)](#) with predictor (7a) proposed by [Henderson \(1950\)](#). Predictor (14a) can be treated as a generalization of the predictor (7a) because (i) the general linear Model (13) covers the assumptions of the general linear mixed Model (6), (ii) the predictor (14a) is the predictor of θ , while the predictor (7a) is the predictor of θ^s (where (5c) holds) and (iii) $\hat{\theta}^s = \mathbf{a}^T \mathbf{Y}_s + b$ considered in Theorem 1 is of the same form as $\hat{\theta} = \mathbf{g}^T \mathbf{Y}_s$ considered in Theorem 2, because $b = 0$ under unbiasedness of the predictor.

Secondly, let us compare predictors of θ given by Equation (14a), proposed by [Royall \(1976\)](#), and given by Equation (12a). Although (14a) is derived under the general linear Model (13), we consider its formula under the special case of the model - the general linear mixed Model (1). Hence, the predictor (14a) simplifies to:

$$\begin{aligned} \hat{\theta}_{BLUP} &= \gamma_s^T \mathbf{Y}_s + \gamma_r^T \mathbf{X}_r \tilde{\beta}(\delta) + \\ &+ \gamma_r^T (\mathbf{R}_{rs}(\delta) + \mathbf{Z}_r \mathbf{G}(\delta) \mathbf{Z}_s^T) \mathbf{V}_{ss}^{-1}(\delta) (\mathbf{Y}_s - \mathbf{X}_s \tilde{\beta}(\delta)). \end{aligned} \quad (15)$$

If $\mathbf{R}_{rs} = 0$ then predictors (15) are (12a) identical and their MSEs (given by Equations (14b) and (12d), respectively) are equal. Being more precise, if $\mathbf{R}_{rs} = 0$, then

$$g_1(\delta) = g_1^s(\delta) + g_4^s(\delta) \quad (16)$$

and

$$g_2(\delta) = g_2^s(\delta). \quad (17)$$

To sum up, the predictor (12a) is the BLUP of θ only for models where $\mathbf{R}_{rs} = 0$.

4. Empirical Best Linear Unbiased Predictors

In the previous section, we assumed that model parameters are known. Now we will take into account additional variability resulting from their estimation and its influence on the MSE and MSE estimation.

Let us start our considerations from the predictor proposed by [Henderson \(1950\)](#). The Best Linear Unbiased Predictor (BLUP) (7a) depends on the variance parameters δ that are unknown in practical applications. Replacing δ by an estimator $\hat{\delta}$, we obtain the two-stage predictor $\hat{\theta}_{EBLUP}^s = \hat{\theta}_{BLUP}^s(\hat{\delta})$ called empirical (estimated) best linear unbiased predictor EBLUP. In the simulation we will use the Restricted Maximum Likelihood (REML) method, known as robust on nonnormality ([Jiang 1996](#)) to estimate δ . Under some weak assumptions presented by [Kackar and Harville \(1981\)](#):

- (i) an estimator of the vector of model parameters is any even, translation-invariant estimator, for example the REML estimator,
- (ii) the distributions of $\mathbf{v}$ and $\mathbf{e}_s$ are both symmetric around 0,
- (iii) the expectation of the predictor is finite, the EBLUP remains unbiased.

The MSE of EBLUP is greater than the MSE of BLUP due to the estimation of the model parameters. The problem of estimating accuracy of the EBLUP is widely discussed in the literature and results discussed below can be directly used for our case just by changing the sizes of the matrices to cover the longitudinal Model (1). The first paper where the problem of the approximate formula (based on the Taylor expansion) of the MSE of the EBLUP is studied and its estimator is proposed is [Kackar and Harville \(1984\)](#). The authors do not study the orders of the neglected terms in the approximation of the MSE and the order of the bias of the MSE estimator. The pioneering results are presented by [Prasad and Rao \(1990\)](#). They assume, inter alia, block-diagonal variance-covariance matrix, normality of random effects and random components and unbiasedness of δ which allows obtaining the order of MSE approximation and proving the asymptotically unbiasedness of their MSE estimator. The problem is also studied in detail by [Datta and Lahiri \(2000\)](#), but for a larger class of estimators δ (including maximum likelihood and restricted maximum likelihood estimators), where unbiasedness is not required. The more general mixed model is studied by [Das et al. \(2004\)](#), who, based on different assumptions obtained different asymptotic results. Jackknife and weighted jackknife estimators of MSE of EBLUP are proposed by [Jiang et al. \(2002\)](#), [Chen and Lahiri \(2002, 2003\)](#). The parametric bootstrap method is used to estimate MSE of EBLUP by, inter alia, [Butar and Lahiri \(2003\)](#), [González-Manteiga et al. \(2007, 2008\)](#) As shown, for example, by [Schmid and Münnich \(2014\)](#), the parametric bootstrap method of MSE estimation is preferable in the case of different predictors as well.

If δ in the formula of the predictor (14a) proposed by [Royall \(1976\)](#) is replaced by an estimator $\hat{\delta}$, we obtain $\hat{\theta}_{EBLUP} = \hat{\theta}_{BLUP}(\hat{\delta})$ - the empirical version of the predictor. It remains unbiased under some weak assumptions presented by [Żądło \(2004\)](#) ((i) and (iii) the same as mentioned above in the case of empirical version of the predictor proposed by [Henderson \(1950\)](#), in assumption (ii) symmetric distribution around zero of $\mathbf{e}$ instead of $\mathbf{e}_s$ is assumed). For the empirical version of the predictor studied by [Royall \(1976\)](#), the MSE estimator using Taylor's series expansion was proposed by [Żądło \(2009\)](#), who

generalized the results of [Datta and Lahiri \(2000\)](#) obtained for the empirical version of the predictor considered by [Henderson \(1950\)](#). In this section, we will propose a parametric bootstrap MSE estimator by a generalization of the results presented by [Butar and Lahiri \(2003\)](#). For (i) the more general predictor - we will study the empirical version of the [Royall \(1976\)](#) predictor instead of the empirical version of the [Henderson \(1950\)](#) predictor and (ii) for the more general model (the model proposed in Section 1).

[Butar and Lahiri \(2003\)](#) consider two superpopulation models assumed for sample data. We assume the following superpopulation models, but for the whole population:

- Model 1: Model (1) with an additional assumption of normality of random effects and random components
- Model 2: bootstrap model:

$$\left\{ \begin{array}{l} \mathbf{Y}_k^* = \mathbf{X}_k \hat{\boldsymbol{\beta}} + \mathbf{Z}_k \mathbf{v}_k^* + \mathbf{e}_k^* \\ E_*(\mathbf{e}_k^*) = 0 \\ E_*(\mathbf{v}_k^*) = 0 \\ D_*^2 \begin{bmatrix} \mathbf{v}_k^* \\ \mathbf{e}_k^* \end{bmatrix} = \begin{bmatrix} \mathbf{G}_k(\hat{\boldsymbol{\delta}}) & 0 \\ 0 & \mathbf{R}_k(\hat{\boldsymbol{\delta}}) \end{bmatrix} \end{array} \right., \quad (18)$$

where $\mathbf{Y}_1^*, \dots, \mathbf{Y}_k^*, \dots, \mathbf{Y}_K^*$ are independent, $\mathbf{v}_k^*$ and $\mathbf{e}_k^*$ are generated from multivariate normal distributions, $\hat{\boldsymbol{\delta}}$ is an estimator of $\boldsymbol{\delta}$ that satisfies the regularity conditions (RC) given in the [Appendix](#) (Section 7), $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\delta}})$, where $\hat{\boldsymbol{\beta}}(\boldsymbol{\delta})$ is given by Equation (7b).

We propose the following parametric bootstrap MSE estimator:

$$\begin{aligned} \widehat{MSE}(\hat{\theta}_{EBLUP}) &= g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) + \\ &- E_* \left(g_1(\boldsymbol{\delta}^*) + g_2(\boldsymbol{\delta}^*) - \left(g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) \right) \right) + \\ &+ E_* \left(\hat{\theta}_{EBLUP}(Y_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}^*), \boldsymbol{\delta}^*) - \hat{\theta}_{EBLUP}(Y_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}), \boldsymbol{\delta}) \right)^2, \end{aligned} \quad (19)$$

where E_* it is the expectation with respect to Model 2, $\boldsymbol{\delta}^*$ is calculated as $\boldsymbol{\delta}$ but based on $\mathbf{Y}_s^*$ instead of $\mathbf{Y}_s$. The formula of the estimator (19) of the empirical version of [Royall \(1976\)](#) predictor is similar to the MSE estimator of the empirical version of [Henderson \(1950\)](#) predictor proposed by [Butar and Lahiri \(2003\)](#). If $g_1(\cdot)$, $g_2(\cdot)$ and $\hat{\theta}_{EBLUP}$ in (19) are replaced by $g_1^s(\cdot)$, $g_2^s(\cdot)$ and $\hat{\theta}_{EBLUP}^s$ respectively, we obtain the [Butar and Lahiri \(2003\)](#) MSE estimator.

We will prove that the estimator (19) is asymptotically unbiased and that the bias is of order $o(K^{-1})$. The proof will be a generalization of the proof presented by [Butar and Lahiri \(2003\)](#). [Butar and Lahiri \(2003\)](#) propose another MSE estimator which approximates the parametric bootstrap MSE estimator but its bias, [Butar and Lahiri \(2003\)](#), is of a higher order – the same order as the bias of the naive MSE estimator.

Lemma 1. Under Model 1 and the regularity conditions (RC) presented in the [Appendix](#) (Section 7), we have

$$\begin{aligned} E\left(g_1(\boldsymbol{\delta}) - \mathbf{B}_{\boldsymbol{\delta}}^T(\boldsymbol{\delta}) \frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} + g_3(\boldsymbol{\delta})\right) &= g_1(\boldsymbol{\delta}) + o(K^{-1}), \\ E(g_2(\boldsymbol{\delta})) &= g_2(\boldsymbol{\delta}) + o(K^{-1}), \\ E(g_3(\boldsymbol{\delta})) &= g_3(\boldsymbol{\delta}) + o(K^{-1}), \end{aligned}$$

where

$$g_3(\boldsymbol{\delta}) = \text{trace}\left(\frac{\partial \mathbf{c}^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \mathbf{V}_{ss} \left(\frac{\partial \mathbf{c}^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right)^T \boldsymbol{\Sigma}(\boldsymbol{\delta})\right) \quad (20)$$

and

$$\mathbf{c}^T = \boldsymbol{\gamma}_r^T \mathbf{V}_{rs} \mathbf{V}_{ss}^{-1} = \boldsymbol{\gamma}_r^T (\mathbf{R}_{rs} + \mathbf{Z}_r \mathbf{G} \mathbf{Z}_s^T) \mathbf{V}_{ss}^{-1},$$

$\boldsymbol{\Sigma}(\boldsymbol{\delta}) = E(\boldsymbol{\delta} - \boldsymbol{\delta})(\boldsymbol{\delta} - \boldsymbol{\delta})^T$, $\mathbf{B}_{\boldsymbol{\delta}}(\boldsymbol{\delta})$ is defined in the regularity condition (f) in the [Appendix](#) (Section 7).

Proof. The proof results directly (assuming our regularity conditions, our model and replacing $\mathbf{m}^T \mathbf{G} \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}$ by $\boldsymbol{\gamma}_r^T \mathbf{V}_{rs} \mathbf{V}_{ss}^{-1}$) from the proof of the theorem of [Datta and Lahiri \(2000, 624\)](#) (called theorem A.2). The difference between Lemma 1 and the theorem presented in [Žađlo \(2009, 110\)](#) results from the assumed model (including the number of blocks in the covariance matrix and the sizes of the matrices).

Remark 1. Under normality of $\mathbf{Y}$ the $g_3(\boldsymbol{\delta})$ given by Equation (20) approximates the difference between the MSE of EBLUP and the MSE of BLUP for the predictor considered by [Royall \(1976\)](#) (the predictor presented in Theorem 2) - see [Žađlo \(2009, 107\)](#). Applying results presented by [Žađlo \(2009\)](#) to RC presented in the [Appendix](#) (Section 7), it can be proved that the order of approximation is $o(K^{-1})$. It is worth noting that Equation (20) is the generalization of:

$$g_3^s(\boldsymbol{\delta}) = \text{trace}\left(\frac{\partial \mathbf{b}^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \mathbf{V}_{ss} \left(\frac{\partial \mathbf{b}^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right)^T \boldsymbol{\Sigma}(\boldsymbol{\delta})\right), \quad (21)$$

where

$$\mathbf{b}^T = \mathbf{m}^T \mathbf{G} \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}, \quad (22)$$

presented by [Datta and Lahiri \(2000\)](#), which approximates the difference between the MSE of EBLUP and the MSE of BLUP for the predictor considered by [Henderson \(1950\)](#) (the predictor presented in Theorem 1). Moreover, if we consider the problem of prediction of Equation (10), then Equation (22) is given by:

$$\mathbf{b}^T = \boldsymbol{\gamma}_r^T \mathbf{Z}_r \mathbf{G} \mathbf{Z}_s^T \mathbf{V}_{ss}^{-1}.$$

Then, for models with $R_{rs} = 0$ we obtain

$$g_3^s(\boldsymbol{\delta}) = g_3(\boldsymbol{\delta}), \quad (23)$$

where $g_3(\boldsymbol{\delta}) = g_3^s(\boldsymbol{\delta})$, are given by Equations (21) and (20), respectively.

Lemma 2. Under Model 1, Model 2 and regularity conditions (RC) presented in [Appendix](#) (Section 7):

- (i) $E_*(g_1(\boldsymbol{\delta}^*)) = g_1(\boldsymbol{\delta}) + \mathbf{B}_{\boldsymbol{\delta}}^T(\boldsymbol{\delta}) \frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} - g_3(\boldsymbol{\delta}) + o_p(K^{-1})$,
- (ii) $E_*(g_2(\boldsymbol{\delta}^*)) = g_2(\boldsymbol{\delta}) + o_p(K^{-1})$,
- (iii) $E_*(\hat{\theta}_{EBLUP}(\mathbf{Y}_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}^*), \boldsymbol{\delta}^*) - \hat{\theta}_{EBLUP}(\mathbf{Y}_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}), \boldsymbol{\delta}))^2 = g_4(\boldsymbol{\delta}) + o_p(K^{-1})$,
- (iv) $E(g_4(\boldsymbol{\delta})) = g_3(\boldsymbol{\delta}) + o(K^{-1})$,

where $g_4(\boldsymbol{\delta}) = \text{trace} \left(\frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} (\mathbf{Y}_s - \mathbf{X}_s \hat{\boldsymbol{\beta}}) (\mathbf{Y}_s - \mathbf{X}_s \hat{\boldsymbol{\beta}})^T \left(\frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \right)^T \Sigma(\boldsymbol{\delta}) \right)$.

Proof. Parts (i) and (ii) in Lemma 2 follow from (i) and (ii) in Lemma 1 (but under Model 2) using $E_*(\boldsymbol{\delta}^* - \boldsymbol{\delta}) = O_p(K^{-1})$ (as [Butar and Lahiri 2003, 74](#)). The proof of (iii) we obtain using

$$\begin{aligned} & \hat{\theta}_{EBLUP}(\mathbf{Y}_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}^*), \boldsymbol{\delta}^*) - \hat{\theta}_{EBLUP}(\boldsymbol{\delta}) = \\ & = (\boldsymbol{\delta}^* - \boldsymbol{\delta}) \frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} (\mathbf{Y}_s - \mathbf{X}_s \hat{\boldsymbol{\beta}}) + O_p(K^{-1}) \end{aligned}$$

under Model 2, which results directly from Equation (25) in [Żądło \(2009, 108\)](#), which is the direct generalization of Equation (A.2) in [Datta and Lahiri \(2000, 623\)](#) used by [Butar and Lahiri \(2003, 74\)](#). The proof of part (iv) results from the RC, see [Butar and Lahiri \(2003, 75\)](#), which implies $\boldsymbol{\delta} - \boldsymbol{\delta} = o_p(1)$, $\Sigma(\boldsymbol{\delta}) = O(K^{-1})$, $\hat{\boldsymbol{\beta}}(\boldsymbol{\delta}) = \boldsymbol{\beta} + o_p(1)$, $\Sigma(\boldsymbol{\delta}) = \Sigma(\boldsymbol{\delta}) + o_p(K^{-1})$. Hence, $\frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} = \frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} + o_p(1)$ and:

$$\begin{aligned} & \frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} (\mathbf{Y}_s - \mathbf{X}_s \hat{\boldsymbol{\beta}}) (\mathbf{Y}_s - \mathbf{X}_s \hat{\boldsymbol{\beta}})^T \left(\frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \right)^T \Sigma(\boldsymbol{\delta}) = \\ & = \frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} (\mathbf{Y}_s - \mathbf{X}_s \boldsymbol{\beta}) (\mathbf{Y}_s - \mathbf{X}_s \boldsymbol{\beta})^T \left(\frac{\partial c^T(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} \right)^T \Sigma(\boldsymbol{\delta}) + \\ & + o_p(K^{-1}). \end{aligned}$$

Finally, using the expressions for $g_3(\boldsymbol{\delta})$ and $g_4(\boldsymbol{\delta})$ we get the part (iv) of Lemma 2.

Theorem 3. Under Model 1, Model 2 and the RC, we have:

$$E(\widehat{MSE}(\hat{\theta}_{EBLUP})) - MSE(\hat{\theta}_{EBLUP}) = o(K^{-1}),$$

where $\widehat{MSE}(\hat{\theta}_{EBLUP})$ is given by Equation (19).

Proof. Using Lemma 1 and Lemma 2 we get:

$$\begin{aligned}
 E(\widehat{MSE}(\hat{\theta}_{EBLUP})) &= E\left[g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) + \right. \\
 &\quad - E_*\left(g_1(\boldsymbol{\delta}^*) + g_2(\boldsymbol{\delta}^*) - (g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}))\right) + \\
 &\quad + E_*\left(\hat{\theta}_{EBLUP}(Y_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}^*), \boldsymbol{\delta}^*) + \right. \\
 &\quad \left. \left. - \hat{\theta}_{EBLUP}(Y_s, \hat{\boldsymbol{\beta}}(\boldsymbol{\delta}), \boldsymbol{\delta})\right)^2\right] = \\
 &= E\left[g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) - g_1(\hat{\boldsymbol{\delta}}) - \mathbf{B}_{\boldsymbol{\delta}}^T(\boldsymbol{\delta}) \frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} + g_3(\hat{\boldsymbol{\delta}}) + \right. \\
 &\quad \left. - g_2(\hat{\boldsymbol{\delta}}) + g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) + g_4(\hat{\boldsymbol{\delta}}) + o_p(K^{-1})\right] \\
 &= E\left[g_1(\hat{\boldsymbol{\delta}}) + g_2(\hat{\boldsymbol{\delta}}) + g_3(\hat{\boldsymbol{\delta}}) + g_4(\hat{\boldsymbol{\delta}}) + \right. \\
 &\quad \left. - \mathbf{B}_{\boldsymbol{\delta}}^T(\boldsymbol{\delta}) \frac{\partial g_1(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}} + o_p(K^{-1})\right] = \\
 &= g_1(\boldsymbol{\delta}) + g_2(\boldsymbol{\delta}) + g_3(\boldsymbol{\delta}) + o(K^{-1}) \\
 &= MSE(\hat{\theta}_{EBLUP}) + o(K^{-1}),
 \end{aligned} \tag{24}$$

where the last equality in Equation (24) for the empirical version of [Royall \(1976\)](#)'s BLUP was proved by [Żądło \(2009, 110\)](#).

5. Real Data Analyses

We will show an application of the proposed method together with other MSE estimators for a real data set. To analyze statistical properties of our method, we will also present Monte Carlo simulation studies, taking the problem of model misspecification into account as well.

In the following analyses, prepared using R ([R Core Team 2019](#)), we consider the real population data for $N = 378$ Polish counties called poviats (NUTS 4) for $M = 3$ periods - for the years 2011–2013. Two poviats were excluded from the analysis - the first because of the lack of data, the second (Warsaw) as an outlier. Investments in companies (in hundreds of million PLN) and the number of new companies registered (in hundreds) are the variable of interest and the auxiliary variable, respectively. The data are divided into $D = 28$ domains in the following way. Firstly, the population of poviats (NUTS 4) is divided into 16 voivodships (NUTS 2). Secondly, poviats in each voivodship are divided into two groups according to the type of poviat (city counties and land counties), but only if the sizes of both groups are at least 3. Domain sizes range from 3 to 37 (with the mean: 13.5). The problem of prediction of domain totals in the last period is considered.

In the first period, poviats are divided into two strata. The first stratum consists of poviats from domains that consist only of city counties. Other poviats belong to the second stratum. In the first period, a simple random sample without replacement is drawn from each stratum, (optimal allocation is used), in which the overall sample size is $n = 38$.

Sample sizes in domains range from 0 to 8 (with the mean: 1.36). In 9 domains zero sample sizes are observed. The same elements are in the samples in other periods (the balanced panel sample). This gives the division of the population in each period into the sample and the set of non-sampled elements.

5.1. Application

We mimic the real data analysis for the considered sample data set. We would like to find the model with the best goodness-of-fit measured by AIC and BIC criteria. Of course different measures of goodness-of-fit, including conditional AIC proposed by [Vaida and Blanchard \(2005\)](#) for clustered data, can be used as well. In the example, we do not group the profiles and hence the subscript k is omitted. We study models with one auxiliary variable, with and without random effects, with and without constants that belong to the following two classes. Firstly,

$$Y_{ijd} = \beta_1 x_{ijd} + \beta_2 + v_1 + v_2 + e_{ijd},$$

where $i = 1, 2, \dots, N$; $d = 1, 2, \dots, D$, $j = 1, 2, \dots, M_{id}$; v_1 , v_2 and e_{ijd} are mutually independent, $v_1 \sim N(0, \sigma_1^2)$, $v_2 \sim N(0, \sigma_2^2)$ and $e_{ijd} \sim N(0, \sigma_e^2)$. Random effects v_1 and v_2 (where $v_1 \neq v_2$) can be domain specific (v_d), time specific (v_t), domain-and-time specific (v_{dt}) or profile specific (v_{id}). Secondly,

$$Y_{ijd} = (\beta_1 + v_1)x_{ijd} + (\beta_2 + v_2)e_{ijd},$$

where $i = 1, 2, \dots, N$; $d = 1, 2, \dots, D$, $j = 1, 2, \dots, M_{id}$; v_1 , v_2 and e_{ijd} are mutually independent, $v_1 \sim N(0, \sigma_1^2)$, $v_2 \sim N(0, \sigma_2^2)$ and $e_{ijd} \sim N(0, \sigma_e^2)$. Random effects v_1 and v_2 (where $v_1 \neq v_2$ or $v_1 = v_2$) can be, for example, domain specific (v_d), time specific (v_t), domain-and-time specific (v_{dt}) or profile specific (v_{id}).

The model with the smallest AIC and BIC criteria is given by

$$Y_{ijd} = (\beta_1 + v_{id})x_{ijd} + e_{ijd}. \quad (25)$$

REML estimates of model parameters are as follows: $\hat{\beta}_1 = 0.4194$, $\hat{\sigma}_v^2 = 0.0778$ and $\hat{\sigma}_e^2 = 1.5194$. Based on the results of permutation tests we can claim that model parameters are statistically significant (p-values for tests of β_1 and σ_v^2 are zero). The model belongs to the class of random regression coefficients models considered by [Moura and Holt \(1999\)](#) and [Hobza and Morales \(2013\)](#).

Remark 2. Firstly, in Model (25) the independence of random components (e_{ijd}) is assumed, which means that $\mathbf{R}_{rs} = \mathbf{0}$ and hence, predictors (12a) and (14a) are identical and the equalities (16), (17) and (23) are true. Secondly, the model belongs to the class of mixed linear models with independent profile-specific random components, widely discussed by, for example, [Verbeke and Molenbergh \(2000\)](#). In this class of models, nonzero covariances between the variables of interest are observed only within profiles, and hence for the balanced panels samples we have $\mathbf{V}_{rs} = \mathbf{0}$ and (because $\mathbf{G}$ is diagonal)

$\mathbf{Z}_r \mathbf{G} \mathbf{Z}_s^T = \mathbf{0}$. This implies that

$$g_3(\boldsymbol{\delta}) = g_3^s(\boldsymbol{\delta}) = 0,$$

where $g_3(\boldsymbol{\delta})$ and $g_3^s(\boldsymbol{\delta})$ are given by Equations (21) and (20), respectively.

In Table 1 we present values for the following MSE estimators of the empirical version of the predictor (14a):

- the MSE estimator based on the Taylor expansion proposed by [Zadło \(2009\)](#), which is - for the considered model - equivalent to the MSE estimator of the empirical version of the predictor (12a) originally proposed by [Datta and Lahiri \(2000\)](#) (*Taylor*),
- the MSE estimator based on delete-one-profile jackknife applied for the empirical version of the predictor (14a), based on the idea presented by [Jiang et al. \(2002\)](#) (where delete-one-domain jackknife for empirical version of the predictor (7a) was studied) (*jack*),
- the MSE estimator based on delete-one-profile weighted-jackknife applied for the empirical version of the predictor (14a), based on the idea presented by [Chen and Lahiri \(2002, 2003\)](#) (where weighted delete-one-domain jackknife for the empirical version of the predictor (7a) was studied) (*w-jack*),
- the parametric bootstrap MSE estimator studied by [González-Manteiga et al. \(2007, 2008\)](#) (*boot1*),
- the proposed parametric bootstrap MSE estimator given by (19) (*boot2*).

Jackknife and weighted jackknife MSE estimators are adapted for the considered predictor by: (i) in formulae presented by [Jiang et al. \(2002, 1,787\)](#) and [Chen and Lahiri \(2002, 474\)](#) replacing $g_1^s(\cdot)$ and $g_1^s(\cdot)$ (see Equations (7e) and (7f) with $g_1(\cdot)$ and $g_2(\cdot)$ (see Equations (14c) and (14d)) and (ii) deleting profiles instead of domains in the case of estimation of $\boldsymbol{\delta}$ (in our model the number of blocks in $\mathbf{V}$ matrix is equal to the number of profiles).

In Table 1 we present results for all observed domain sample sizes. If the same sample size is observed in many domains, we present results only for two of them – with the smallest and the largest MSE estimates. For most of domains, the values of all MSE estimators except *jack* were very similar, which suggests similar stochastic properties (studied in the next section).

Table 1. Values of the MSE estimators for selected domains.

d	n_d	N_d	<i>Taylor</i>	<i>jack</i>	<i>w-jack</i>	<i>boot1</i>	<i>boot2</i>
13	0	14	36.4	32.2	36.4	36.7	36.5
22	0	3	643.6	612.7	642.0	707.0	643.3
12	1	4	8.1	7.7	8.0	7.8	8.1
6	1	3	798.1	766.9	796.0	824.1	797.8
27	2	4	87.0	83.4	86.8	85.4	87.0
19	2	18	94.5	71.4	94.6	94.2	94.7
11	3	21	85.6	65.4	85.7	89.6	85.8
28	4	21	109.5	87.4	109.7	117.4	109.8
17	6	31	326.1	242.9	326.6	335.2	326.9
8	8	19	145.2	116.1	145.3	142.8	145.5

All of the considered MSE estimators are based on the normal mixed model where the normality of random effects and random components is assumed, which is equivalent (logical biconditional) to normality $\mathbf{Y}$ (see Remark 3).

Remark 3. Assuming (3) (including independence of $\mathbf{v}$ and $\mathbf{e}$ and full column rank of $\mathbf{Z}$), $\mathbf{v}$ and $\mathbf{e}$ have multivariate normal distributions if and only if $\mathbf{Y}$ has the multivariate normal distribution. The above statement is true because both implications are true. The implication, that if $\mathbf{v}$ and $\mathbf{e}$ in Equation (3) have multivariate normal distributions, then $\mathbf{Y}$ has the multivariate normal distribution, is a standard result (see e.g., Muirhead (2005), 6 theorem 1.2.6 and 14 theorem 1.2.14). The implication, that if $\mathbf{Y}$ in (3) has the multivariate normal distribution, then $\mathbf{v}$ and $\mathbf{e}$ have multivariate normal distribution, results from the following two statements. Firstly, if $\mathbf{Y}$ in (3) has the multivariate normal distribution, then two random vectors of the same size $\mathbf{Z}\mathbf{v}$ and $\mathbf{e}$ have multivariate normal distributions (see e.g., Muirhead (2005), 14 theorem 1.2.13). Secondly, if $\mathbf{Z}\mathbf{v}$ has multivariate normal distribution and $\mathbf{Z}$ is of full rank, then $\mathbf{v}$ has the multivariate normal distribution. It is true, because there exists a linear transformation of $\mathbf{Z}\mathbf{v}$ to $\mathbf{v}$ (e.g., $((\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T)(\mathbf{Z}\mathbf{v}) = \mathbf{v}$) and then (e.g., Muirhead (2005), 6 theorem 1.2.6) normality of $\mathbf{Z}\mathbf{v}$ implies normality of $\mathbf{v}$.

Fig. 1. Q-Q plots of predicted random components and random effects.

The crucial point of the proofs of asymptotic unbiasedness of MSE estimators, where normality of $\mathbf{Y}$ (normality of random effects and random components) is required, is the approximate decomposition of the MSE of the EBLUP into: the MSE of the BLUP and the additional component usually denoted by $g_3(\cdot)$ (see Equation (20) for our case). The proof of the decomposition for the empirical version of the predictor studied by [Henderson \(1950\)](#) is considered by, for example, [Kackar and Harville \(1984, 855\)](#) and [Robinson \(1991, 19\)](#) and for the empirical version of the predictor studied by [Royall \(1976\)](#) and by [Harville and Jeske \(1992\)](#) in Section 2. Then, based on the decomposition (under normality of $\mathbf{Y}$), $g_3(\cdot)$ component is estimated using different methods and, finally, the MSE estimator is obtained. The assumption of normality of $\mathbf{Y}$ is used in the case of the MSE estimator based on Taylor's expansion by [Datta and Lahiri \(2000, 623\)](#) and [Żądło \(2009\)](#), for the weighted jackknife MSE estimator by [Chen and Lahiri \(2003, 908\)](#) and for the parametric bootstrap MSE estimators by [Butar and Lahiri \(2003, 66\)](#) and in our proposal (see Remark 1). In case of the jackknife MSE estimator the proof is different, but normality of $\mathbf{Y}$ is also required, as shown by [Jiang et al. \(2002, 1,803\)](#). For our data, the assumption of normality of $\mathbf{Y}$ (or equivalently the normality of $\mathbf{v}$ and $\mathbf{e}$ – see Remark 3) is not met – using the Shapiro test we checked normality using residuals $\mathbf{Y}_s - \mathbf{X}_s\hat{\beta}$ after the Cholesky transformation (p-value equals 0). Q-Q plots of predicted random effects $\hat{\mathbf{v}} = \tilde{\mathbf{v}}(\hat{\boldsymbol{\delta}})$ (where $\tilde{\mathbf{v}}(\boldsymbol{\delta})$ is given by Equation (7c)) and predicted random components (conditional residuals) $\hat{\mathbf{e}}_s = \mathbf{Y}_s - \mathbf{X}_s\hat{\beta} - \mathbf{Z}_s\hat{\mathbf{v}}$ are presented in [Figure 1](#). Hence, in the next section we will study the biases and MSEs of the considered MSE estimators under normality and different non-normal cases. We are interested in comparing the behavior of our estimator with its competitors under different distributions.

5.2. Simulation Study

In the model-based simulation study prepared in R ([R Core Team 2019](#)), we analyze the same MSE estimators, the same data, the same model and the same division into the sampled and non-sampled sets as in the previous section. The number of iteration equals $L = 5,000$.

Bootstrap MSE estimators are computed based on 200 replications.

The values of variable of interest are generated based on Model (25) (with parameters computed based on the whole population data), where random effects and random components are generated using the following distributions:

- normal,
- scaled t-Student with 3 degrees of freedom,
- shifted exponential,
- shifted log-normal (where the third standardized moment equals 3),
- shifted gamma (where the third standardized moment equals 4),
- shifted Pareto (where the third standardized moment equals 5).

Distributions with positive asymmetry are usually used in Monte Carlo simulation studies in economic applications (e.g., [Białek 2014](#)). The true MSEs are computed based on the following formula: $MSE = L^{-1} \sum_{l=1}^L (\hat{\theta}_l - \theta_l)^2$, where the number of Monte Carlo iterations equals $L = 5000$, $\hat{\theta}_l$ and θ_l are computed for l th iteration as Equations (14a) and (5a), respectively.

Fig. 2. Relative biases of MSE estimators for $D = 28$ domains (in percent).

In [Figure 2](#) we present relative biases of the MSE estimators computed as $rb(\widehat{MSE}) = MSE^{-1}L^{-1}\sum_{l=1}^L(\widehat{MSE}_l - MSE)$, where $\widehat{MSE}_l$ is the value of the MSE estimator computed for the l th iteration. If the normality assumption is met, the relative biases of the estimators, except *jack*, are between approximately -5% and 4%. For other distributions, absolute biases are larger, but - in our opinion - acceptable for most of the domains. Moreover, median relative biases for all of the MSE estimators except *jack* are very close to zero.

Even though for non-normal distributions the increase of the absolute biases is in our opinion and this acceptable, we observe a large increase of relative RMSEs (see [Figure 3](#)) computed as $rRMSE(\widehat{MSE}) = MSE^{-1}\left(L^{-1}\sum_{l=1}^L(\widehat{MSE}_l - MSE)^2\right)^{0.5}$. Values of relative RMSEs for normal distribution are between approximately 18% and 29% (except *jack*), but for other distributions they even exceed 100% for some cases. It is interesting to note that although large absolute biases are observed for the *jack* MSE estimator (see [Figure 2](#)) it is the MSE estimator with the smallest RMSEs for non-normal distributions (see [Figure 3](#)).

Summing up the results of the Monte Carlo simulation study presented in the previous two paragraphs, we can state that all of the considered MSEs are robust on the lack of

Fig. 3. Relative RMSEs of MSE estimators for $D = 28$ domains (in percent).

normality taking into account their biases, but not if their MSEs are considered. We observe a large decrease of their accuracies both for the heavy-tailed scaled t-Student distribution and for positively skewed distributions, where for the stronger asymmetry a larger increase of the MSE is observed. Interesting results are observed for the jackknife MSE estimator for which slightly smaller MSEs are received in non-normal cases. Furthermore, the proposed parametric bootstrap MSE estimator, jackknife and weighted jackknife MSE estimators (that we adapted for the considered predictor) have very similar properties compared with the MSE estimator-based on the Taylor expansion. However, their advantage over the Taylor expansion-based MSE estimator is that they do not require derivation of the $g_3(\delta)$ component of the MSE, which can be problematic for complex superpopulation models.

6. Conclusion

In the article, we show that the empirical version of the best linear unbiased predictor proposed by Royall (1976) is a generalization of the empirical version of the predictor

studied by [Henderson \(1950\)](#), together with the condition when they are equivalent. We generalize the parametric bootstrap MSE estimator proposed by [Butar and Lahiri \(2003\)](#) and prove its asymptotic unbiasedness. The proof of asymptotic unbiasedness requires normality of random effects and random components, which is a limitation of the method but is also a typical assumption for other MSE estimators. This is the reason why, in the application and in the simulation study, the properties of the proposed MSE estimator are compared with other MSE estimators, showing that they have very similar properties both under normality and non-normal cases. However, our method has the advantage that it does not require derivation of the $g_3(\cdot)$ component of the MSE, as the MSE estimator is based on Taylor's expansion, which can be crucial for models with a complex covariance structure.

7. Appendix

We assume the following regularity conditions, which will be referred to as (RC). The following regularity conditions (a), (b), (e), (f), (g) are direct generalizations of the respective regularity conditions proposed by [Butar and Lahiri \(2003\)](#) for the proposed longitudinal model (i.e., the definition sizes of the matrices cover the case of longitudinal data and the assumption that population and subpopulations can change in time). We replace regularity conditions (c) and (d) proposed by [Butar and Lahiri \(2003\)](#) with the following regularity conditions (c) and (d) to cover both the more general predictor (the empirical version of the predictor proposed by [Royall 1976](#)) and the proposed model. The regularity conditions (RC) are:

- (a) The elements of matrices $\mathbf{X}_k$ and $\mathbf{Z}_k$ are uniformly bounded such that $\{\mathbf{X}_s^T \mathbf{V}_{ss}^{-1} \mathbf{X}_s\} = [O(K)]_{p \times p}$,
- (b) $\sup_{1 \leq d \leq D} n_k < \infty$ and $\sup_{1 \leq d \leq D} r_k < \infty$,
- (c) $\mathbf{X}_r^T \gamma_r - \mathbf{X}_s^T \mathbf{V}_{ss}^{-1} \mathbf{V}_{sr} \gamma_r = [O(1)]_{p \times 1}$,
- (d) $\frac{\partial}{\partial \delta_c} \mathbf{X}_s^T \mathbf{V}_{ss}^{-1} \mathbf{V}_{sr} \gamma_r = [O(1)]_{p \times 1}$ where $c = 1, 2, \dots, q$,
- (e) $\mathbf{R}_{sk}(\boldsymbol{\delta}) = \sum_{c=0}^q \boldsymbol{\delta}_j \mathbf{C}_{kc} \mathbf{C}_{kc}^T$ and $\mathbf{G}_k(\boldsymbol{\delta}) = \sum_{c=0}^q \boldsymbol{\delta}_j \mathbf{F}_{kc} \mathbf{F}_{kc}^T$ where $\boldsymbol{\delta}_0 = 1$, $\mathbf{C}_{kc}$ and $\mathbf{F}_{kc}$ ($k = 1, 2, \dots, K$; $c = 0, 1, \dots, q$) are known matrices of the order $n_k \times r_k$ and $r_k \times r_k$, respectively, and the elements are uniformly bounded known constants such that $\mathbf{R}_{sk}(\boldsymbol{\delta})$ and $\mathbf{G}_k(\boldsymbol{\delta})$ ($k = 1, 2, \dots, K$) are all positive definite matrices. In special cases, some of $\mathbf{C}_{kc}$ and $\mathbf{F}_{kc}$ may be null matrices.
- (f) $\hat{\boldsymbol{\delta}}$ is an estimator of $\boldsymbol{\delta}$ that satisfies (i) $\hat{\boldsymbol{\delta}} - \boldsymbol{\delta} = O_p(K^{-1/2})$, (ii) $\hat{\boldsymbol{\delta}} - \hat{\boldsymbol{\delta}}_{ML} = O_p(K^{-1})$ (where $\hat{\boldsymbol{\delta}}_{ML}$ is the maximum-likelihood estimator of $\boldsymbol{\delta}$), (iii) $\hat{\boldsymbol{\delta}}(\mathbf{Y}_s) = \hat{\boldsymbol{\delta}}(-\mathbf{Y}_s)$, (iv) $\hat{\boldsymbol{\delta}}(\mathbf{Y}_s + \mathbf{X}_s \mathbf{b}) = \hat{\boldsymbol{\delta}}(\mathbf{Y}_s)$, for any $\mathbf{b} \in R^p$ and for all $\mathbf{Y}_s$. Assume that $E(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = \mathbf{B}_{\boldsymbol{\delta}}(\boldsymbol{\delta}) + o(K^{-1})$, which means that the approximate (to the order $o(K^{-1})$) formula of the bias of $\boldsymbol{\delta}$ is known.
- (g) $E(\hat{\beta}(\boldsymbol{\delta}) - \beta)(\boldsymbol{\delta} - \boldsymbol{\delta})^T = o(K^{-1})$.

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