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Existence of a weak bounded solution for nonlinear degenerate elliptic equations in Musielak-Orlicz spaces

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ABSTRACT. In this paper, we show the existence of solutions for the nonlinear elliptic equations of the form

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f, \\ u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega), \end{cases}$$

where $a(x, s, \xi) \cdot \xi \geq \bar{\varphi}_x^{-1}(\varphi(x, h(|s|)))\varphi(x, |\xi|)$ and $h : \mathbb{R}^+ \rightarrow]0, 1]$ is a continuous decreasing function with unbounded primitive. The second term f belongs to $L^N(\Omega)$ or to $L^m(\Omega)$, with $m = \frac{rN}{r+1}$ for some $r > 0$ and φ is a Musielak function satisfying the Δ_2 -condition.

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1. Introduction

In recent years, there has been an increasing interest in the analysis of various mathematical problems in the spaces generated by modular functions. These problems have many considerations in applications. Classical Sobolev-spaces, variable exponent Sobolev-spaces and Orlicz-Sobolev spaces are particular types of Musielak Sobolev-spaces which are defined by a more

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general function $\varphi(x, t)$ which depends also on the space variable. This x -dependance creates some difficulties in dealing with these problems. Good news arising from the functional analysis of the setting concerning the density results of smooth functions in the modular topology in Musielak-Orlicz-Sobolev spaces under sufficient assumptions (see [2]) on the Musielak function $\varphi(x, t)$, namely,

(Φ_1) The function φ (resp. $\bar{\varphi}$) is locally integrable, that is for any constant number $c > 0$ and for any compact set $K \subset \Omega$ we have $\int_K \varphi(x, c) dx < \infty$.

(Φ_2) There exists a function $\varphi_0 : [0, 1/2] \times [0, \infty) \rightarrow [0, \infty)$ such that $\varphi_0(\cdot, s)$ and $\varphi_0(x, \cdot)$ are nondecreasing functions and for all $x, y \in \bar{\Omega}$ with $|x - y| < \frac{1}{2}$ and for any constant $c > 0$

$$\varphi(x, s) \leq \varphi_0(|x - y|, s) \varphi(y, s) \text{ with } \limsup_{\varepsilon \rightarrow 0^+} \varphi_0(\varepsilon, c\varepsilon^{-N}) < \infty.$$

These new results are necessary to study the partial differential equations in Musielak setting. Let Ω be a bounded subset of $\mathbb{R}^N$, $N \geq 2$ and φ is a Musielak function satisfying the Δ_2 -condition. Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \rightarrow W^{-1} L_{\bar{\varphi}}(\Omega)$ be an operator of Leray-Lions type of the form:

$$A(u) := -\operatorname{div} a(x, u, \nabla u).$$

In this paper we prove an existence result of bounded solutions in the setting of Musielak spaces for the elliptic problem

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where the vector $a(\cdot, \cdot, \cdot)$ satisfies

$$a(x, s, \xi) \cdot \xi \geq \bar{\varphi}_x^{-1}(\varphi(x, h(|s|))) \varphi(x, |\xi|). \quad (1.2)$$

Existence of solutions for problem (1.1) has been vigorously studied in the special cases mentioned above, see for instance [4, 6, 9]. And then in [19], Youssfi has proved the existence of bounded solutions for problem (1.1) in Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ under the condition

$$a(x, s, \xi) \cdot \xi \geq \bar{M}^{-1}(M(h(|s|))) M(|\xi|). \quad (1.3)$$

Our purpose in this study, is to prove the existence of weak bounded solution for problem (1.1) in the setting of Musielak-Orlicz-Sobolev spaces under a more general condition than (1.3). Our result generalizes the work in [19].

This paper is organized as follows, in section 2, we recall some well-known preliminaries, results and properties of Musielak-Orlicz-Sobolev spaces. Section 3 is devoted to basic assumptions and problem setting. Finally, we give the proof of the main result in section 4.

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. For further definitions and properties we refer the reader to [8, 16].

2.1. Musielak-Orlicz function. Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 2$ and let φ be real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions

(a) $\varphi(x, \cdot)$ is an N-function, i.e., convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$ and

$$\limsup_{t \rightarrow 0} \frac{\varphi(x, t)}{t} = 0 \quad \text{for almost all } x \in \Omega,$$

$$\liminf_{t \rightarrow \infty} \frac{\varphi(x, t)}{t} = \infty \quad \text{for almost all } x \in \Omega.$$

(b) $\varphi(\cdot, t)$ is a measurable function.

A function $\varphi(x, t)$, which satisfies the conditions (a) and (b), is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x, t)$ we put $\varphi_x(t) = \varphi(x, t)$ and we associate its nonnegative reciprocal function with respect to t and φ_x^{-1} that is,

$$\varphi_x^{-1}(\varphi(x, t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering:

(c) If there exists two positive constants c and T such that for almost all $x \in \Omega$

$$\varphi(x, t) \leq \gamma(x, ct) \quad \text{for } t \geq T,$$

then we write $\varphi \prec \gamma$ and we say that γ dominates φ globally if $T = 0$ and near infinity if $T > 0$.

(d) If for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t \rightarrow 0} \left(\sup_{x \in \Omega} \frac{\varphi(x, ct)}{\gamma(x, t)} \right) = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} \left(\sup_{x \in \Omega} \frac{\varphi(x, ct)}{\gamma(x, t)} \right) = 0,$$

then we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

We recall that the Musielak function φ is said to satisfy the Δ_2 -condition (or doubling) if for some $k > 0$, and a non-negative function c , integrable on Ω , we have

$$\varphi(x, 2t) \leq k\varphi(x, t) + c(x) \quad \text{for all } x \in \Omega \text{ and all } t \geq 0.$$

2.2. Musielak-Orlicz-Sobolev spaces. For a Musielak function φ and a measurable function $u : \Omega \rightarrow \mathbb{R}$ we define the functional

$$\varrho_{\varphi, \Omega}(u) = \int_{\Omega} \varphi(x, |u(x)|) dx.$$

The set $K_{\varphi}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \varrho_{\varphi, \Omega}(u) < \infty \right\}$ is called the Musielak class (or the Musielak-Orlicz class or generalized Orlicz class). The Musielak space (or Musielak-Orlicz space or generalized Orlicz space) $L_{\varphi}(\Omega)$ is the vector space generated by $K_{\varphi}(\Omega)$, that is the smallest linear space containing the set $K_{\varphi}(\Omega)$. Equivalently

$$L_{\varphi}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \varrho_{\varphi, \Omega}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\}.$$

For a Musielak function φ we put

$$\bar{\varphi}(x, s) = \sup_{t \geq 0} \{st - \varphi(x, t)\}.$$

$\bar{\varphi}$ is called the Musielak function complementary to φ (or conjugate of φ) in the sense of Young with respect to s .

we say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n \rightarrow \infty} \varrho_{\varphi, \Omega} \left(\frac{u_n - u}{\lambda} \right) = 0.$$

In the space $L_\varphi(\Omega)$ we can define two norms, the first is called the Luxemburg norm, that is

$$\|u\|_{\varphi, \Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \varphi(x, \frac{|u(x)|}{\lambda}) dx \leq 1 \right\}$$

and the second so-called the Orlicz norm, that is

$$\|u\|_{\varphi, \Omega} = \sup_{\|v\|_{\bar{\varphi}} \leq 1} \int_{\Omega} |u(x) v(x)| dx,$$

where $\bar{\varphi}$ is the Musielak function complementary to φ . These two norms are equivalent and we have a Musielak class $K_\varphi(\Omega)$ is a convex subset of the Musielak space $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and if $\varphi \in \Delta_2$, we have $(E_{\bar{\varphi}}(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfy the Δ_2 -condition for large values of t or for all values of t , according to whether Ω has finite measure or not.

We define

$$W^1 L_\varphi(\Omega) = \left\{ u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \forall |\alpha| \leq 1 \right\}$$

$$W^1 E_\varphi(\Omega) = \left\{ u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \forall |\alpha| \leq 1 \right\}$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$, $|\alpha| = |\alpha_1| + \dots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1 L_\varphi(\Omega)$ is called the Musielak-Sobolev space. For $u \in W^1 L_\varphi(\Omega)$, let

$$\bar{\varrho}_{\varphi, \Omega}(u) = \sum_{|\alpha| \leq 1} \varrho_{\varphi, \Omega}(D^\alpha u) \text{ and } \|u\|_{\varphi, \Omega}^1 = \inf \left\{ \lambda > 0 : \bar{\varrho}_{\varphi, \Omega} \left(\frac{u}{\lambda} \right) \leq 1 \right\}$$

these functionals are convex modular and a norm on $W^1 L_\varphi(\Omega)$ respectively. The pair $\langle W^1 L_\varphi(\Omega), \|u\|_{\varphi, \Omega}^1 \rangle$ is a Banach space if φ satisfy the following condition

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x \in \Omega} \varphi(x, 1) > c.$$

The space $W^1 L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha| \leq 1} L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\bar{\varphi}})$ closed.

We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth function with compact support in Ω and by $\mathfrak{D}(\bar{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω . We have $\mathfrak{D}(\Omega) \subset E_\varphi(\Omega)$ and $(E_\varphi(\Omega))^*$ is

isomorphic to $L_\varphi(\Omega)$, then the space $W_0^1 L_\varphi(\Omega)$ is well defined as the $\sigma(\Pi L_\varphi, \Pi E_{\bar{\varphi}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1 L_\varphi(\Omega)$ and the space $W_0^1 E_\varphi(\Omega)$ as the (norm) closure of the Schwarz space $\mathfrak{D}(\Omega)$ in $W^1 L_\varphi(\Omega)$.

For two complementary Musielak functions φ and $\bar{\varphi}$ we have

i) The Young inequality:

$$ts \leq \varphi(x, t) + \bar{\varphi}(x, s) \text{ for all } t, s \geq 0, x \in \Omega.$$

ii) The Hölder inequality:

$$\left| \int_{\Omega} u(x) v(x) dx \right| \leq 2 \|u\|_{\varphi, \Omega} \|v\|_{\bar{\varphi}, \Omega}, \text{ for all } u \in L_\varphi(\Omega), v \in L_{\bar{\varphi}}(\Omega).$$

We say that a sequence of function u_n converges to u for the modular convergence in $W^1 L_\varphi(\Omega)$ (respectively in $W_0^1 L_\varphi(\Omega)$) if we have

$$\lim_{n \rightarrow \infty} \bar{\varrho}_{\varphi, \Omega} \left(\frac{u_n - u}{\lambda} \right) = 0, \text{ for some } \lambda > 0.$$

Define the following space of distributions

$$W^{-1} L_{\bar{\varphi}}(\Omega) = \left\{ f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in L_{\bar{\varphi}}(\Omega) \right\}$$

and

$$W^{-1} E_{\bar{\varphi}}(\Omega) = \left\{ f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in E_{\bar{\varphi}}(\Omega) \right\}.$$

2.3. Some technical lemmas.

Lemma 2.1. [2] Assume that Ω satisfies the segment property and φ is Musielak function satisfies assumptions (Φ_1) and (Φ_2) and $\bar{\varphi}$ satisfies (Φ_1) , then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathfrak{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathfrak{D}(\bar{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution in $W^{-1} L_{\bar{\varphi}}(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined.

Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property (see [13]) if there exist a locally finite open covering O_i of the boundary $\partial\Omega$ of Ω and a corresponding vectors y_i such that if $x \in \bar{\Omega} \cap O_i$ for some i , then $x + ty_i \in \Omega$ for $0 < t < 1$.

Lemma 2.2. [5] (The Nemytskii operator) Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak functions. Let $f : \Omega \times \mathbb{R}^{p_1} \rightarrow \mathbb{R}^{p_2}$ be a Caratheodory function such that

$$|f(x, s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2 |s|),$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}^{p_1}$, where k_1, k_2 are real positive constant and $c \in E_\psi(\Omega)$. Then the Nemytskii operator N_f , defined by $N_f(u)(x) = f(x, u(x))$ is continuous from $\left(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2}) \right)^{p_1} = \Pi \left\{ u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2} \right\}$ into $(L_\psi)^{p_2}$ for the modular convergence.

Furthermore, if $c \in E_\gamma(\Omega)$ and $\gamma \prec \prec \psi$ then N_f is strongly continuous from $\left(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^{p_1}$ into $\left(E_\gamma(\Omega)\right)^{p_2}$.

Lemma 2.3. *If $f_n \in L^1(\Omega)$ with $f_n \rightarrow f \in L^1(\Omega)$ a.e. in Ω , $f_n, f \geq 0$ a.e. in Ω and $\int_\Omega f_n(x) dx \rightarrow \int_\Omega f(x) dx$, then $f_n \rightarrow f$ in $L^1(\Omega)$.*

2.4. Decreasing rearrangement. We recall the definition of decreasing rearrangement of a real-valued measurable function u in a measurable subset Ω of $\mathbb{R}^N$ having finite measure. Let $|E|$ stands the Lebesgue measure of a subset E of Ω . The distribution function of u , denoted by μ_u , is a map which informs about the content of level sets of u , that is

$$\mu_u(t) = \left| \{x \in \Omega : |u(x)| > t\} \right|, \quad t \geq 0.$$

The decreasing rearrangement of u is defined as the generalized inverse function of μ_u , that is the function $u^* : [0, |\Omega|] \rightarrow [0, +\infty]$, defined as

$$u^*(s) = \inf \left\{ t \geq 0 : \mu_u(t) \leq s \right\}, \quad s \in [0, |\Omega|].$$

In other words, u^* is the (unique) non-increasing, right-continuous function in $[0, +\infty)$ equidistributed with u . Furthermore, for every $t \geq 0$

$$u^*(\mu_u(t)) \leq t. \quad (2.1)$$

We also recall that

$$u^*(0) = \text{ess sup } |u|. \quad (2.2)$$

3. Assumptions and main result

Through this paper Ω will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and φ is a doubling Musielak function. There exists an Orlicz function q (see remark 3.1 below) such that

$$q(t) \leq \varphi(x, t).$$

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \rightarrow W^{-1} L_{\bar{\varphi}}(\Omega)$ be the mapping (non-everywhere defined) given by

$$A u := -\text{div } a(x, u, \nabla u),$$

We are interested, in proving the existence of bounded solutions in the setting of Musielak spaces to the nonlinear problem

$$\begin{cases} A u := -\text{div } a(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Caratheodory function satisfying, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$ the following conditions

$$a(x, s, \xi) \cdot \xi \geq \bar{\varphi}_x^{-1}(\varphi(x, h(|s|))) \varphi(x, |\xi|) \quad (3.2)$$

where $h : \mathbb{R} \rightarrow \mathbb{R}_+^*$ is a continuous decreasing function such that : $h(0) \leq 1$ and its primitive $H(s) = \int_0^s h(t) dt$ is unbounded. there exist a function $a_0(x) \in E_{\bar{\varphi}}(\Omega)$ and some positive constants k_1, k_2, k_3 and k_4 and a Musielak function $P \prec \prec \varphi$ such that

$$|a(x, s, \xi)| \leq a_0(x) + k_1 \bar{P}_x^{-1}(\varphi(x, k_2|s|)) + k_3 \bar{\varphi}_x^{-1}(\varphi(x, k_4|\xi|)), \quad (3.3)$$

$$(a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) > 0. \quad (3.4)$$

For that concerns the right hand side the data f , we assume one of the following two assumptions: Either

$$f \in L^N(\Omega), \quad (3.5)$$

or

$$\begin{cases} f \in L^m(\Omega) \text{ with } m = \frac{rN}{r+1} \text{ for some } r > 0, \\ \text{and } \int_0^{+\infty} \left(\frac{t}{q(t)} \right)^r dt < +\infty. \end{cases} \quad (3.6)$$

Remark 3.1. Notice that, in particular case, in variable exponent spaces when $\varphi(x, t) = t^{p(x)}$, the Orlicz function $q(t)$ plays the role of t^{p^-} , where $p^- = \operatorname{ess\,inf}_{x \in \Omega} p(x)$. Moreover, $q(t)$ is an N -function satisfying the Δ_2 -condition, for its construction see ([14], [11, Lemma A.4]).

Definition 3.1. Let $f \in L^N(\Omega)$, we say that a function $u \in W_0^1 L_\varphi(\Omega)$ is a weak solution of (3.1), if $a(\cdot, u, \nabla u) \in (L_{\bar{\varphi}}(\Omega))^N$ and

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx$$

holds for all $v \in \mathfrak{D}(\Omega)$.

Lemma 3.1. Let φ be a Musielak function and q is an Orlicz function such that $q(t) \leq \varphi(x, t)$ for all $(x, t) \in \Omega \times \mathbb{R}^+$ then

$$\bar{q}^{-1}(q(t)) \leq \bar{\varphi}_x^{-1}(\varphi(x, t)) \text{ for all } (x, t) \in \Omega \times \mathbb{R}^+,$$

where $\bar{q}$ and $\bar{\varphi}$ are the complementary functions of q and φ respectively.

Proof. Let $s, t \in \mathbb{R}^+$ and $x \in \Omega$. We have $q(t) \leq \varphi(x, t)$, then

$$st - q(t) \geq st - \varphi(x, t).$$

Passing to the sup over $t \geq 0$

$$\sup_{t \geq 0} \{st - q(t)\} \geq \sup_{t \geq 0} \{st - \varphi(x, t)\}.$$

That means

$$\bar{q}(s) \geq \bar{\varphi}(x, s) := \bar{\varphi}_x(s), \text{ for all } s \in \mathbb{R}^+.$$

It follows that for all $s \in \mathbb{R}^+$,

$$\bar{q}^{-1}(s) \leq \bar{\varphi}_x^{-1}(s)$$

Taking $s = q(t)$, since $\bar{\varphi}_x^{-1}$ is increasing, we have $\forall t \in \mathbb{R}^+$,

$$\bar{q}^{-1}(q(t)) \leq \bar{\varphi}_x^{-1}(q(t)) \leq \bar{\varphi}_x^{-1}(\varphi(x, t)).$$

Remark 3.2. Note that from $\int_{\Omega} \varphi(x, c) dx < \infty$ and the fact that $q(t) \leq \varphi(x, t) \iff \bar{\varphi}(x, t) \leq \bar{q}(t)$, the assumption (Φ_1) is automatically verified for φ and $\bar{\varphi}$.

The next theorem is the main result of this paper.

Theorem 3.1. Suppose that the assumptions (Φ_2) , (3.2), (3.3), (3.4) and either (3.5) or (3.6) are satisfied and assume that $\int_{\Omega} \varphi(x, c) dx < \infty$ for all constant $c > 0$. Then the following problem

$$\begin{cases} \Delta u := -\operatorname{div} a(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

has at least one weak solution in $W_0^1 L_{\varphi}(\Omega) \cap L^{\infty}(\Omega)$.

4. Proof of main result.

The proof of theorem (3.1) is divided into six steps.

Step 1: Approximate problems. Let us define $T_k(s) = \max(-k, \min(k, s))$, $k > 0$ and the function $G_k(s) = s - T_k(s)$. For $n \in \mathbb{N}^*$ denoting by m^* either N or m according as we assume (3.5) or (3.6), and let $\{f_n\} \subset W^{-1} E_{\bar{\varphi}}(\Omega)$ be a sequence of smooth functions such that

$$f_n \rightarrow f \text{ strongly in } L^{m^*}(\Omega)$$

and

$$\|f_n\|_{m^*} \leq \|f\|_{m^*},$$

consider the operator:

$$A_n(u) := -\operatorname{div} a(x, T_n(u), \nabla u)$$

the operator A_n satisfies the conditions of [8], then the approximate problem has at least a solution u_n such that

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v dx = \int_{\Omega} f_n v dx \quad (4.1)$$

holds for all $v \in W_0^1 L_{\varphi}(\Omega)$.

Step 2: Preliminary results.

Lemma 4.1. Let $K(t) = \frac{q(t)}{t}$ and $\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|$, for all $t > 0$. We have for almost every $t > 0$:

$$h(t) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{1/N}} \frac{-\mu'_n(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1} \left(\frac{\int_{\{|u_n|>t\}} |f_n| dx}{\bar{q}^{-1}(q(1))NC_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} \right) \quad (4.2)$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$.

Proof. Testing by $v = T_k(G_t(u_n))$ in (4.1), one has

$$\int_{\{t < |u_n| \leq t+k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \leq k \int_{\{|u_n| > t\}} |f_n| \, dx.$$

Then (3.2) yields

$$\frac{1}{k} \int_{\{t < |u_n| \leq t+k\}} \bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \varphi(x, |\nabla u_n|) \, dx \leq \int_{\{|u_n| > t\}} |f_n| \, dx.$$

Let $k \rightarrow 0^+$ we have

$$-\frac{d}{dt} \int_{\{|u_n| > t\}} \bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \varphi(x, |\nabla u_n|) \, dx \leq \int_{\{|u_n| > t\}} |f_n| \, dx, \quad (4.3)$$

for almost every $t > 0$. We use the properties of the function $Q(t) = \frac{1}{K^{-1}(t)}$ which is decreasing and convex (see [17]). By lemma (3.1), $\varphi(x, |\nabla u_n|) \geq q(|\nabla u_n|) \geq 0$ and $\bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \geq \bar{q}^{-1}(q(h(|u_n|))) \geq 0$, then

$$\bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \varphi(x, |\nabla u_n|) \geq \bar{q}^{-1}(q(h(|u_n|))) q(|\nabla u_n|)$$

Using the fact that Q is decreasing and then Jensen's inequality yield

$$\begin{aligned} & Q \left(\frac{\int_{\{t < |u_n| \leq t+k\}} \bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \varphi(x, |\nabla u_n|) \, dx}{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) |\nabla u_n| \, dx} \right) \\ & \leq Q \left(\frac{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) q(|\nabla u_n|) \, dx}{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) |\nabla u_n| \, dx} \right) \\ & = Q \left(\frac{\int_{\{t < |u_n| \leq t+k\}} K(|\nabla u_n|) \bar{q}^{-1}(q(h(|u_n|))) |\nabla u_n| \, dx}{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) |\nabla u_n| \, dx} \right) \\ & \leq \frac{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) \, dx}{\int_{\{t < |u_n| \leq t+k\}} \bar{q}^{-1}(q(h(|u_n|))) |\nabla u_n| \, dx} \\ & \leq \frac{\bar{q}^{-1}(q(h(t))) (-\mu_n(t+k) + \mu_n(t))}{\bar{q}^{-1}(q(h(t+k))) \int_{\{t < |u_n| \leq t+k\}} |\nabla u_n| \, dx}. \end{aligned}$$

Since $\frac{\bar{q}^{-1}(q(h(t)))}{\bar{q}^{-1}(q(1))} \leq 1$, using the convexity of Q and then letting $k \rightarrow 0^+$, we get for almost every $t > 0$,

$$\begin{aligned} & \frac{\bar{q}^{-1}(q(1))}{\bar{q}^{-1}(q(h(t)))} Q \left(\frac{-\frac{d}{dt} \int_{\{|u_n|>t\}} \bar{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \varphi(x, |\nabla u_n|) dx}{\bar{q}^{-1}(q(1)) \left(-\frac{d}{dt} \int_{\{|u_n|>t\}} |\nabla u_n| dx\right)} \right) \\ & \leq \frac{-\mu'_n(t)}{-\frac{d}{dt} \int_{\{|u_n|>t\}} |\nabla u_n| dx}. \end{aligned}$$

Now we recall the following inequality from [17]:

$$-\frac{d}{dt} \int_{\{|u_n|>t\}} |\nabla u_n| dx \geq NC_N^{1/N} \mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t > 0. \quad (4.4)$$

Thanks to (4.3), (4.4) and the monotonicity of the function Q we obtain

$$\begin{aligned} & \frac{1}{\bar{q}^{-1}(q(h(t)))} \\ & \leq \frac{-\mu'_n(t)}{\bar{q}^{-1}(q(1)) NC_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} K^{-1} \left(\frac{\int_{\{|u_n|>t\}} |f_n| dx}{\bar{q}^{-1}(q(1)) NC_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} \right). \end{aligned}$$

Using the inequality in Orlicz spaces (see [19])

$$q(t) \leq t \bar{q}^{-1}(q(t)) \leq 2q(t) \quad \text{for all } t \geq 0$$

and the fact that $0 < h(t) \leq 1$, (4.2) follows.

step 3: Uniform L^∞ -estimation. From (4.2), using the rearrangement techniques, we prove that there exists a constant c_∞ such that

$$\|u_n\|_\infty \leq c_\infty. \quad (4.5)$$

Proof. If we assume (3.5), by Hölder's inequality one has

$$\int_{\{|u_n|>t\}} |f_n| dx \leq \|f\|_N \mu_n(t)^{1-\frac{1}{N}}.$$

Inequality (4.2) becomes

$$h(t) \leq \frac{2q(1)(-\mu'_n(t))}{\bar{q}^{-1}(q(1)) NC_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1)) NC_N^{1/N}} \right).$$

Then, integrating between 0 and s , we get

$$\int_0^s h(t)dt \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right) \int_0^s \frac{-\mu'_n(t)}{\mu_n(t)^{1-\frac{1}{N}}} dt.$$

So, one has

$$H(s) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right) \int_0^s \frac{-\mu'_n(t)}{\mu_n(t)^{1-\frac{1}{N}}} dt.$$

Hence, a change of variables yields

$$H(s) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right) \int_{\mu_n(s)}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

By (2.1) we get

$$H(u_n^*(\sigma)) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right) \int_{\sigma}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

So that

$$H(u_n^*(0)) \leq \frac{2q(1)|\Omega|^{\frac{1}{N}}}{\bar{q}^{-1}(q(1))C_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right).$$

Thanks to (2.2) and the fact that $\lim_{s \rightarrow +\infty} H(s) = +\infty$, we conclude that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, if we denote by H^{-1} the inverse function of H , one has:

$$\|u_n\|_\infty \leq H^{-1} \left(\frac{2q(1)|\Omega|^{\frac{1}{N}}}{\bar{q}^{-1}(q(1))C_N^{\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_N}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \right) \right). \quad (4.6)$$

We now assume that (3.6) is filled. Then, using again Hölder's inequality we have

$$\int_{\{|u_n|>t\}} |f_n| dx \leq \|f\|_m \mu_n(t)^{1-\frac{1}{m}}.$$

Inequality (4.2) becomes

$$h(s) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \frac{-\mu'_n(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} \mu_n(t)^{\frac{1}{m}-\frac{1}{N}}} \right).$$

Integrating between 0 and s , we get

$$H(s) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_0^s \frac{-\mu'_n(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1} \left(\frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} \mu_n(t)^{\frac{1}{m}-\frac{1}{N}}} \right) dt.$$

Then, a change of variables gives

$$H(s) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_{\mu_n(s)}^{|\Omega|} K^{-1} \left(\frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} \sigma^{\frac{1}{m} - \frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1 - \frac{1}{N}}}.$$

By virtue of (2.1) we get

$$H(u_n^*(\tau)) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_{\tau}^{|\Omega|} K^{-1} \left(\frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} \sigma^{\frac{1}{m} - \frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1 - \frac{1}{N}}}.$$

Then, by (2.2) we obtain

$$H(\|u_n\|_{\infty}) \leq \frac{2q(1)}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_0^{|\Omega|} K^{-1} \left(\frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} \sigma^{\frac{1}{m} - \frac{1}{N}}} \right) \frac{d\sigma}{\sigma^{1 - \frac{1}{N}}}.$$

A change of variables gives

$$H(\|u_n\|_{\infty}) \leq \frac{2q(1)c_0^r \|f\|_m^r}{(\bar{q}^{-1}(q(1)))^{r+1} N^r C_N^{\frac{r+1}{N}}} \int_{\lambda}^{+\infty} r t^{-r-1} K^{-1}(t) dt,$$

where $\lambda = \frac{c_0 \|f\|_m}{\bar{q}^{-1}(q(1))NC_N^{\frac{1}{N}} |\Omega|^{\frac{1}{rN}}}$. Then, by an integration by parts we obtain that

$$H(\|u_n\|_{\infty}) \leq \frac{2q(1)c_0^r \|f\|_m^r}{(\bar{q}^{-1}(q(1)))^{r+1} N^r C_N^{\frac{r+1}{N}}} \left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty} \left(\frac{s}{q(s)} \right)^r ds \right).$$

The assumption made on H guarantees that the sequence $\{u_n\}$ is uniformly bounded in $L^{\infty}(\Omega)$. Indeed, denoting by H^{-1} the inverse function of H , one has

$$\|u_n\|_{\infty} \leq H^{-1} \left(\frac{2q(1)c_0^r \|f\|_m^r}{(\bar{q}^{-1}(q(1)))^{r+1} N^r C_N^{\frac{r+1}{N}}} \left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty} \left(\frac{s}{q(s)} \right)^r ds \right) \right). \quad (4.7)$$

Consequently, in both cases the sequence $\{u_n\}$ is uniformly bounded in $L^{\infty}(\Omega)$, so that in the sequel, we will denote by c_{∞} the constant appearing either in (4.6) or in (4.7), that is :

$$\|u_n\|_{\infty} \leq c_{\infty}. \quad (4.8)$$

Step 4: Estimation in $W_0^1 L_{\varphi}(\Omega)$. Now we will obtain an estimate in $W_0^1 L_{\varphi}(\Omega)$ under either (3.5) or (3.6). Let m^* denotes either N or m according as we assume (3.5) or (3.6). Testing (4.1) by u_n , it follows

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx = \int_{\Omega} f_n u_n dx.$$

Then by (3.2) and (4.5), we obtain

$$\int_{\Omega} \bar{\varphi}_x^{-1}(\varphi(x, h(|T_n(u_n)|))) \varphi(x, |\nabla u_n|) dx \leq c_{\infty} \|f\|_{m^*} |\Omega|^{1 - \frac{1}{m^*}}.$$

By lemma (3.1) we have

$$\begin{aligned} \int_{\Omega} \bar{q}^{-1}(q(h(|T_n(u_n)|))) \varphi(x, |\nabla u_n|) dx &\leq \int_{\Omega} \bar{\varphi}_x^{-1}(\varphi(x, h(|T_n(u_n)|))) \varphi(x, |\nabla u_n|) dx \\ &\leq c_{\infty} \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}}. \end{aligned}$$

Finally, since the function h is decreasing and $T_n(u_n) \leq T_n(c_{\infty}) \leq c_{\infty}$, we get

$$\int_{\Omega} \varphi(x, |\nabla u_n|) dx \leq \frac{c_{\infty} \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}}}{\bar{q}^{-1}(q(h(c_{\infty})))}. \quad (4.9)$$

Which implies, by Poincaré's inequality from [3], that the sequence $\{u_n\}$ is bounded in $W_0^1 L_{\varphi}(\Omega)$. And then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function u in $W_0^1 L_{\varphi}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^1 L_{\varphi}(\Omega) \text{ for } \sigma(\Pi L_{\varphi}, \Pi E_{\bar{\varphi}}) \quad (4.10)$$

and the compact embedding gives

$$u_n \rightarrow u \quad \text{in } E_{\varphi}(\Omega) \text{ strongly and a.e. in } \Omega. \quad (4.11)$$

Step 5: Almost everywhere convergence of the gradients. The next lemma will be used later, so proving it first.

Lemma 4.2. *The sequence $\{a(x, T_n(u_n), \nabla u_n)\}$ is bounded in $(L_{\bar{\varphi}}(\Omega))^N$.*

Proof. We will use the dual norm of $(L_{\bar{\varphi}}(\Omega))^N$. Let $\phi \in (E_{\varphi}(\Omega))^N$ such that $\|\phi\|_{\varphi} = 1$. By (3.4) we have

$$\left(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \frac{\phi}{k_4}) \right) \cdot \left(\nabla u_n - \frac{\phi}{k_4} \right) \geq 0.$$

Thanks to (3.3), (4.5), (4.9) and Young's inequality we get

$$\begin{aligned} &\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \phi dx \\ &\leq k_4 \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx - k_4 \int_{\Omega} a(x, T_n(u_n), \frac{\phi}{k_4}) \cdot \nabla u_n dx \\ &\quad + \int_{\Omega} a(x, T_n(u_n), \frac{\phi}{k_4}) \cdot \phi dx \\ &\leq k_4 c_{\infty} \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}} + k_4 (1 + k_1 + k_3) \frac{c_{\infty} \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}}}{\bar{q}^{-1}(q(h(c_{\infty})))} \\ &\quad + (1 + k_4) \int_{\Omega} \bar{\varphi}(x, a_0(x)) dx + k_1 (1 + k_4) \int_{\Omega} \bar{\varphi}(x, \bar{P}_x^{-1}(\varphi(x, k_2 c_{\infty}))) dx \\ &\quad + (k_3 (1 + k_4) + (1 + k_1 + k_3)) \int_{\Omega} \varphi(x, 1) dx. \end{aligned}$$

Since $a_0(x) \in E_{\bar{\varphi}}(\Omega)$, $\int_{\Omega} \bar{\varphi}(x, a_0(x)) dx < +\infty$ and we have $\int_{\Omega} \varphi(x, 1) dx < +\infty$. For the remaining integral in the last inequality, recall that $P \prec\prec \varphi \Leftrightarrow \bar{\varphi} \prec\prec \bar{P}$, and using the fact

that

$$\bar{\varphi} \prec \prec \bar{P} \Rightarrow \forall \varepsilon > 0, \exists h \in L^1(\Omega) : \quad \bar{\varphi}(x, t) \leq \bar{P}(x, \varepsilon t) + h(x).$$

Thus, taking $\varepsilon \leq 1$, and since $\bar{P}$ is increasing, we get

$$\begin{aligned} \int_{\Omega} \bar{\varphi}(x, \bar{P}_x^{-1}(\varphi(x, k_2 c_{\infty}))) dx &\leq \int_{\Omega} \bar{P}(x, \varepsilon \bar{P}_x^{-1}(\varphi(x, k_2 c_{\infty}))) dx + \int_{\Omega} h(x) dx \\ &\leq \int_{\Omega} \bar{P}(x, \bar{P}_x^{-1}(\varphi(x, k_2 c_{\infty}))) dx + \int_{\Omega} h(x) dx \\ &\leq \int_{\Omega} \varphi(x, k_2 c_{\infty}) dx + \int_{\Omega} h(x) dx \\ &< +\infty. \end{aligned}$$

This ends the proof.

From (4.5) and (4.11) we obtain that $u \in W_0^1 L_{\varphi}(\Omega) \cap L^{\infty}(\Omega)$, so that by lemma 2.1 there exists a sequence $\{v_j\}$ in $\mathfrak{D}(\Omega)$ such that $v_j \rightarrow u$ in $W_0^1 L_{\varphi}(\Omega)$ as $j \rightarrow \infty$ for the modular convergence and almost everywhere in Ω .

For $s > 0$, we denote by χ_j^s the characteristic function of the set

$$\Omega_j^s = \{x \in \Omega : |\nabla v_j(x)| \leq s\}$$

and by χ^s the characteristic function of the set $\Omega^s = \{x \in \Omega : |\nabla u(x)| \leq s\}$. Testing by $u_n - v_j$ in (4.1), we obtain

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n - \nabla v_j) dx = \int_{\Omega} f_n(u_n - v_j) dx \quad (4.12)$$

Denote by $\epsilon_i(n, j)$, ($i = 0, 1, \dots$), various sequences of real numbers which tend to 0 when n and $j \rightarrow \infty$, i.e.

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \epsilon_i(n, j) = 0.$$

For the right-hand side of (4.12), we have

$$\int_{\Omega} f_n(u_n - v_j) dx = \epsilon_0(n, j). \quad (4.13)$$

The left-hand side of (4.12) is written as

$$\begin{aligned} &\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n - \nabla v_j) dx \\ &= \int_{\Omega} \left(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s) \right) \cdot (\nabla u_n - \nabla v_j \chi_j^s) dx \\ &\quad + \int_{\Omega} a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) dx \\ &\quad - \int_{\Omega \setminus \Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j dx \end{aligned} \quad (4.14)$$

We will pass to the limit over n and j , for s fixed, in the second and the third terms of the right-hand side of (4.14). By Lemma 4.2, we deduce that there exists $\xi_0 \in (L_{\bar{\varphi}}(\Omega))^N$ and

up to a subsequence $a(x, T_n(u_n), \nabla u_n) \rightharpoonup \xi_0$ weakly in $(L_{\bar{\varphi}}(\Omega))^N$ for $\sigma(\Pi L_{\bar{\varphi}}, \Pi E_{\varphi})$. Since $\nabla v_j \chi_{\Omega \setminus \Omega_j^s} \in (E_{\varphi}(\Omega))^N$, we have by letting $n \rightarrow \infty$,

$$- \int_{\Omega \setminus \Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j \, dx \rightarrow - \int_{\Omega \setminus \Omega_j^s} \xi_0 \cdot \nabla v_j \, dx.$$

Using the modular convergence of v_j , we get as $j \rightarrow \infty$

$$- \int_{\Omega \setminus \Omega_j^s} \xi_0 \cdot \nabla v_j \, dx \rightarrow - \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx.$$

Hence, we have proved that the third term

$$- \int_{\Omega \setminus \Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j \, dx = - \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx + \epsilon_1(n, j). \quad (4.15)$$

For the second term, as $n \rightarrow \infty$, we have

$$\int_{\Omega} a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx \rightarrow \int_{\Omega} a(x, u, \nabla v_j \chi_j^s) \cdot (\nabla u - \nabla v_j \chi_j^s) \, dx,$$

since $a(x, T_n(u_n), \nabla v_j \chi_j^s) \rightarrow a(x, u, \nabla v_j \chi_j^s)$ strongly in $(E_{\bar{\varphi}}(\Omega))^N$ as $n \rightarrow \infty$ by lemma 2.2 and (4.11), while $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_{\varphi}(\Omega))^N$ by (4.10). And since $\nabla v_j \chi_j^s \rightarrow \nabla u \chi^s$ strongly in $(E_{\varphi}(\Omega))^N$ as $j \rightarrow \infty$, we obtain

$$\int_{\Omega} a(x, u, \nabla v_j \chi_j^s) \cdot (\nabla u - \nabla v_j \chi_j^s) \, dx \rightarrow 0$$

as $j \rightarrow \infty$. So that

$$\int_{\Omega} a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx = \epsilon_2(n, j). \quad (4.16)$$

Then, from (4.13), (4.15) and (4.16), we obtain

$$\begin{aligned} & \int_{\Omega} \left(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s) \right) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx \\ &= \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx + \epsilon_3(n, j). \end{aligned} \quad (4.17)$$

On the other hand

$$\begin{aligned} & \int_{\Omega} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u \chi^s)) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &= \int_{\Omega} \left(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s) \right) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx \\ &+ \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla v_j \chi_j^s - \nabla u \chi^s) \, dx \\ &- \int_{\Omega} a(x, T_n(u_n), \nabla u \chi^s) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &+ \int_{\Omega} a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx. \end{aligned}$$

Proceeding as above, we obtain

$$\begin{aligned} \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla v_j \chi_j^s - \nabla u \chi^s) \, dx &= \epsilon_4(n, j), \\ \int_{\Omega} a(x, T_n(u_n), \nabla u \chi^s) \cdot (\nabla u_n - \nabla u \chi^s) \, dx &= \epsilon_5(n, j), \\ \int_{\Omega} a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) \, dx &= \epsilon_6(n, j). \end{aligned}$$

Then, by (4.17) we have

$$\begin{aligned} &\int_{\Omega} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u \chi^s)) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &= \epsilon_7(n, j) + \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx. \end{aligned}$$

For $r \leq s$, we write

$$\begin{aligned} 0 &\leq \int_{\Omega^r} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \\ &\leq \int_{\Omega^s} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \\ &= \int_{\Omega^s} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u \chi^s)) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &\leq \int_{\Omega} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u \chi^s)) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &\leq \epsilon_7(n, j) + \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx. \end{aligned}$$

Passing at first to the limit superior over n and then over j , it follows

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \int_{\Omega^r} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \\ &\leq \int_{\Omega \setminus \Omega^s} \xi_0 \cdot \nabla u \, dx. \end{aligned}$$

Letting $s \rightarrow +\infty$ in the previous inequality, we conclude that as $n \rightarrow \infty$,

$$\int_{\Omega^r} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0. \quad (4.18)$$

Define A_n by

$$A_n = (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u).$$

As a consequence of (4.18), one has $A_n \rightarrow 0$ strongly in $L^1(\Omega^r)$, extracting a subsequence, still denoted by $\{u_n\}$, we get $A_n \rightarrow 0$ a.e in Ω^r . Then, there exists a subset Ω_0 of Ω^r , of zero measure, such that: $A_n(x) \rightarrow 0$ for all $x \in \Omega^r \setminus \Omega_0$. On the other hand, using (3.2), (3.3) and

then lemma 3.1 we obtain for all $x \in \Omega^r \setminus \Omega_0$,

$$\begin{aligned} A_n(x) &\geq \bar{\varphi}_x^{-1}(\varphi(x, h(c_\infty)))\varphi(x, |\nabla u_n(x)|) \\ &\quad - c_1(x) \left(1 + \bar{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n(x)|)) + |\nabla u_n(x)|\right) \\ &\geq \bar{q}^{-1}(q(h(c_\infty)))\varphi(x, |\nabla u_n(x)|) \\ &\quad - c_1(x) \left(1 + \bar{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n(x)|)) + |\nabla u_n(x)|\right) \end{aligned} \quad (4.19)$$

where c_∞ is the constant which appears in (4.5) and $c_1(x)$ is a constant which does not depend on n . For x fixed the sequence $\{\nabla u_n(x)\}$ is bounded in $\mathbb{R}^N$, else, if $\{\nabla u_n(x)\}$ is unbounded, there exists a subsequence still denoted by $\{\nabla u_n(x)\}$ which tends to $+\infty$ and we have $\varphi(x, |\nabla u_n(x)|) \rightarrow +\infty$ and recall the inequality

$$\varphi(x, t) \leq t \bar{\varphi}_x^{-1}(\varphi(x, t)) \leq 2\varphi(x, t),$$

for all $t \geq 0$, which implies that

$$\bar{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n(x)|)) \leq \frac{2\varphi(x, k_4 |\nabla u_n(x)|)}{k_4 |\nabla u_n(x)|}.$$

Since φ satisfies the Δ_2 -condition, there exists a positive constant k and a positive L^1 -function c such that

$$\bar{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n(x)|)) \leq \frac{2\varphi(x, k_4 |\nabla u_n(x)|)}{k_4 |\nabla u_n(x)|} \leq \frac{2k\varphi(x, |\nabla u_n(x)|)}{k_4 |\nabla u_n(x)|} + c(x).$$

Consequently, the right term in the inequality (4.19) goes to $+\infty$ as n goes to $+\infty$, which is a contradiction with $\lim_{n \rightarrow \infty} A_n = 0$. Thus, $\{\nabla u_n(x)\}$ is bounded, then for a subsequence $\{u_{n'}(x)\}$, we have

$$\begin{aligned} \nabla u_{n'}(x) &\rightarrow \xi \quad \text{in } \mathbb{R}^N, \\ (a(x, u(x), \xi) - a(x, u(x), \nabla u(x))) \cdot (\xi - \nabla u(x)) &= 0. \end{aligned}$$

Since $a(x, s, \xi)$ is strictly monotone, we have $\xi = \nabla u(x)$, and so $\nabla u_n(x) \rightarrow \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } \Omega^r.$$

Consequently, as r is arbitrary, we deduce

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } \Omega. \quad (4.20)$$

Step 6: Passing to the limit. Let v be a function in $\mathfrak{D}(\Omega)$. Taking v as test function in (4.1), one has

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v \, dx = \int_{\Omega} f_n v \, dx.$$

Lemma 4.2, (4.11) and (4.20) imply that

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } (L_{\bar{\varphi}}(\Omega))^N \text{ for } \sigma(\Pi L_{\bar{\varphi}}, \Pi E_{\varphi}),$$

so, passing to the limit in the previous equality to obtain

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx.$$

Moreover, from (4.5) and (4.11) we have $u \in W_0^1 L_{\varphi}(\Omega) \cap L^{\infty}(\Omega)$. This completes the proof of theorem 3.1.

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