Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 6(1), 2020, Pages 34–41

ISSN: Online 2351-8227 - Print 2605-6364

DOI: 10.2478/mjpaa-2020-0003

Hypercyclicity of Composition Operators on Orlicz Function Spaces

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ABSTRACT. In this paper, we discuss the hypercyclic properties of composition operators on Orlicz function spaces. We give some different conditions under which a composition operator on Orlicz spaces is hypercyclic or not. Similarly, multiplication operators are considered. It is shown that there is no hypercyclic multiplication operator on Orlicz spaces.

Mathematics Subject Classification (2010): 47A16, 47B33, 47B38.

Key words and phrases: Composition operator, Multiplication operators, Orlicz function spaces, Hypercyclic operators.

1. Introduction and preliminaries

Let (Ω, S, μ) be a complete σ -finite Borel measure space. Let denote be $L^0 = L^0(\Omega, \mu)$ the set of all equivalence classes of complex valued measurable functions defined on Ω . For $1 \leq p < \infty$, $L^p(\Omega, \mu) = L^p(\Omega, S, \mu)$ is the set of all $f \in L^0$ such that

$$\|f\|_p = \left(\int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

Received : 31 December 2019 - Accepted: 28 February 2020.

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For $p = \infty$, $L^\infty(\Omega, \mu)$ is the set of all equivalence classes of essentially bounded measurable functions $f : \Omega \rightarrow \mathbb{C}$ with the norm

$$\|f\|_\infty = \text{ess sup}_\Omega |f| := \inf\{\alpha \in \mathbb{C} : \mu(\{x \in \Omega : |f(x)| > \alpha\}) = 0\}.$$

Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function, i.e. continuous, convex, increasing function, $\Phi(0) = 0$, $\lim_{x \rightarrow \infty} \Phi(x) = \infty$ and $\lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty$ (resp. $\lim_{x \rightarrow 0} \frac{\Phi(x)}{x} = 0$). Then, we define the Orlicz spaces as follows

$$L^\Phi(\Omega, \mu) = \left\{ f \in L^0 : \int_\Omega \Phi(\lambda|f|) d\mu < \infty \text{ for some } \lambda > 0 \right\}.$$

This space endowed with the Luxemburg norm, which is defined by

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left(\frac{|f|}{\lambda} \right) d\mu \leq 1 \right\}.$$

is a Banach space.

Now, we give some notations and we briefly review certain properties which will be used later.

- If $\Phi(t) = t^p$, then $L^\Phi(\Omega, \mu) = L^p(\Omega, \mu)$.
- If $A \in S$ and $0 < \mu(A) < \infty$, then

$$\|\chi_A\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(A)}\right)},$$

where $\Phi^{-1}(t) = \inf\{r > 0 : \Phi(r) > t\}$ is the right-inverse of Φ .

- The Orlicz function Φ is called to satisfy the Δ_2 -condition, written $\Phi \in \Delta_2$, if there are constants $k > 0$ and $t_0 \geq 0$ such that $\Phi(2t) \leq k\Phi(t)$, for all $t \geq t_0$.
- The measurable transformation $T : \Omega \rightarrow \Omega$ is said to be measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in S$.
- A subset $A \in S$ is said to be an atom with $\mu(A) > 0$, if for every $E \in S$ such that $E \subset A$, then $\mu(E) = 0$ or $\mu(E) = \mu(A)$.
- A measurable transformation $T : \Omega \rightarrow \Omega$ is called non-singular if

$$\mu T^{-1}(A) = \mu(T^{-1}(A)) = 0,$$

whenever $\mu(A) = 0$ for all $A \in S$. This condition means that the measure μT^{-1} is absolutely continuous with respect to μ (It is usually denoted by $\mu T^{-1} \ll \mu$). Then the Radon-Nikodym theorem assures the existence of a unique non-negative Borel measurable function $h \in L^1(\Omega, \mu)$ on Ω such that

$$\mu T^{-1}(A) = \int_A h(t) d\mu(t) \text{ for all } A \in S.$$

In this case h is said to be the Radon-Nikodym derivative and denoted by $\frac{d\mu T^{-1}}{d\mu}$.

- A non-singular measurable transformation T induces a well-defined composition operator C_T from $L^\Phi(\mu)$ into itself defined by $C_T f(x) = f(T(x))$ for $x \in \Omega$ and $f \in L^\Phi(\Omega, \mu)$. Note that if T is not non-singular transformation, then C_T is not well-defined (see for instance [20, page 18]).

• Let $u : \Omega \rightarrow \mathbb{C}$ be a function such that $u.f \in L^\Phi(\Omega, \mu)$ for every $f \in L^\Phi(\Omega, \mu)$. If u is a function in $L^\infty(\Omega, \mu)$, then the multiplication operator M_u from $L^\Phi(\Omega, \mu)$ into itself defined by $M_u f = u.f$ for any $f \in L^\Phi(\Omega, \mu)$ is a bounded operator (see [13]).

In the literature corresponding to the Orlicz spaces, the basic properties of these spaces were considered by Rao and Ren in [17]. Cui, Hudzic, Kumar and Maligranda [7] and Komar [15] proved some properties of composition operators on Orlicz spaces. In [13] Komal and Gupta established certain properties of multiplication operators on Orlicz spaces. Finally, Chen and Du [5] presented new characterizations of disjoint topological transitivity on Orlicz spaces. Next, a bounded linear operator T on Banach space X is (weakly) hypercyclic if there exists a vector $x \in X$ such that the orbit of x under T defined by

$$\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$$

is (weakly)dense in X . Every such vector x is called (weakly) hypercyclic vector for T .

In recent years, great attention has been focused on the study of hypercyclic operators. The first example of a hypercyclic operator is λB for $|\lambda| > 1$, $\lambda \in \mathbb{C}$ on the space $l^2(\mathbb{N})$, where B is the backward shifts. For more details, we refer the interested reader to [2, 18], and the references therein.

2. Hypercyclic Composition Operators on $L^\Phi(\Omega, \mu)$

In this section, we discuss about hypercyclic properties of composition operators on Orlicz function spaces.

Lemma 2.1. *If T is a non-singular transformation and $\frac{d\mu T^{-1}}{d\mu} = h$, then for all $n \in \mathbb{N}$, T^n is also non-singular transformation and $\frac{d\mu T^{-n}}{d\mu} = \prod_{j=0}^{n-1} h \circ T^{-j}$. Moreover, if T is invertible and its inverse is also non-singular transformation, then for all $n \in \mathbb{N}$, $\frac{d\mu T^n}{d\mu} = (\prod_{j=1}^n h \circ T^j)^{-1}$.*

Proof. We use induction on n . This is true for $n = 1$. Suppose it is true for n , we prove for $n + 1$ is true. Let $E \in S$ and $\mu(E) < \infty$, so

$$\mu T^{-(n+1)}(E) = \mu T^{-n}(T^{-1}(E)) = \int_{T^{-1}(E)} \prod_{j=0}^{n-1} h \circ T^{-j} d\mu$$

Using a change of variable, we get

$$\mu T^{-(n+1)}(E) = \int_E \prod_{j=1}^n h \circ T^{-j} d\mu T^{-1} = \int_E h \prod_{j=1}^n h \circ T^{-j} d\mu = \int_E \prod_{j=0}^n h \circ T^{-j} d\mu$$

By uniqueness of Radon-Nikodym derivative theorem, we have

$$\frac{d\mu T^{-n}}{d\mu} = \prod_{j=0}^{n-1} h \circ T^{-j}.$$

For $E \in S$, we obtain

$$\int_E h(T(x)) d\mu T(x) = \int_{T(E)} h(r) d\mu(r) = \mu(T^{-1}(T(E))) = \mu(E).$$

So

$$\frac{d\mu}{d\mu T} = h \circ T \text{ or } \frac{d\mu T}{d\mu} = \frac{1}{h \circ T}.$$

For the rest of proof we follow the same steps as before.

In [12] the authors presented the hypercyclicity criterion. This criterion was independently improved by Gethner and Shapiro [8]. Eventually, the most general example of this criterion was expressed by Bés [3].

Hypercyclicity Criterion (see [2]). Suppose that T is a continuous linear operator on a separable Banach space X , for which the sequence of non-negative powers (T^n) tends pointwise to zero on a dense subset of X . Suppose further that there is a (possibly different) dense subset Y of X , and a (possibly discontinuous) map $S : Y \rightarrow Y$ such that $TS = \text{identity}$ on Y , and (S^n) tends pointwise to zero on Y , then T is hypercyclic.

Remark 1. In [16] Kufner, John and Fucik proved that $L^\Phi(\Omega, \mu)$ is a separable Banach space, if Φ satisfies the Δ_2 -condition. Since hypercyclicity has sense only if the underlying space is separable, then we consider only the case where Φ satisfies the Δ_2 -condition.

Theorem 2.1. Let $T : \Omega \rightarrow \Omega$ be a non-singular measurable transformation, its inverse is measurable and non-singular transformation. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ be an Orlicz function such that Φ satisfies the Δ_2 -condition and $C_T \in B(L^\Phi(\Omega, \mu))$. If

$$\prod_{j=0}^{n-1} h \circ T^{-j} \rightarrow 0 \text{ a.e. as } n \rightarrow \infty \text{ and } \left(\prod_{j=0}^n h \circ T^j \right)^{-1} \rightarrow 0, \text{ a.e., as } n \rightarrow \infty$$

Then C_T is hypercyclic on $L^\Phi(\Omega, \mu)$.

Proof. For $A \in S$ and $0 < \mu(A) < \infty$, we have

$$\|C_T^n \chi_A\|_\Phi = \|\chi_{T^{-n}(A)}\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(T^{-n}(A))}\right)}.$$

Therefore, by Lemma 2.1, we get

$$\|C_T^n \chi_A\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\int_A \prod_{j=0}^{n-1} h \circ T^{-j}(t) d\mu(t)}\right)}.$$

Since Φ is an increasing function, so by [9, Lemma 2.3.9], Φ^{-1} is continuous and $\Phi \in \Delta_2$ implies that $\lim_{t \rightarrow \infty} \Phi^{-1}(t) = \infty$. Consider the fact that $\prod_{j=0}^{n-1} h \circ T^{-j}$ is Rodon-Nykodym derivative and μ is σ -finite measure, hence as in [19], we obtain

$$\int_A \prod_{j=0}^{n-1} h \circ T^{-j}(t) d\mu(t) < \infty.$$

Then by Lebesgue Convergence theorem, we have

$$\|C_T^n \chi_A\|_\Phi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, for all simple functions s , it follows

$$\|C_T^n s\|_\Phi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly, we find that

$$\|C_T^{-n} s\|_\Phi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since simple functions are dense in $L^\Phi(\Omega, \mu)$, thus by Hypercyclic Criterion, C_T is hypercyclic on $L^\Phi(\Omega, \mu)$.

Example 2.1. Let $\Omega = [0, 1]$ and μ be a Lebesgue measure on $[0, 1]$. Consider $T(x) = 1 - \sqrt{1-x}$. So T be a non-singular transformation and $h = 2(1-x)$ by [10], we have:

$$\prod_{j=0}^{n-1} h \circ T^{-j}(x) = \prod_{j=0}^{n-1} 2(1-x)^{2^j} = 2^n \cdot (1-x)^{2^n-1},$$

and

$$\prod_{j=1}^n h \circ T^j(x) = \prod_{j=1}^n 2 \sqrt[n]{1-x} = 2^n \cdot (1-x)^{1-(\frac{1}{2})^{n+1}}.$$

Since

$$\lim_{n \rightarrow \infty} 2^n \cdot (1-x)^{2^n-1} = 0 \text{ and } \lim_{n \rightarrow \infty} \left(2^n \cdot (1-x)^{1-(\frac{1}{2})^{n+1}} \right)^{-1} = 0.$$

Thus, by Theorem 2.1, C_T is hypercyclic.

Next, For an Orlicz function Φ define

$$b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\} \text{ and } a_\Phi = \inf\{u > 0 : \Phi(u) > 0\}.$$

Clearly, $0 \leq a_\Phi \leq b_\Phi \leq \infty$.

Theorem 2.2. If $0 < a_\Phi \leq b_\Phi < \infty$, then there is no weakly hypercyclic composition operator on $L^\Phi(\Omega, \mu)$.

Proof. If $0 < a_\Phi = b_\Phi < \infty$, then

$$\Phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a_\Phi, \\ \infty & \text{if } a_\Phi < t < \infty. \end{cases}$$

Let $f \in L^\Phi(\Omega, \mu)$, So there exists $\lambda > 0$ such that $\int_\Omega \Phi(\lambda|f|)d\mu < \infty$. By the definition of Φ , we must have $0 < \lambda|f| \leq a_\Phi$, that is, $|f| \leq \frac{a_\Phi}{\lambda}$. This means $f \in L^\infty(\Omega, \mu)$, and $\|f\|_\Phi \geq \frac{\|f\|_\infty}{a_\Phi}$. Now, suppose $f \in L^\infty(\Omega, \mu)$, so there is $\lambda > 0$ such that $\lambda|f| \in [0, a_\Phi]$, it follows that $\int_\Omega \Phi(\lambda|f|)d\mu = 0$, and $f \in L^\Phi(\Omega, \mu)$. Thus $\|f\|_\Phi \leq \frac{\|f\|_\infty}{a_\Phi}$. Hence, in this case, $L^\Phi(\Omega, \mu) = L^\infty(\Omega, \mu)$ with $\|f\|_\infty = a_\Phi \|f\|_\Phi$.

Next, suppose that $0 < a_\Phi < b_\Phi < \infty$. Choose $\epsilon > 0$ such that $0 < \epsilon < a_\Phi$ and $\Phi(\epsilon) = 0$. Since (Ω, S, μ) be a σ -finite measure space, there exists disjoint subsets Ω_n of Ω with $\mu(\Omega_n) < \infty$

such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Suppose that $f \in L^{\infty}(\Omega, \mu)$, so there is $\lambda > 0$ such that $\frac{\|f\|_{\infty}}{\lambda} \leq \epsilon$. Since Φ be an increasing function, we have

$$\int_{\Omega_n} \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq \Phi(\epsilon)\mu(\Omega_n) = 0.$$

So, for all $n \in \mathbb{N}$ we obtain $\int_{\Omega_n} \Phi\left(\frac{|f|}{\lambda}\right) d\mu = 0$. As $\int_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) d\mu = \sum_{n=1}^{\infty} \int_{\Omega_n} \Phi\left(\frac{|f|}{\lambda}\right) d\mu$. Then $\int_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) d\mu = 0$, thus $f \in L^{\Phi}(\Omega, \mu)$, $L^{\Phi}(\Omega, \mu) \subseteq L^{\infty}(\Omega, \mu)$ and $\|f\|_{\Phi} \leq \frac{\|f\|_{\infty}}{\epsilon}$.

Conversely, suppose that $f \in L^{\Phi}(\Omega, \mu)$. So there exists $\lambda > 0$ such that $\int_{\Omega} \Phi(\lambda|f|)d\mu \leq \infty$. For all $n \in \mathbb{N}$, we define

$$A_n = \{x \in \Omega : |f(x)| > n\}, \quad B_n = \{x \in \Omega : \Phi(\lambda|f(x)|) > \Phi(\lambda n)\}.$$

Since Φ be an increasing function, we have $A_n \subseteq B_n$. By assumption we have $b_{\Phi} < \infty$, so there exists $x_0 \neq 0$, $n_0 \in \mathbb{N}$ such that $\Phi(x_0) = \infty$ and $\lambda n_0 > x_0$. Consequently, $\Phi(\lambda n_0) = \infty$ and

$$\infty > \int_{\Omega} \Phi(\lambda|f|)d\mu \geq \int_{B_{n_0}} \Phi(\lambda|f|)d\mu \geq \int_{B_{n_0}} \Phi(\lambda n_0)d\mu.$$

Since $\mu(A_{n_0}) \leq \mu(B_{n_0})$, $\mu(A_{n_0}) = 0$, $f \in L^{\infty}(\Omega, \mu)$ and $\|f\|_{\infty} \leq n_0$. Hence, $\mu(B_{n_0}) = 0$. Thus in this case $L^{\Phi}(\Omega, \mu) = L^{\infty}(\Omega, \mu)$ and $\frac{\|f\|_{\infty}}{n_0} \leq \|f\|_{\Phi} \leq \frac{\|f\|_{\infty}}{\epsilon}$. On the other hand for all $f \in L^{\infty}(\Omega, \mu)$, we have $\|f \circ T\|_{\infty} \leq \|f\|_{\infty}$, so $\|C_T\|_{\infty} \leq 1$, and C_T cannot be weakly hypercyclic on $L^{\infty}(\Omega, \mu)$ and also on $L^{\Phi}(\Omega, \mu)$.

Proposition 2.1. *If C_T is (weakly) hypercyclic on $L^{\Phi}(\Omega, \mu)$, then:*

- a) $T^{-1}(S) = S$ i.e. $S = \{T^{-1}(A) : A \in S\}$.
- b) T is not a non-singular measure-preserving transformation.
- c) $L^{\Phi}(X_{\epsilon}, \mu T^{-1})$ is infinite dimensional for each $\epsilon > 0$, where $X_{\epsilon} = \left\{x \in \Omega : \frac{d\mu T^{-1}}{d\mu}(x) \geq \epsilon\right\}$.
- d) $N(h, \epsilon) = \{x \in \Omega : h = \frac{d\mu T^{-1}}{d\mu}(x) > \epsilon\}$ consists of infinitely many atoms for each ϵ .
- d) $h \geq 1$.

Proof. a) Suppose $T^{-1}(S) \neq S$. By [14, Corollary 4.5], since C_T has not dense range, so is not (weakly) hypercyclic.

b) Let T be a non-singular measure-preserving transformation, then by [14, Corollary 4.8] C_T is an isometry. Thus, by [1, Corollary 3.2] C_T can not be (weakly) hypercyclic on $L^{\Phi}(\Omega, \mu)$.

c) Suppose that $L^{\Phi}(X_{\epsilon}, \mu T^{-1})$ is finite dimensional for each $\epsilon > 0$, then by [14, Theorem 3.1] C_T is compact. Thus C_T is not (weakly) hypercyclic.

d) Suppose that $N(h, \epsilon)$ consists of finitely many atoms, so by [10, Proposition 2.1] C_T is compact, therefore C_T is not (weakly) hypercyclic on $L^{\Phi}(\Omega, \mu)$.

e) Let $h < 1$, so by [7, Remark 3], $\|C_T\|_{\Phi} \leq 1$. Hence C_T can not be (weakly) hypercyclic on $L^{\Phi}(\Omega, \mu)$.

3. Hypercyclic Multiplication Operators on $L^\Phi(\Omega, \mu)$

In this section, we show that there is no hypercyclic multiplication operators on Orlicz function spaces. For this we use the same method as the one used by that authors in [6].

Theorem 3.1. *Let (Ω, S, μ) be a regular σ -finite measure space. If $M_u \in B(L^\Phi(\Omega, \mu))$, then M_u is not hypercyclic on $L^\Phi(\Omega, \mu)$.*

Proof. Since $\|M_u\|_\Phi = \|u\|_\infty$, we have two cases, $\|u\|_\infty \leq 1$ and $\|u\|_\infty > 1$. We prove that M_u is not hypercyclic in this both cases. If $\|u\|_\infty \leq 1$, then $\|M_u\|_\Phi \leq 1$. So M_u is not hypercyclic. Now, we suppose that $\|u\|_\infty > 1$. Since μ is regular, therefore $\mu(\Omega) = \sup\{\mu(K) : K \text{ is compact}, K \subseteq \Omega\}$, so there is a compact set $K \subset \Omega$ such that $\mu(K) > 0$. On the other hand, $\lim_{x \rightarrow \infty} \Phi(x) = \infty$, there exists $\varepsilon > 0$ such that $\Phi(\frac{1}{\varepsilon}) > \frac{2}{\mu(K)}$ or $\mu(K) > \frac{2}{\Phi(\frac{1}{\varepsilon})}$. Moreover, u is continuous, so we may assume $u(x) > 1$ for all $x \in K$. Since the hypercyclic vectors for M_u are dense in $L^\Phi(\Omega, \mu)$, then there exists a hypercyclic vector f for M_u such that for sufficiently large $n \in \mathbb{N}$, we have:

$$\|f - 2\chi_K\|_\Phi < \varepsilon \quad \text{and} \quad \|M_u^n f\|_\Phi < \varepsilon.$$

Let $W = \{x \in K : |f(x)| < 1\}$ and suppose $\mu(W) \neq 0$. Then

$$\varepsilon > \|f - 2\chi_K\|_\Phi \geq \|\chi_W \cdot (f - 2\chi_K)\|_\Phi \geq \|\chi_W\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(W)}\right)}.$$

Therefore in the both cases $\mu(W) < \frac{1}{\Phi(\frac{1}{\varepsilon})}$. On the other hand, suppose $Q = K - W$, and $\mu(Q) \neq 0$.

$$\varepsilon > \|M_u^n f\|_\Phi \geq \|\chi_Q \cdot (u^n f)\|_\Phi \geq \|\chi_Q\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(Q)}\right)}.$$

Hence $\mu(Q) = \mu(K - W) = \frac{1}{\Phi(\frac{1}{\varepsilon})}$. So in all cases $\mu(K) = \mu(W) + \mu(Q) < \frac{2}{\Phi(\frac{1}{\varepsilon})}$, which is a contradiction.

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