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Leray-Schauder's solution for a nonlocal problem in a fractional Orlicz-Sobolev space

ATHMANE BOUMAZOURH¹, MOHAMMED SRATI²

ABSTRACT. Via Leray-Schauder's nonlinear alternative, we obtain the existence of a weak solution for a non-local problem driven by an operator of elliptic type in a fractional Orlicz-Sobolev space, with homogeneous Dirichlet boundary conditions.

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1. Introduction

In this paper, we aim to investigate the existence of solution for a class of quasilinear elliptic problem driven by a nonlocal integro-differential operator with homogeneous Dirichlet boundary data. In other words, we study the following problem

$$(P_a) \begin{cases} (-\Delta)_a^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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The Author(s): ^{1,2} *Department of Mathematics, Sidi Mohamed Ben Abdallah University, Fez, P.O. Box 1796, Atlas 30000 Fez, Morocco.*

¹ *e-mail: athmane.boumazourh@gmail.com*

² *e-mail: mohammed.srati@usmba.ac.ma (Corresponding Author) .*

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying some growth conditions, and $(-\Delta)_a^s$ is the nonlocal fractional a -Laplacian operator of elliptic type which defined as

$$(-\Delta)_a^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{dy}{|x - y|^{N+s}},$$

for all $x \in \mathbb{R}^N$, where $a : \mathbb{R} \rightarrow \mathbb{R}$ which will be specified later.

Recently, a great attention has been given to the study of nonlocal fractional operators of elliptic type (see, for instance [4, 6, 5, 7]). This type of operators arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Levy processes, see for example [13, 18, 17]).

The study of quasilinear elliptic equations which involves homogeneous operators is based on the theory of Sobolev spaces $W^{1,p}(\Omega)$ and their corresponding fractional Sobolev spaces $W^{s,p}(\Omega)$ in order to find weak solutions. However, in the case of non-homogeneous differential operators, when we try to slot some conditions (growth conditions for example), problems have no correspondence in the classical Lebesgue and Sobolev spaces. This leads to use a more general class of functional spaces which called Orlicz-Sobolev spaces. These spaces consists of functions that have weak derivatives and satisfy certain integrability conditions. Since (P_a) is a problem which involves the fractional a -Laplacian operator, the most appropriate functional framework to deal with it is the fractional Orlicz -Sobolev space.

When $a(t) = t^{p-2}$, the problem (P_a) is reduced to the fractional p -Laplacian problem

$$(P_p) \quad \begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p -Laplacian operator which, up to normalization, defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy.$$

One typical feature of problem (P_p) is the nonlocality, in the sense that the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the value of u in a neighborhood of x , but on the value of u on the entire of $\mathbb{R}^N$. In recent years, the problem (P_p) has been studied in many papers, we refer to [2, 16, 21], in which the authors have used different methods to get the existence of solutions for (P_p) . For example in [24], by using Schauders fixed point theorem, Xiang et al studied the existence of solutions the problem (P_p) .

For the problems involving fractional p -Laplace operator, we refer the reader to the works [8, 9, 10, 11]. they use different methods to establish the existence of solutions.

This paper is divided into four sections. In the second section, we recall some properties of fractional Orlicz Sobolev spaces. In the third section, applying several important properties of fractional a -Laplacian operator, we obtain the existence of weak solutions of problem (P_a) .

2. Some preliminaries results

In this section, we briefly review the definitions and list some basic properties of the fractional Orlicz-Sobolev space. We refer the reader to [1, 3, 12, 14, 23, 19] for further reference and for some of the proofs of the results in this section.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. We assume that $a : \mathbb{R} \rightarrow \mathbb{R}$ in (P_a) is such that : $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by :

$$\varphi(t) = \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau.$$

Then, Φ , is N -function, see [1], i.e. $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, convex, increasing function, with $\frac{\Phi(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ and $\frac{\Phi(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$.

For the function Φ introduced above we define the Orlicz space :

$$L_\Phi(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable } \int_\Omega \Phi(\lambda|u(x)|) dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The conjugate N -function of Φ is defined by $\bar{\Phi}(t) = \int_0^t \bar{\varphi}(\tau) d\tau$, where $\bar{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\bar{\varphi}(t) = \sup \{s : \varphi(s) \leq t\}$. Furthermore, it is possible to prove a Hölder type inequality, that is

$$\left| \int_\Omega uv dx \right| \leq 2 \|u\|_\Phi \|v\|_{\bar{\Phi}} \quad \forall u \in L_\Phi(\Omega) \text{ and } v \in L_{\bar{\Phi}}(\Omega). \quad (2.1)$$

Throughout this paper, we assume that

$$1 < \varphi^- := \inf_{t \geq 0} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^+ := \sup_{t \geq 0} \frac{t\varphi(t)}{\Phi(t)} < +\infty. \quad (2.2)$$

The above relation implies that $\Phi \in \Delta_2$ i.e. Φ satisfies the global Δ_2 -condition (see [22]) :

$$\Phi(2t) \leq K\Phi(t) \text{ for all } t \geq 0,$$

where K is a positive constant.

Furthermore, we assume that Φ satisfies the following condition

$$\text{the function } [0, \infty) \ni t \mapsto \Phi(\sqrt{t}) \text{ is convex.} \quad (2.3)$$

The above relation assures that $L_\Phi(\Omega)$ is an uniformly convex space (see [22]).

Definition 2.1. Let A, B be two N-function. A is stronger (resp essentially stronger) than B , $A \succ B$ (resp $A \succ\succ B$) in symbols, if

$$B(x) \leq A(ax), \quad x \geq x_0 \geq 0,$$

for some (resp for each) $a > 0$ and x_0 (depending on a).

Remark 1. (see. [1, Section 8.5]). $A \succ\succ B$ is equivalent to the condition

$$\lim_{x \rightarrow \infty} \frac{B(\lambda x)}{A(x)} = 0,$$

for all $\lambda > 0$.

Now, we defined the fractional Orlicz-Sobolev space $W^s L_\Phi(\Omega)$ as follows

$$W^s L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) : \int_\Omega \int_\Omega \Phi \left(\frac{\lambda |u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} < \infty \text{ for some } \lambda > 0 \right\}.$$

This space is equipped with the norm,

$$\|u\|_{s,\Phi} = \|u\|_\Phi + [u]_{s,\Phi},$$

where $[.]_{s,\Phi}$ is the Gagliardo seminorm, defined by

$$[u]_{s,\Phi} = \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega \Phi \left(\frac{|u(x) - u(y)|}{\lambda |x - y|^s} \right) \frac{dx dy}{|x - y|^N} \leq 1 \right\}.$$

To deal with the problem under consideration, we choose

$$W_0^s L_\Phi(\Omega) = \left\{ u \in W^s L_\Phi(\mathbb{R}^N) : u = 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega \right\}.$$

which can be equivalently renormed by setting $\|.\| = [.]_{s,\Phi}$. By [3], $W_0^s L_\Phi(\Omega)$ is Banach space, also separable (resp. reflexive) space if and only if $\Phi \in \Delta_2$ (resp. $\Phi \in \Delta_2$ and $\bar{\Phi} \in \Delta_2$). Furthermore if $\Phi \in \Delta_2$ and $\Phi(\sqrt{t})$ is convex, then the space $W^s L_\Phi(\Omega)$ is uniformly convex.

Lemma 2. (see. [12]). Let Φ be an N-function which satisfies the global Δ_2 -condition. Then we have,

$$\bar{\Phi}(\varphi(t)) \leq c\Phi(t) \text{ for all } t \geq 0$$

where $c > 0$.

Proposition 3. (cf. [3]) Assume condition (2.2) is satisfied. Then the following relations holds true,

$$[u]_{s,\Phi}^{\varphi^-} \leq \varphi(u) \leq [u]_{s,\Phi}^{\varphi^+}, \quad \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} > 1,$$

$$[u]_{s,\Phi}^{\varphi^+} \leq \varphi(u) \leq [u]_{s,\Phi}^{\varphi^-}, \quad \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} < 1,$$

$$\text{where } \varphi(u) = \int_\Omega \int_\Omega \Phi \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}.$$

In this paper, we assume the following conditions :

$$\int_0^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau < \infty, \quad (2.4)$$

and

$$\int_1^\infty \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau = \infty. \quad (2.5)$$

We define the inverse Sobolev conjugate N-function of Φ as follows,

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau. \quad (2.6)$$

Theorem 4. (cf. [3]) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$ -regularity and bounded boundary. If (2.4), (2.5) and (2.2) hold, then*

$$W^s L_\Phi(\Omega) \hookrightarrow L_{\Phi_*}(\Omega). \quad (2.7)$$

Theorem 5. (cf. [3]) *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $C^{0,1}$ -regularity with bounded boundary. If (2.4), (2.5) and (2.2) hold, then*

$$W^s L_\Phi(\Omega) \hookrightarrow L_B(\Omega), \quad (2.8)$$

is compact for all $B \prec\prec \Phi_$.*

Theorem 6. (cf. [3]) *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then,*

$$C_0^2(\Omega) \subset W^s L_\Phi(\Omega).$$

Finally, the proof of our main result is based on the following fixed point theorem:

Theorem 7. (Schauder fixed point [15]). *Let T be a continuous and compact mapping from a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda Tx \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point.

3. Mains Results

In this section we prove the existence of solutions to problem (P_a) . Our approach is based on some properties of the operator $(-\Delta)_a^s u(\cdot)$ and on the nonlinear Leray Schauder alternative.

Let g an odd increasing homeomorphism function from $\mathbb{R}$ onto $\mathbb{R}$. By setting

$$G(s) = \int_0^s g(t) dt, \quad \bar{G}(s) = \int_0^s \bar{g}(t) dt, \quad (3.1)$$

where $\bar{g}(t) = \sup \{s : g(s) \leq t\}$, we obtain complementary N-functions which define corresponding Orlicz spaces L_G and $L_{\bar{G}}$.

Then, we suppose that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying:

(f₁) $\left\{ \begin{array}{l} \text{There exists nonnegative constant } c_1 \text{ and } c_2 \text{ such that} \\ |f(x, t)| \leq b(x) + c_2 g(|t|) \text{ for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega, \end{array} \right.$
 where $b \in L_{\overline{G}}(\Omega)$. We will also assume that

$$(f_2) \quad 1 < q^- = \inf_{s>0} \frac{tg(t)}{G(s)} \leq q^+ = \sup_{s>0} \frac{tg(t)}{G(s)} < +\infty.$$

$$(f_3) \quad \lim_{t \rightarrow \infty} \frac{G(kt)}{\Phi_*(t)} = 0 \text{ for all } k > 0.$$

Remark 8. Condition (f₃) implies that $G \prec\prec \Phi_*$. So by Theorem 5, the following embedding :

$$W_0^s L_\Phi(\Omega) \hookrightarrow L_G(\Omega)$$

is compact, that is,

$$\|u\|_G \leq c_0 \|u\| \text{ for all } u \in W_0^s L_\Phi(\Omega)$$

where c_0 is a positive constant.

Definition 3.1. We say that $u \in W_0^s L_\Phi(\Omega)$ is a weak solution of problem (P_a) if

$$\int_{\Omega \times \Omega} a \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{v(x) - v(y)}{|x - y|^s} \frac{dx dy}{|x - y|^N} = \int_{\Omega} f(x, u) v dx,$$

for all $v \in W_0^s L_\Phi(\Omega)$.

Theorem 9. Assume that f satisfies (f₁)-(f₃) and we assume that Φ satisfies (2.2)- (2.5). Then, if $\varphi^- > 2$ the problem (P_a) has at least one weak solution $u \in W_0^s L_\Phi(\Omega)$.

First, we discuss the fractional a-Laplace operator

$$\mathcal{T}u := (-\Delta)_a^s u(x) = 2p.v \int_{\mathbb{R}^N} a \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{dy}{|x - y|^{N+s}}.$$

To simplify the notation, we set

$$h(u) := \frac{u(x) - u(y)}{|x - y|^s},$$

and the dual space of $(W_0^s L_\Phi(\Omega), ||\cdot||)$ is denoted by $((W_0^s L_\Phi(\Omega))^*, ||\cdot||_*)$.

Lemma 10.

- 1) $\mathcal{T} : W_0^s L_\Phi(\Omega) \longrightarrow (W_0^s L_\Phi(\Omega))^*$ is continuous, bounded and strictly monotone operator.
- 2) $\mathcal{T}$ is a mapping of type (S_+) , that is, for a sequence $\{u_n\}$ that converges weakly to u in $W_0^s L_\Phi(\Omega)$ and

$$\limsup_{n \rightarrow \infty} \langle \mathcal{T}u_n, u_n - u \rangle \leq 0, \quad (3.2)$$

we have $u_n \longrightarrow u$ in $W_0^s L_\Phi(\Omega)$.

- 3) $\mathcal{T} : W_0^s L_\Phi(\Omega) \longrightarrow (W_0^s L_\Phi(\Omega))^*$ is a homomorphism.

Proof. 1) Consider the following functional :

$$J(u) = \int_{\Omega \times \Omega} \Phi \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N}.$$

We know that (see [3]), $J \in C^1(W_0^s L_\Phi(\Omega), \mathbb{R})$, and the fractional a -Laplace operator is the derivative operator of J in the weak sense. We denote $\mathcal{T} = J' : W_0^s L_\Phi(\Omega) \longrightarrow (W_0^s L_\Phi(\Omega))^*$, then

$$\langle \mathcal{T}u, v \rangle = \int_{\Omega \times \Omega} a \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{v(x) - v(y)}{|x - y|^s} \frac{dx dy}{|x - y|^N}.$$

Obviously that $\mathcal{T}$ is continuous and bounded. Let $u, v \in W_0^s L_\Phi(\Omega)$, with $u \neq v$ and let $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$. Since $a(|t|)t$ is increasing and :

$$\langle \mathcal{T}u - \mathcal{T}v, u - v \rangle = \int_{\Omega} \int_{\Omega} (a(|h(u)|)h(u) - a(|h(v)|)h(v)) (h(u) - h(v)) \frac{dx dy}{|x - y|^N} > 0,$$

So, $\mathcal{T} : W_0^s L_\Phi(\Omega) \longrightarrow (W_0^s L_\Phi(\Omega))^*$ is strictly monotone.

2) Since u_n converges weakly to u in $W_0^s L_\Phi(\Omega)$, then $\{|u_n|\}$ is a bounded sequence of real numbers, this and Proposition 3, imply that $\{J(u_n)\}$ is bounded, then we have for a subsequence noted also u_n ,

$$J(u_n) \longrightarrow c.$$

Since J is weak lower semi continuous, we get

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n) = c.$$

On the other hand, by the convexity of J , we have

$$J(u) \geq J(u_n) + \langle J'(u_n), u_n - u \rangle. \quad (3.3)$$

Taking into account (3.2), we obtain

$$J(u) = c.$$

Since $\left\{ \frac{u_n + u}{2} \right\}$ converges weakly to u in $W_0^s L_\Phi(\Omega)$, so since J is sequentially weakly lower semicontinuous :

$$c = J(u) \leq \liminf_{n \rightarrow \infty} J \left(\frac{u_n + u}{2} \right). \quad (3.4)$$

Assume by contradiction that $\{u_n\}$ does not converge to u in $W_0^s L_\Phi(\Omega)$. Hence, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and there exists $\varepsilon_0 > 0$ such that

$$\left\| \frac{u_n - u}{2} \right\| \geq \frac{\varepsilon_0}{2},$$

by Proposition 3, we have

$$J \left(\frac{u_n - u}{2} \right) \geq \max \left\{ \varepsilon_0^{\varphi^-}, \varepsilon_0^{\varphi^+} \right\}.$$

On the other hand, by the conditions (2.2) and (2.3), we apply Lemma 2.1 in [20] to obtain

$$\frac{1}{2}J(u_n) + \frac{1}{2}J(u) - J\left(\frac{u_n + u}{2}\right) \geq J\left(\frac{u_n - u}{2}\right) \geq \max\{\varepsilon_0^-, \varepsilon_0^+\}. \quad (3.5)$$

It follows from (3.5) that

$$J(u) - \max\{\varepsilon_0^-, \varepsilon_0^+\} \geq \limsup_{n \rightarrow \infty} J\left(\frac{u_n + u}{2}\right). \quad (3.6)$$

Thus, from (3.4) and (3.6) we obtain a contradiction, and hence $\{u_n\}$ converges strongly to u .

3) First, we assume that the operator $\mathcal{T} : W_0^s L_\Phi(\Omega) \rightarrow (W_0^s L_\Phi(\Omega))^*$ is invertible. By the Minty-Browder theorem (see [25]), it suffices to prove that $\mathcal{T}$ is strictly monotone, hemicontinuous and coercive in the sense of monotone operators.

From 1) the operator, $\mathcal{T} : W_0^s L_\Phi(\Omega) \rightarrow (W_0^s L_\Phi(\Omega))^*$ is strictly monotone. Let $u \in W_0^s L_\Phi(\Omega)$, with $\|u\| > 1$, By (2.2) and Proposition 3, we have $\varphi^- \Phi(t) \leq \varphi(t)t$ for all $t > 0$ and

$$\begin{aligned} \frac{\langle \mathcal{T}u, u \rangle}{\|u\|} &\geq \frac{\varphi^- J(u)}{\|u\|} \\ &\geq \varphi^- \|u\|^{\varphi^- - 1}. \end{aligned}$$

Thus

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle \mathcal{T}u, u \rangle}{\|u\|} = \infty,$$

so, $\mathcal{T}$ is coercive.

Now, from 1), $\mathcal{T}$ is hemicontinuous. Thus, in view of the Minty-Browder theorem, there exists $\mathcal{T}^{-1} : (W_0^s L_\Phi(\Omega))^* \rightarrow W_0^s L_\Phi(\Omega)$ and it is bounded.

Let us prove that $\mathcal{T}^{-1}$ is continuous by showing that its is sequentially continuous. Let $\{u_n\} \subset (W_0^s L_\Phi(\Omega))^*$ be a sequence strongly converging to $u \in (W_0^s L_\Phi(\Omega))^*$ and let $v_n = \mathcal{T}^{-1}(u_n)$ and $v = \mathcal{T}^{-1}(u)$. Then, $\{v_n\}$ bounded in $W_0^s L_\Phi(\Omega)$, so, it converges weakly to a certain $v_0 \in W_0^s L_\Phi(\Omega)$. Since u_n converges strongly to u , we have

$$\lim_{n \rightarrow \infty} \langle \mathcal{T}(v_n), v_n - v_0 \rangle = \lim_{n \rightarrow \infty} \langle u_n, v_n - v_0 \rangle = 0,$$

that is,

$$\int_{\Omega} \int_{\Omega} a(h(v_n)) h(v_n) (h(v_n) - h_{x,y}(v_0)) \frac{dx dy}{|x - y|^N} = 0. \quad (3.7)$$

By applying 2), using (3.7) and the fact that v_n converges weakly to v_0 in $W_0^s L_\Phi(\Omega)$, we conclude that v_n converge strongly to v_0 in $W_0^s L_\Phi(\Omega)$. Thus, this complete the proof. $\square$

Proof of Theorem 9 From Remark 8, $W_0^s L_\Phi(\Omega)$ compactly embedded in $L_G(\Omega)$. Denote by i the compact injection of $W_0^s L_\Phi(\Omega)$ in $L_G(\Omega)$ and $i^* : L_{\overline{G}}(\Omega) \rightarrow (W_0^s L_\Phi(\Omega))^*$, $i^*v = v \circ i$ for all $v \in L_{\overline{G}}(\Omega)$, its adjoint. Its follows from the assumption (f₁) that the Nemytskii operator N_f generated by f , $(N_f u)(x) = f(x, u(x))$ is well defined from $L_G(\Omega)$ into $L_{\overline{G}}(\Omega)$, continuous and bounded. In order to prove that problem (P_a) has a weak solution $u \in W_0^s L_\Phi(\Omega)$, it's sufficient to prove that the equation

$$\mathcal{T}u = (i^* N_f i)u \quad (3.8)$$

has a solution in $W_0^s L_\Phi(\Omega)$. Indeed, if $u \in W_0^s L_\Phi(\Omega)$ is a solution of (3.8), then for all $v \in W_0^s L_\Phi(\Omega)$, one has

$$\langle \mathcal{T}u, v \rangle = \langle (i^* N_f i)u, v \rangle = \langle N_f(iu), iv \rangle,$$

which can be rewritten :

$$\int_{\Omega \times \Omega} a \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{v(x) - v(y)}{|x - y|^s} \frac{dx dy}{|x - y|^N} = \int_{\Omega} f(x, u) v dx,$$

so, $u \in W_0^s L_\Phi(\Omega)$ is a weak solution of problem (P_a) .

By Lemma (10), $\mathcal{T}$ is a homeomorphism from $W_0^s L_\Phi(\Omega)$ into $(W_0^s L_\Phi(\Omega))^*$. The equation (3.8) is equivalent to

$$u = \mathcal{T}^{-1}(i^* N_f i)u. \quad (3.9)$$

Therefore, proving that the problem (P_a) has a weak solution in $W_0^s L_\Phi(\Omega)$ remains to show that the compact operator :

$$\mathcal{L} := \mathcal{T}^{-1}(i^* N_f i) : W_0^s L_\Phi(\Omega) \longrightarrow (W_0^s L_\Phi(\Omega))^*,$$

has a fixed point.

By Theorem 7, to prove that $\mathcal{L}$ admit a fixed point it sufficient to verifies the following property: There exists constant $R > 0$ such that

$$S = \{u \in W_0^s L_\Phi(\Omega) : u = \lambda \mathcal{L}u \text{ for some } \lambda \in [0, 1]\} \subset B(0, R).$$

For $\lambda = 0$, the only solution of equation $u = \lambda \mathcal{L}u$ is $u = 0$.

For $\lambda \in (0, 1]$, let $u \in W_0^s L_\Phi(\Omega)$ which satisfies $u = \lambda \mathcal{T}^{-1}[(i^* N_f i)u]$. Then, we have

$$\langle \mathcal{T}u, u \rangle = \lambda \langle (i^* N_f i)u, u \rangle.$$

If $\|u\| < 1$, by (f_1) and Proposition 3, we have $p^- \Phi(t) \leq \varphi(t)t$ for all $t > 0$ and

$$\begin{aligned} p^- \|u\|^{p^-} &\leq p^- \int_{\Omega \times \Omega} \Phi \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^N} \\ &\leq \langle \mathcal{T}u, u \rangle \\ &\leq \langle (i^* N_f i)u, u \rangle \\ &\leq \|i^*\|_* \|N_f(iu)\|_{L_{\bar{G}}} \|u\|_G \\ &\leq c_0 \|i^*\|_* \|N_f(iu)\|_{L_{\bar{G}}} \|u\|. \end{aligned} \quad (3.10)$$

In order to estimate $\|N_f(iu)\|_{L_{\bar{G}}}$, we deduce first from assumption (f_1) that

$$|(N_f v)(x)| \leq c_1 g(|v|) + b(x),$$

so, by Lemma 2 we have

$$\|N_f v\|_{L_{\bar{G}}} \leq c_2 (c_1 \|v\|_{L_G} + \|b\|_{L_{\bar{G}}}), \quad (3.11)$$

where c_2 is a positive constant. By taking (3.11) for $v = iu$, $u \in W_0^s L_\Phi(\Omega)$, we have

$$\|N_f v\|_{L_{\bar{G}}} \leq c_2 (c_1 \|i\|_* \|u\| + \|b\|_{L_{\bar{G}}}). \quad (3.12)$$

In particular, if $u \in W_0^s L_\Phi(\Omega)$, for some $\lambda \in (0, 1]$, we conclude from (3.9) – (3.10) – (3.12) that

$$\begin{aligned} p^- \|u\|^{p^-} &\leq c_0 c_2 \|i^*\|_* (c_1 \|i\|_* \|u\| + \|b\|_{L_{\overline{G}}}) \|u\|. \\ &\leq c_3 \|u\|^2 + c_4 \|u\|, \end{aligned} \quad (3.13)$$

where $c_2 = c_0 c_1 c_2 \|i^*\|_* \|i\|_*$, and $c_3 = c_0 c_2 \|i^*\|_* \|b\|_{L_{\overline{G}}}$. Applying Young's inequality, we get

$$c_3 \|u\|^2 \leq \frac{1}{4} \|u\|^{p^-} + 4^{\frac{2}{p^- - 2}} c_3^{\frac{p^-}{p^- - 2}},$$

and

$$c_4 \|u\| \leq \frac{1}{4} \|u\|^{p^-} + 4^{\frac{1}{p^- - 1}} c_4^{\frac{p^-}{p^- - 1}}.$$

Inserting these two inequalities into (3.13), one has

$$\frac{1}{2} \|u\|^{p^-} \leq 4^{\frac{2}{p^- - 2}} c_3^{\frac{p^-}{p^- - 2}} + 4^{\frac{1}{p^- - 1}} c_4^{\frac{p^-}{p^- - 1}}.$$

Set

$$R_1 = (2^{\frac{p^- + 2}{p^- - 2}} c_3^{\frac{p^-}{p^- - 2}} + 2^{\frac{p^- + 1}{p^- - 1}} c_4^{\frac{p^-}{p^- - 1}})^{\frac{1}{p^-}},$$

then $\|u\| \leq R_1$. Similarly if $\|u\| \geq 1$ we have $\|u\| \leq R_2$ where

$$R_2 = (2^{\frac{p^+ + 2}{p^+ - 2}} c_3^{\frac{p^+}{p^+ - 2}} + 2^{\frac{p^+ + 1}{p^+ - 1}} c_4^{\frac{p^+}{p^+ - 1}})^{\frac{1}{p^+}}$$

this implies that the set S is bounded. $\square$

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