Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 6(1), 2020, Pages 1–15

ISSN: Online 2351-8227 - Print 2605-6364

DOI: 10.2478/mjpaa-2020-0001

Limit cycles of Liénard polynomial systems type by averaging method

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ABSTRACT. We apply the averaging theory of first and second order for studying the limit cycles of generalized polynomial Liénard systems of the form

$$\dot{x} = y - l(x)y, \quad \dot{y} = -x - f(x) - g(x)y - h(x)y^2,$$

where $l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x)$, $f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x)$, $g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x)$ and $h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x)$ where $l_k(x)$ has degree m and $f_k(x)$, $g_k(x)$ and $h_k(x)$ have degree n for each $k = 1, 2$, and ϵ is a small parameter.

Mathematics Subject Classification (2010): 34C29, 34C25, 47H11.

Key words and phrases: Limit cycle, Liénard differential system, Periodic orbit, Averaging theory.

1. Introduction and statement of the main results

The second part of the 16th Hilbert's problem [14] wants to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree. The 16th problem, except the one related with the Riemann hypothesis, seems to be the most elusive of Hilbert's problems.

Following Smale [28] many authors consider an easier and special class of polynomial differential systems, the *Liénard polynomial differential systems*

$$\dot{x} = y, \quad \dot{y} = -x - f(x)y, \tag{1.1}$$

Received : 27 November 2019 - Accepted: 25 February 2020.

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where $f(x)$ is a polynomial in the variable x of degree n .

In 1977 Lins et al. [17] stated the conjecture that if $f(x)$ has degree $n \geq 1$ then system (1.1) has at most $[n/2]$ limit cycles. They proved this conjecture for $n = 1, 2$. The conjecture for $n = 3$ has been proved by Chengzi and Llibre in [15]. For $n \geq 5$ the conjecture is not true, see De Maesschalck and Dumortier [7] and Dumortier et al. [10]. So it remains to know if the conjecture is true or not for $n = 4$.

The following class of polynomial differential systems

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \quad (1.2)$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m , respectively is a generalisation of the Linard polynomial differential systems (1.1).

Many results on limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point, that are so called small amplitude limit cycles, see [22]. We denote by $\hat{H}(m, n)$ the maximum number of small amplitude limit cycles for systems of the form (1.2). The values of $\hat{H}(m, n)$ give a lower bound for the maximum number $H(m, n)$ (i.e. the Hilbert number) of limit cycles that the differential equation (1.2) with m and n fixed can have.

- In 1928 Linard [16] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equations (1.2) has a unique limit cycle.
- In 1973, Rychkov [26] proved that if $m = 1$ and $F(x)$ is an odd polynomial of degree five, then equations (1.2) have at most two limit cycles.
- In 1977, Lins et al. [17] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.
- Dumortier et al. in [11] and [8] proved that $H(3, 1) = 1$.
- In 1997, Dumortier and Li [9] proved that $H(2, 2) = 1$.
- In 1998 Coppel [6] proved that $H(2, 1) = 1$.

Blows, Lloyd [2] and Lynch ([23],[24]) have used inductive arguments in order to prove the following results:

- If g is odd then $\hat{H}(m, n) = [n/2]$.
- If f is even then $\hat{H}(m, n) = n$, whatever g is.
- If f is odd then $\hat{H}(m, 2n + 1) = [(m - 2)/2] + n$.
- If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

Christopher and Lynch [5] developed a new algebraic method for determining the Liapunov quantities of system (1.2) and proved the following:

- $\hat{H}(m, 2) = [(2m + 1)/3]$.
- $\hat{H}(2, n) = [(2n + 1)/3]$.
- $\hat{H}(m, 3) = 2[(3m + 2)/8]$ for all $1 < m \leq 50$.
- $\hat{H}(3, n) = 2[(3n + 2)/8]$ for all $1 < n \leq 50$.
- $\hat{H}(4, k) - \hat{H}(k, 4)$ for $k = 6, 7, 8, 9$ and $\hat{H}(5, 6) = \hat{H}(6, 5)$.

In 1998 Gasull and Torregrosa [13] obtained upper bounds for $\hat{H}(7, 6)$, $\hat{H}(6, 7)$, $\hat{H}(7, 7)$, and $\hat{H}(4, 20)$. In 2006 Yu and Han in [30] give some accurate values of $\hat{H}(m, n) = \hat{H}(n, m)$ for $n = 4, m = 10, 11, 12, 13$; $n = 5, m = 6, 7, 8, 9$; $n = 6, m = 5, 6$.

There are many results using the averaging theory studying the maximum number of limit cycles $\tilde{H}(m, n)$ which can bifurcate from the periodic orbits of a linear center $\dot{x} = y, \dot{y} = -x$, perturbed inside the class of generalized polynomial Linard differential systems:

- In 2010, Llibre et al. [18] compute the maximum number of limit cycles $\tilde{H}_k(m, n)$ of systems (1.2), using the averaging theory of order k , for $k = 1, 2, 3$.
- In 2012, Llibre and Valls [21] studied using the averaging theory of first and second order the following systems

$$\dot{x} = y - g_1(x) - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y,$$

where $g_1(x), f_1(x), g_2(x)$ and $f_2(x)$ have degree k, l, m and n , respectively.

- In 2013, Llibre and Valls studied using the averaging theory of third order the following systems (see [19] and [20] respectively)

$$\begin{aligned} \dot{x} &= y - g_1(x), \quad \dot{y} = -x - g_2(x) - f_2(x)y, \\ \dot{x} &= y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y, \end{aligned}$$

where $g_1(x), f_1(x), g_2(x)$ and $f_2(x)$ have degree k, l, m and n , respectively.

- In 2016, using the averaging theory of first and second order, Mellahi et al. [25] studied the maximum number of limit cycles bifurcating from the periodic orbits of the linear center, $\dot{x} = -y, \dot{y} = x$ perturbed inside a class of generalized Kukles polynomial differential systems

$$\begin{cases} \dot{x} &= -y + l(x), \\ \dot{y} &= x - f(x) - g(x)y - h(x)y^2 - d_0y^3. \end{cases}$$

- In 2019, Boulfoul et al. [3] introduced the more generalized Kukles systems

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x - f(x) - g(x)y - h(x)y^2 - l(x)y^3. \end{cases}$$

- In 2014, García et al. [12] studied the maximum number of small amplitude limit cycles for the more generalized polynomial Linard differential systems

$$\dot{x} = y, \quad \dot{y} = -x - f(x) - g(x)y - h(x)y^2, \tag{1.3}$$

where $f(x), g(x)$, and $h(x)$ have degree n .

In this paper using the averaging theory of first and second order, we study the maximum number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, perturbed inside the class of the more generalized polynomial Linard differential systems

$$\begin{cases} \dot{x} &= y - l(x)y, \\ \dot{y} &= -x - f(x) - g(x)y - h(x)y^2, \end{cases} \tag{1.4}$$

where $l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x)$, $f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x)$, $g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x)$ and $h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x)$ where $l_k(x)$ has degree m , $f_k(x)$, $g_k(x)$ and $h_k(x)$ have degree n for each $k = 1, 2$, and ϵ is a small parameter.

Note that when $l(x) = h(x) = 0$, coincides with generalized polynomial Linard differential

systems (1.2).

In what follows we present our main results.

Theorem 1. *For $|\epsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Linard differential systems (1.4) bifurcating from the periodic orbits of linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of first order is*

$$\tilde{H}_1(m, n) = \left\lfloor \frac{n}{2} \right\rfloor$$

limit cycles. Moreover, this upper bound is reached.

Section 3 contains the proof of Theorem 1.

Theorem 2. *For $|\epsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Linard differential systems (1.4) bifurcating from the periodic orbits of linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of second order is*

$$\tilde{H}_2(m, n) = \max \left\{ n, \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 \right\}$$

limit cycles.

Section 4 contains the proof of Theorem 2.

In [12], it has been shown that there exist generalized Linard equation (1.3), having at least n limit cycles. The result in Theorem 2 improves this lower estimate ($\tilde{H}_2(m, n) > n$ for all $n \geq 3$ and $m > n + 2$).

Note that we do not know if the upper bound $\tilde{H}_2(m, n)$ for the number of limit cycles of the polynomial differential system (1.4), which bifurcate from the periodic solutions of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order is reached. We prove that the upper bound is reached for $1 \leq n \leq 5$ and $1 \leq m \leq 5$ in the second order averaging theory (see Example in Section 4).

2. Averaging theory of first and second order

We consider the differential system

$$\dot{x} = \epsilon F_1(t, x) + \epsilon^2 F_2(t, x) + \epsilon^3 R(t, x, \epsilon), \quad (2.1)$$

where $F_1, F_2 : \mathbb{R} \times D \rightarrow \mathbb{R}$, $R : \mathbb{R} \times D \times (-\epsilon_f, \epsilon_f) \rightarrow \mathbb{R}$ are continuous functions, T -periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following hypotheses (i), (ii) hold.

- (i): $F_1(t, \cdot) \in C^2(D)$, $F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R are locally Lipschitz with respect to x , and R is twice differentiable with respect to ϵ .

We define $F_{k0} : D \rightarrow \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T} \int_0^T F_1(s, z) ds,$$

$$F_{20}(z) = \frac{1}{T} \int_0^T [D_z F_1(s, z) \cdot y_1(s, z) + F_2(s, z)] ds,$$

where

$$y_1(s, z) = \int_0^s F_1(t, z) dt.$$

(ii): For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f) \setminus \{0\}$, there exists $a \in V$ such that $F_{10}(a) + \epsilon F_{20}(a) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$. The expression $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$ means that the Brouwer degree of the function $F_{10} + \epsilon F_{20} : V \rightarrow \mathbb{R}$ at the fixed point a is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \epsilon F_{20}$ at a is not zero.

Then, for $|\epsilon| > 0$ sufficiently small there exists a T -periodic solution $\varphi(., \epsilon)$ of the equation (2.1) such that $\varphi(0, \epsilon) \rightarrow a$ when $\epsilon \rightarrow 0$.

- If F_{10} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the averaging theory of first order.
 - If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the averaging theory of second order.

For a general introduction to averaging theory see ([27], [4], [29]).

3. Proof of Theorem 1

This proof is based on the first order averaging theory, by using the coordinates (θ, r) with

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{and} \quad r > 0. \quad (3.1)$$

In this context, we take

$$f_1(x) = \sum_{i=0}^n a_i x^i, \quad g_1(x) = \sum_{i=0}^n b_i x^i, \quad h_1(x) = \sum_{i=0}^n c_i x^i \quad \text{and} \quad l_1(x) = \sum_{i=0}^m e_i x^i, \quad (3.2)$$

then system (1.4) with $k=1$ can be written as

$$\begin{cases} \dot{r} = -\epsilon \left(\sum_{i=0}^n P_{i,1}(\theta, r) r^i + \sum_{i=0}^m e_i R_{i+1}(\theta) r^{i+1} \right), \\ \dot{\theta} = -1 - \frac{\epsilon}{r} \left(\sum_{i=0}^n Q_{i,1}(\theta, r) r^i - \sum_{i=0}^m e_i T_i(\theta) r^{i+1} \right), \end{cases} \quad (3.3)$$

where

$$P_{i,1}(\theta, r) = a_i R_i(\theta) + b_i T_i(\theta) r + c_i S_i(\theta) r^2,$$

$$R_j(\theta) = \cos^j \theta \sin \theta,$$

$$T_j(\theta) = \cos^j \theta \sin^2 \theta = \cos^j \theta - \cos^{j+2} \theta,$$

$$S_j(\theta) = \cos^j \theta \sin^3 \theta = \cos^j \theta \sin \theta - \cos^{j+2} \theta \sin \theta,$$

$$Q_{i,1}(\theta, r) = a_i \cos^{i+1} \theta + b_i R_{i+1}(\theta) r + c_i T_{i+1}(\theta) r^2.$$

Now taking θ as the new independent variable, system (3.3) becomes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \epsilon F_1(\theta, r) + O(\epsilon^2),$$

where

$$F_1(\theta, r) = \sum_{i=0}^n P_{i,1}(\theta, r) r^i + \sum_{i=0}^m e_i R_{i+1}(\theta) r^{i+1}. \quad (3.4)$$

Hence

$$F_{10}(r) = \frac{1}{2\pi} \left(\sum_{i=0}^n r^i (a_i J_i(2\pi) + b_i r \tilde{I}_i(2\pi) + c_i r^2 \tilde{J}_i(2\pi)) + \sum_{i=0}^m e_i J_{i+1}(2\pi) r^{i+1} \right), \quad (3.5)$$

where

$$J_k(2\pi) = \int_0^{2\pi} R_k(\theta) d\theta, \quad \tilde{I}_k(2\pi) = \int_0^{2\pi} T_k(\theta) d\theta, \quad \tilde{J}_k(2\pi) = \int_0^{2\pi} S_k(\theta) d\theta.$$

Now using the expressions of the integrals in appendix A, we obtain

$$F_{10}(r) = r \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} r^{2i} b_{2i} \tilde{I}_{2i}(2\pi) = r \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} r^{2i} \frac{\alpha_i}{2^{i+1}(i+1)!} b_{2i}, \quad (3.6)$$

where

$$\alpha_k = 3.5 \dots (2k-1), \quad \alpha_{k+1} = (2k+1)\alpha_k.$$

Then the polynomial $F_{10}(r)$ has at most $\lfloor \frac{n}{2} \rfloor$ positive roots. We can choose the coefficients b_{2i} and, in addition, these coefficients appear multiplied by nonzero constants, it is possible to reach this upper bound. Hence, Theorem 1 is proved.

4. Proof of Theorem 2

As before we apply the change of variables (3.1). If we write $f_1(x)$, $g_1(x)$, $h_1(x)$ and $l_1(x)$ as in (3.2), and

$$f_2(x) = \sum_{i=0}^n p_i x^i, \quad g_2(x) = \sum_{i=0}^n q_i x^i, \quad h_2(x) = \sum_{i=0}^n s_i x^i \quad \text{and} \quad l_2(x) = \sum_{i=0}^m v_i x^i,$$

then system (1.4) with $k = 1, 2$ takes the form

$$\begin{cases} \dot{r} &= -\epsilon \left(\sum_{i=0}^n P_{i,1}(\theta, r) r^i + \sum_{i=0}^m e_i R_{i+1}(\theta) r^{i+1} \right) - \epsilon^2 \left(\sum_{i=0}^n P_{i,2}(\theta, r) r^i \right. \\ &\quad \left. + \sum_{i=0}^m v_i R_{i+1}(\theta) r^{i+1} \right), \\ \dot{\theta} &= -1 - \frac{\epsilon}{r} \left(\sum_{i=0}^n Q_{i,1}(\theta, r) r^i - \sum_{i=0}^m e_i T_i(\theta) r^{i+1} \right) - \frac{\epsilon^2}{r} \left(\sum_{i=0}^n Q_{i,2}(\theta, r) r^i \right. \\ &\quad \left. - \sum_{i=0}^m v_i T_i(\theta) r^{i+1} \right), \end{cases} \quad (4.1)$$

where

$$\begin{aligned} P_{i,2}(\theta, r) &= p_i R_i(\theta) + q_i T_i(\theta) r + s_i S_i(\theta) r^2, \\ Q_{i,2}(\theta, r) &= p_i \cos^{i+1} \theta + q_i R_{i+1}(\theta) r + s_i T_{i+1}(\theta) r^2. \end{aligned}$$

Taking θ as the new independent variable, system (4.1) becomes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \epsilon F_1(\theta, r) + \epsilon^2 F_2(\theta, r) + O(\epsilon^3), \quad (4.2)$$

where

$$F_1(\theta, r) = \sum_{i=0}^n P_{i,1}(\theta, r) r^i + \sum_{i=0}^m e_i R_{i+1}(\theta) r^{i+1}, \quad (4.3)$$

$$\begin{aligned} F_2(\theta, r) &= \sum_{i=0}^n P_{i,2}(\theta, r) r^i + \sum_{i=0}^m v_i R_{i+1}(\theta) r^{i+1} - \frac{1}{r} \left(\sum_{i=0}^n P_{i,1}(\theta, r) r^i \right. \\ &\quad \left. + \sum_{i=0}^m e_i R_{i+1}(\theta) r^{i+1} \right) \left(\sum_{i=0}^n Q_{i,1}(\theta, r) r^i - \sum_{i=0}^m e_i T_i(\theta) r^{i+1} \right). \end{aligned} \quad (4.4)$$

To compute $F_{20}(r)$, we need that $F_{10}(r)$ be identically zero. Then from (3.6) we consider

$$b_{2i} = 0, \quad \forall 0 \leq i \leq \left[\frac{n}{2} \right]. \quad (4.5)$$

First, using (4.5) and, by substituting in (4.3) we obtain

$$F_1(\theta, r) = \sum_{i=0}^n r^i \left(a_i R_i(\theta) + c_i r^2 S_i(\theta) \right) + \sum_{i=0}^{\left[\frac{n-1}{2} \right]} r^{2i+2} b_{2i+1} T_{2i+1}(\theta) + \sum_{i=0}^m e_i r^{i+1} R_{i+1}(\theta).$$

Then

$$\begin{aligned} \frac{dF_1(\theta, r)}{dr} &= \sum_{i=0}^n r^{i-1} \left(i a_i R_i(\theta) + (i+2) c_i r^2 S_i(\theta) \right) + \sum_{i=0}^{\left[\frac{n-1}{2} \right]} (2i+2) b_{2i+1} r^{2i+1} T_{2i+1}(\theta) \\ &\quad + \sum_{i=0}^m (i+1) e_i r^i R_{i+1}(\theta). \end{aligned} \quad (4.6)$$

Again, using the integrals of appendix A, we obtain

$$\begin{aligned}
y(\theta, r) &= \int_0^\theta F_1(s, r) ds \\
&= \sum_{i=0}^n r^i (a_i J_i(\theta) + c_i r^2 \tilde{J}_i(\theta)) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2i+2} b_{2i+1} \sum_{l=0}^{i+1} \tilde{\gamma}_{i,l} \sin((2l+1)\theta) \\
&\quad + \sum_{i=0}^m e_i r^{i+1} J_{i+1}(\theta),
\end{aligned} \tag{4.7}$$

where

$$\tilde{\gamma}_{i,l} = \begin{cases} \gamma_{i,l} - \gamma_{i+1,l}, & 0 \leq l \leq i, \\ -\gamma_{i+1,i+1}, & l = i+1. \end{cases}$$

Now, we determine the corresponding function

$$F_{20}(r) = F_{20}^1(r) + F_{20}^2(r),$$

with

$$\begin{aligned}
F_{20}^1(r) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{dF_1(\theta, r)}{dr} \cdot y(\theta, r) d\theta, \\
F_{20}^2(r) &= \frac{1}{2\pi} \int_0^{2\pi} F_2(\theta, r) d\theta.
\end{aligned}$$

In the following Lemmas we obtain some results of the integrals $F_{20}^1(r)$ and $F_{20}^2(r)$.

Lemma 3. *The integral $F_{20}^1(r)$ is a polynomial in the variable r is*

$$\begin{aligned}
F_{20}^1(r) &= \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2s+2k+1} \left(a_{2s} b_{2k+1} \left(s \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} C_{s,l} - \frac{(k+1) \alpha_{k+s+1}}{2^{k+s+1} (2s+1) (k+s+2)!} \right) \right. \\
&\quad \left. + c_{2s} b_{2k+1} r^2 \left((s+1) \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} K_{s,l} - \frac{(k+1) (4k+10s+15) \alpha_{k+s+1}}{2^{k+s+2} (2s+1) (2s+3) (k+s+3)!} \right) \right) \\
&\quad + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} e_{2s+1} b_{2k+1} r^{2s+2k+3} \left(\frac{-(k+1) \alpha_{k+s+2}}{2^{k+s+2} (2s+3) (k+s+3)!} + (s+1) \right. \\
&\quad \left. \times \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} C_{s+1,l} \right).
\end{aligned} \tag{4.8}$$

Proof. From (4.6) and (4.7) we have

$$\begin{aligned}
F_{20}^1(r) &= \sum_{i=0}^n \sum_{j=0}^n r^{i+j-1} \left(i a_i a_j \int_0^{2\pi} R_i(\theta) J_j(\theta) d\theta + i a_i c_j r^2 \int_0^{2\pi} R_i(\theta) \tilde{J}_j(\theta) d\theta \right. \\
&\quad \left. + (i+2) c_i a_j r^2 \int_0^{2\pi} S_i(\theta) J_j(\theta) d\theta + (i+2) c_i c_j r^4 \int_0^{2\pi} S_i(\theta) \tilde{J}_j(\theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{i+2k+1} \left(a_i b_{2k+1} (i \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} \int_0^{2\pi} R_i(\theta) \sin((2l+1)\theta) d\theta + (2k+2) \right. \\
& \times \int_0^{2\pi} T_{2k+1}(\theta) J_i(\theta) d\theta) + c_i b_{2k+1} r^2 ((i+2) \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} \int_0^{2\pi} S_i(\theta) \sin((2l+1)\theta) d\theta \\
& + (2k+2) \int_0^{2\pi} T_{2k+1}(\theta) \tilde{J}_i(\theta) d\theta) \Big) + \sum_{i=0}^n \sum_{j=0}^m r^{i+j} \left(a_i e_j (i \int_0^{2\pi} R_i(\theta) J_{j+1}(\theta) d\theta \right. \\
& + (j+1) \int_0^{2\pi} R_{j+1}(\theta) J_i(\theta) d\theta) + c_i e_j r^2 ((i+2) \int_0^{2\pi} S_i(\theta) J_{j+1}(\theta) d\theta \\
& + (j+1) \int_0^{2\pi} R_{j+1}(\theta) \tilde{J}_i(\theta) d\theta) \Big) + \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2k+2) b_{2s+1} b_{2k+1} r^{2s+2k+3} \\
& \times \sum_{l=0}^{s+1} \tilde{\gamma}_{s,l} \int_0^{2\pi} T_{2k+1}(\theta) \sin((2l+1)\theta) d\theta + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^m e_i b_{2k+1} r^{i+2k+2} ((2k+2) \\
& \times \int_0^{2\pi} T_{2k+1}(\theta) J_{i+1}(\theta) d\theta + (i+1) \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} \int_0^{2\pi} R_{i+1}(\theta) \sin((2l+1)\theta) d\theta) \\
& + \sum_{i=0}^m \sum_{j=0}^m (j+1) e_i e_j r^{i+j+1} \int_0^{2\pi} R_{j+1}(\theta) J_{i+1}(\theta) d\theta.
\end{aligned}$$

For simplifying expression of the polynomial $F_{20}^1(r)$, using the integrals of appendix A we have

- $\int_0^{2\pi} R_i(\theta) J_j(\theta) d\theta = \int_0^{2\pi} R_i(\theta) \tilde{J}_j(\theta) d\theta = \int_0^{2\pi} S_i(\theta) J_j(\theta) d\theta =$
 $\int_0^{2\pi} S_i(\theta) \tilde{J}_j(\theta) d\theta = \int_0^{2\pi} T_i(\theta) \sin(2l+1)\theta d\theta = 0.$
- $\int_0^{2\pi} T_{2k+1}(\theta) J_j(\theta) d\theta = \begin{cases} \frac{-1}{2s+1} \tilde{I}_{2k+2s+2}(2\pi) = \\ \frac{-\pi \alpha_{k+s+1}}{2^{k+s+1} (2s+1) (k+s+2)!}, j = 2s, \\ 0, j = 2s+1. \end{cases}$
- $\int_0^{2\pi} T_{2k+1}(\theta) \tilde{J}_j(\theta) d\theta = \begin{cases} -\frac{2}{(2s+1)} \tilde{I}_{2k+2s+2}(2\pi) + \frac{2}{(2s+3)} \tilde{I}_{2k+2s+4}(2\pi) = \\ \frac{-\pi (4k+10s+15) \alpha_{k+s+1}}{2^{k+s+2} (2s+1) (2s+3) (k+s+3)!}, j = 2s, \\ 0, j = 2s+1. \end{cases}$
- $\int_0^{2\pi} R_i(\theta) \sin(2l+1)\theta d\theta = \pi C_{i,l}.$
- $\int_0^{2\pi} S_i(\theta) \sin(2l+1)\theta d\theta = \pi K_{i,l}.$

Hence Lemma 1 is proved.

Lemma 4. *The integral $F_{20}^2(r)$ is a polynomial in the variable r is*

$$\begin{aligned}
F_{20}^2(r) &= \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\alpha_s}{2^{s+1}(s+1)!} q_{2s} r^{2s+1} - \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2s+2k+1} (a_{2s} b_{2k+1} \frac{\alpha_{s+k+1}}{2^{s+k+1}(s+k+2)!} \\
&\quad + c_{2s} b_{2k+1} r^2 \frac{3\alpha_{s+k+1}}{2^{s+k+2}(s+k+3)!}) - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} e_{2s+1} b_{2k+1} r^{2s+2k+3} \\
&\quad \times \frac{(s+k)\alpha_{s+k+1}}{2^{s+k+2}(s+k+3)!}.
\end{aligned} \tag{4.9}$$

Proof. Using (4.5) and, substituting in (4.4) we have

$$\begin{aligned}
F_2(\theta, r) &= \sum_{i=0}^n r^i (p_i R_i(\theta) + q_i r T_i(\theta) + s_i r^2 S_i(\theta)) + \sum_{i=0}^m v_i r^{i+1} R_{i+1}(\theta) - \frac{1}{r} \left(\sum_{i=0}^n r^i (a_i \right. \\
&\quad \times R_i(\theta) + c_i r^2 S_i(\theta)) + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2k+2} b_{2k+1} T_{2k+1}(\theta) + \sum_{i=0}^m e_i r^{i+1} R_{i+1}(\theta) \Big) \\
&\quad \times \left(\sum_{i=0}^n r^i (a_i \cos^{i+1}(\theta) + c_i r^2 T_{i+1}(\theta)) + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2k+2} b_{2k+1} R_{2k+2}(\theta) \right. \\
&\quad \left. \left. - \sum_{i=0}^m e_i r^{i+1} T_i(\theta) \right) \right),
\end{aligned}$$

then

$$\begin{aligned}
F_{20}^2(r) &= \sum_{i=0}^n r^i \left(p_i \int_0^{2\pi} R_i(\theta) d\theta + q_i r \int_0^{2\pi} T_i(\theta) d\theta + s_i r^2 \int_0^{2\pi} S_i(\theta) d\theta \right) \\
&\quad - \sum_{i=0}^n \sum_{j=0}^n r^{i+j-1} \left(a_i a_j \int_0^{2\pi} R_i(\theta) \cos^{j+1} \theta d\theta + a_i c_j r^2 \int_0^{2\pi} R_i(\theta) T_{j+1}(\theta) d\theta \right. \\
&\quad \left. + c_i a_j r^2 \int_0^{2\pi} S_i(\theta) \cos^{j+1} \theta d\theta + c_i c_j r^4 \int_0^{2\pi} S_i(\theta) T_{j+1}(\theta) d\theta \right) - \sum_{i=0}^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{i+2k+1} \\
&\quad \times \left(a_i b_{2k+1} \left(\int_0^{2\pi} T_{2k+1}(\theta) \cos^{i+1} \theta d\theta + \int_0^{2\pi} R_{2k+2}(\theta) R_i(\theta) d\theta \right) + c_i b_{2k+1} r^2 \right. \\
&\quad \left. \times \left(\int_0^{2\pi} T_{2k+1}(\theta) T_{i+1}(\theta) d\theta + \int_0^{2\pi} R_{2k+2}(\theta) S_i(\theta) d\theta \right) + \sum_{i=0}^n \sum_{j=0}^m r^{i+j} (a_i e_j \right. \\
&\quad \left. \times \left(\int_0^{2\pi} R_i(\theta) T_j(\theta) d\theta - \int_0^{2\pi} R_{j+1}(\theta) \cos^{i+1} \theta d\theta \right) + c_i e_j r^2 \left(\int_0^{2\pi} S_i(\theta) T_j(\theta) d\theta \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \int_0^{2\pi} R_{j+1}(\theta) T_{i+1}(\theta) d\theta) + \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} b_{2s+1} b_{2k+1} r^{2s+2k+3} \int_0^{2\pi} T_{2s+1}(\theta) \\
& \times R_{2k+2}(\theta) d\theta + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^m e_i b_{2k+1} r^{i+2k+2} \left(- \int_0^{2\pi} R_{2k+2}(\theta) R_{i+1}(\theta) d\theta \right) \\
& + \int_0^{2\pi} T_i(\theta) T_{2k+1}(\theta) d\theta + \sum_{i=0}^m \sum_{j=0}^m e_i e_j r^{i+j+1} \int_0^{2\pi} R_{j+1}(\theta) T_i(\theta) d\theta.
\end{aligned}$$

For an explicit expression of the polynomial $F_{20}^2(r)$, using the integrals of appendix A we have

- $\int_0^{2\pi} R_i(\theta) d\theta = \int_0^{2\pi} S_i(\theta) d\theta = \int_0^{2\pi} \cos^{i+1} \theta R_j(\theta) d\theta = \int_0^{2\pi} \cos^{i+1} \theta S_j(\theta) d\theta =$
 $\int_0^{2\pi} R_i(\theta) T_{j+1}(\theta) d\theta = \int_0^{2\pi} S_i(\theta) T_{j+1}(\theta) d\theta = 0.$
- $\int_0^{2\pi} T_i(\theta) d\theta = \begin{cases} 0, & \text{if } i = 2k+1, \\ \tilde{I}_{2k}(2\pi) = \frac{\pi \alpha_k}{2^{k(k+1)}!}, & \text{if } i = 2k. \end{cases}$
- $\int_0^{2\pi} \cos^{i+1} \theta T_{2k+1}(\theta) d\theta = \int_0^{2\pi} R_{2k+2}(\theta) R_i(\theta) d\theta =$
 $\begin{cases} 0, & \text{if } i = 2s+1, \\ \tilde{I}_{2k+2s+2}(2\pi) = \frac{\pi \alpha_{k+s+1}}{2^{k+s+1}(k+s+2)!}, & \text{if } i = 2s. \end{cases}$
- $\int_0^{2\pi} R_{2k+2}(\theta) S_i(\theta) d\theta = \int_0^{2\pi} T_{2k+1}(\theta) T_{i+1}(\theta) d\theta$
 $= \begin{cases} 0, & \text{if } i = 2s+1, \\ \tilde{I}_{2k+2s+2}(2\pi) - \tilde{I}_{2k+2s+4}(2\pi) = \frac{3\pi \alpha_{k+s+1}}{2^{k+s+2}(k+s+3)!}, & \text{if } i = 2s. \end{cases}$
- $\int_0^{2\pi} T_{2k+1}(\theta) T_i(\theta) d\theta = \begin{cases} 0, & \text{if } i = 2s, \\ \tilde{I}_{2k+2s+2}(2\pi) - \tilde{I}_{2k+2s+4}(2\pi) & \text{if } i = 2s+1. \\ = \frac{3\pi \alpha_{k+s+1}}{2^{k+s+2}(k+s+3)!}, \end{cases}$
- $\int_0^{2\pi} R_{2k+2}(\theta) R_{i+1}(\theta) d\theta = \begin{cases} 0, & \text{if } i = 2s, \\ \tilde{I}_{2k+2s+4}(2\pi) = \frac{\pi \alpha_{k+s+2}}{2^{k+s+2}(k+s+3)!}, & \text{if } i = 2s+1. \end{cases}$

Hence Lemma 2 is proved.

By Lemma 1 and Lemma 2, we have

$$\begin{aligned}
F_{20}(r) &= F_{20}^1(r) + F_{20}^2(r) \\
&= \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} r^{2s+1} \frac{\alpha_s}{2^{s+1}(s+1)!} q_{2s} + \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r^{2s+2k+1} \left(a_{2s} b_{2k+1} \left(s \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} C_{s,l} \right. \right. \\
&\quad \left. \left. - \frac{(2s+k+2)\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(k+s+2)!} \right) + c_{2s} b_{2k+1} r^2 ((s+1) \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} K_{s,l} \right. \\
&\quad \left. \left. - \frac{(12s^2 + 4k^2 + 34s + 10sk + 19k + 24)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(2s+3)(k+s+3)!} \right) \right) + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} r^{2s+2k+3}
\end{aligned}$$

$$\begin{aligned} & \times e_{2s+1} b_{2k+1} \left(-\frac{(2s^2 + 5s + 4sk + 8k + 2k^2 + 3)\alpha_{k+s+1}}{2^{k+s+2}(2s+3)(k+s+3)!} + (s+1) \right. \\ & \left. \times \sum_{l=0}^{k+1} \tilde{\gamma}_{k,l} C_{s+1,l} \right). \end{aligned}$$

Then, to find the real positive roots of $F_{20}(r)$, we must find the zeros of a polynomial in r^2 of degree equal to the $\max\{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 1, \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m-1}{2} \rfloor + 1\}$. Since $\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ takes the values $\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 1$ if n is odd, and the $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor + 1$ if n is even, that is n in both cases. We conclude that $F_{20}(r)$ has at most $\max\left\{n, \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m-1}{2} \rfloor + 1\right\}$ simple positive zeros.

Example. In this Example we prove that the upper bounds $\tilde{H}_2(m, n)$ is reached in the case when $1 \leq n \leq 5$ and $1 \leq m \leq 5$. The computations have been verified with the help of Maple. We denote by $F_{20}^{m,n}(r)$ the function $F_{20}(r)/r$ corresponding, an adequate computation allows to obtain that

- $F_{20}^{m,1}(r) = \frac{1}{2}q_0 - \frac{1}{2}a_0b_1 - \frac{1}{8}c_0b_1r^2$, now if we fixed $a_0 = \frac{1}{2}$, $b_1 = -2$, $c_0 = -4$ and $q_0 = 1$, we have that $F_{20}^{m,1}(r) = 1 - r^2$, that has exactly one positive zero.
- $F_{20}^{m,2}(r) = \frac{1}{2}q_0 - \frac{1}{2}a_0b_1 + (\frac{1}{8}q_2 - \frac{1}{8}c_0b_1 - \frac{1}{8}a_2b_1)r^2 - \frac{1}{48}b_1c_2r^4$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $b_1 = -2$, $c_0 = -4$, $c_2 = 12$, $q_0 = 1$ and $q_2 = 2$, we have that $F_{20}^{m,2}(r) = \frac{1}{2}(1 - r^2)(2 - r^2)$, that has exactly two positive zeros.
- $F_{20}^{1,3}(r) = F_{20}^{2,3}(r) = \frac{1}{2}q_0 - \frac{1}{2}a_0b_1 + (\frac{1}{8}q_2 - \frac{1}{8}c_0b_1 - \frac{1}{8}a_2b_1 - \frac{3}{8}a_0b_3)r^2 + (-\frac{1}{24}e_1b_3 - \frac{1}{48}b_1c_2 - \frac{5}{48}a_2b_3 - \frac{7}{48}c_0b_3)r^4 - \frac{3}{128}c_2b_3r^6$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -\frac{35}{9}$, $b_1 = -2$, $b_3 = \frac{16}{27}$, $c_0 = -4$, $c_2 = 12$, $q_0 = 1$, $q_2 = 2$ and $e_1 = \frac{125}{36}$, we have that $F_{20}^{1,3}(r) = F_{20}^{2,3}(r) = \frac{1}{6}(1 - r^2)(2 - r^2)(3 - r^2)$, that has exactly three positive zeros.
- $F_{20}^{3,3}(r) = F_{20}^{4,3}(r) = F_{20}^{1,3}(r) - \frac{1}{64}e_3b_3r^6$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $b_1 = -2$, $b_3 = \frac{16}{9}$, $c_0 = -4$, $c_2 = 12$, $q_0 = 1$, $q_2 = 2$, $e_1 = \frac{59}{4}$ and $e_3 = -12$, we have that $F_{20}^{3,3}(r) = F_{20}^{4,3}(r) = \frac{1}{6}(1 - r^2)(2 - r^2)(3 - r^2)$, that has exactly three positive zeros.
- $F_{20}^{5,3}(r) = F_{20}^{3,3}(r) - \frac{1}{128}b_3e_5r^8$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $b_1 = -2$, $b_3 = \frac{28}{9}$, $c_0 = -4$, $c_2 = 12$, $q_0 = 1$, $q_2 = 2$, $e_1 = \frac{395}{28}$, $e_3 = -\frac{66}{7}$ and $e_5 = -\frac{12}{7}$, we have that $F_{20}^{5,3}(r) = \frac{1}{24}(1 - r^2)(2 - r^2)(3 - r^2)(4 - r^2)$, that has exactly four positive zeros.
- $F_{20}^{1,4}(r) = F_{20}^{2,4}(r) = \frac{1}{2}q_0 - \frac{1}{2}a_0b_1 + (\frac{1}{8}q_2 - \frac{1}{8}c_0b_1 - \frac{1}{8}a_2b_1 - \frac{3}{8}a_0b_3)r^2 + (\frac{1}{16}q_4 - \frac{1}{24}e_1b_3 - \frac{1}{48}b_1c_2 - \frac{5}{48}a_2b_3 - \frac{7}{48}c_0b_3 - \frac{1}{16}a_4b_1)r^4 + (-\frac{3}{128}c_2b_3 - \frac{1}{128}b_1c_4 - \frac{7}{128}a_4b_3)r^6 - \frac{11}{1280}b_3c_4r^8$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -\frac{44}{9}$, $a_4 = \frac{1161}{308}$, $b_1 = -2$, $b_3 = \frac{16}{27}$, $c_0 = -4$, $c_2 = 12$, $c_4 = -\frac{90}{11}$, $q_0 = 1$, $q_2 = 2$, $q_4 = -2$ and $e_1 = \frac{63433}{44352}$ we have that $F_{20}^{1,4}(r) = F_{20}^{2,4}(r) = \frac{1}{24}(1 - r^2)(2 - r^2)(3 - r^2)(4 - r^2)$, that

has exactly four positive zeros.

- $F_{20}^{3,4}(r) = F_{20}^{4,4}(r) = F_{20}^{1,4}(r) - \frac{1}{64}e_3b_3r^6$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $a_4 = \frac{657}{308}$, $b_1 = -2$, $b_3 = \frac{16}{9}$, $c_0 = -4$, $c_2 = 12$, $c_4 = -\frac{30}{11}$, $q_0 = 1$, $q_2 = 0$, $q_4 = \frac{1417}{462}$, $e_1 = \frac{59}{4}$ and $e_3 = -12$ we have that $F_{20}^{3,4}(r) = F_{20}^{4,4}(r) = \frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, that has exactly four positive zeros.
- $F_{20}^{5,4}(r) = F_{20}^{3,4}(r) - \frac{1}{128}b_3e_5r^8$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $a_4 = \frac{1}{2}$, $b_1 = -2$, $b_3 = \frac{28}{9}$, $c_0 = -4$, $c_2 = 12$, $c_4 = \frac{49}{9}$, $q_0 = 1$, $q_2 = 2$, $q_4 = -1$, $e_1 = \frac{395}{28}$, $e_3 = -\frac{66}{7}$ and $e_5 = -\frac{4853}{630}$ we have that $F_{20}^{5,4}(r) = \frac{1}{24}(1-r^2)(2-r^2)(3-r^2)(4-r^2)$, that has exactly four positive zeros.
- $F_{20}^{1,5}(r) = F_{20}^{2,5}(r) = \frac{1}{2}q_0 - \frac{1}{2}a_0b_1 + (\frac{1}{8}q_2 - \frac{1}{8}c_0b_1 - \frac{1}{8}a_2b_1 - \frac{3}{8}a_0b_3)r^2 + (\frac{1}{16}q_4 - \frac{1}{24}e_1b_3 - \frac{1}{48}b_1c_2 - \frac{5}{48}a_2b_3 - \frac{7}{48}c_0b_3 - \frac{1}{16}a_4b_1 - \frac{5}{16}a_0b_5)r^4 + (-\frac{3}{128}c_2b_3 - \frac{1}{128}b_1c_4 - \frac{7}{128}a_4b_3 - \frac{5}{96}b_5e_1 - \frac{35}{384}a_2b_5 - \frac{55}{384}c_0b_5)r^6 + (-\frac{63}{1280}a_4b_5 - \frac{91}{3840}c_2b_5 - \frac{11}{1280}b_3c_4)r^8 - \frac{9}{1024}c_4b_5r^{10}$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -\frac{1410989}{213444}$, $a_4 = \frac{6121}{693}$, $b_1 = -2$, $b_3 = \frac{16}{27}$, $b_5 = -\frac{704}{6075}$, $c_0 = -4$, $c_2 = 12$, $c_4 = -\frac{90}{11}$, $q_0 = 1$, $q_2 = \frac{683623}{177870}$, $q_4 = -\frac{35777327}{4802490}$ and $e_1 = \frac{63433}{44352}$ we have that $F_{20}^{1,5}(r) = F_{20}^{2,5}(r) = \frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, that has exactly five positive zeros.
- $F_{20}^{3,5}(r) = F_{20}^{4,5}(r) = F_{20}^{1,5}(r) - \frac{1}{64}b_3e_3r^6 - \frac{7}{320}b_5e_3r^8$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -\frac{596927}{30492}$, $a_4 = -3$, $b_1 = -2$, $b_3 = \frac{16}{27}$, $b_5 = -\frac{704}{6075}$, $c_0 = -4$, $c_2 = 12$, $c_4 = -\frac{90}{11}$, $q_0 = 1$, $q_2 = \frac{756589}{25410}$, $q_4 = \frac{334177}{98010}$, $e_1 = \frac{63433}{44352}$ and $e_3 = \frac{2050}{77}$ we have that $F_{20}^{3,5}(r) = F_{20}^{4,5}(r) = \frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, that has exactly five positive zeros.
- $F_{20}^{5,5}(r) = F_{20}^{3,5}(r) - \frac{1}{128}b_3e_5r^8 - \frac{3}{256}b_5e_5r^{10}$, now if we fixed $a_0 = \frac{1}{2}$, $a_2 = -3$, $a_4 = \frac{1}{2}$, $b_1 = -2$, $b_3 = \frac{28}{9}$, $b_5 = 1$, $c_0 = -4$, $c_2 = 12$, $c_4 = \frac{49}{9}$, $q_0 = 1$, $q_2 = \frac{2}{5}$, $q_4 = \frac{1093657}{21870}$, $e_1 = \frac{27767}{810}$, $e_3 = -\frac{34393}{1512}$ and $e_5 = -\frac{607}{180}$ we have that $F_{20}^{5,5}(r) = \frac{1}{120}(1-r^2)(2-r^2)(3-r^2)(4-r^2)(5-r^2)$, that has exactly five positive zeros.

Appendix A. Formulae

In this appendix we recall some formulae that will be used during the paper, see for more details [1]. For $i \geq 0$ we have

$$J_i(\theta) = \int_0^\theta \cos^i t \sin t dt = \frac{1}{i+1}(1 - \cos^{i+1} \theta).$$

$$\tilde{J}_i(\theta) = \int_0^\theta \cos^i t \sin^3 t dt = \frac{2}{(i+1)(i+3)} - \frac{1}{i+1} \cos^{i+1} \theta + \frac{1}{i+3} \cos^{i+3} \theta.$$

$$J_i(2\pi) = \tilde{J}_i(2\pi) = 0.$$

$$I_i(\theta) = \int_0^{2\pi} \cos^i \theta d\theta = \begin{cases} \sum_{l=0}^k \gamma_{k,l} \sin(2l+1)\theta, & \text{if } i=2k+1; \\ \delta_k + \sum_{l=1}^k \beta_{k,l} \sin(2l\theta), & \text{if } i=2k, \end{cases}$$

where

$$\delta_i = \frac{1}{2^{2i}} \binom{2i}{i} \theta, \quad \gamma_{i,l} = \frac{1}{2^{2i}} \binom{2i+1}{i-l} \frac{1}{2l+1}, \quad \beta_{i,l} = \binom{2i}{i+l} \frac{1}{l}.$$

$$I_i(2\pi) = \begin{cases} 0, & \text{if } i=2k+1, \\ \frac{\pi \alpha_k}{2^{k-1} k!}, & \text{if } i=2k, \end{cases}$$

where

$$\alpha_k = 3.5 \dots (2k-1), \quad \alpha_{k+1} = (2k+1) \alpha_k.$$

And

$$I_{2k+2}(2\pi) = \frac{2k+1}{2k+2} I_{2k}(2\pi).$$

$$\tilde{I}_i(2\pi) = I_i(2\pi) - I_{i+2}(2\pi).$$

$$\int_0^{2\pi} \cos^i \theta \sin^j \theta \sin(2l+1)\theta d\theta \neq 0, \text{ if } i \text{ even and } j \text{ odd,}$$

$$\int_0^{2\pi} \cos^i \theta \sin^j \theta \sin(2l+1)\theta d\theta = \begin{cases} 0, & \text{if } i \text{ odd or } j \text{ even;} \\ \pi C_{k,l}, & i=2k, j=1 \text{ and } l \geq 0, \\ \pi K_{k,l}, & i=2k, j=3 \text{ and } l \geq 0, \end{cases}$$

where

$C_{k,l}, K_{k,l}$ are non-zero constants.

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