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A new generalization of two refined Young inequalities and applications

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ABSTRACT. In this paper, we prove that if $a, b > 0$ and $0 \leq \alpha \leq 1$, then for $m = 1, 2, 3, \dots$,

$$\begin{aligned} r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 &\leq r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}} \right) \\ &\leq \left(\alpha a + (1 - \alpha)b \right)^m - \left(a^\alpha b^{1-\alpha} \right)^m, \end{aligned}$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$. This is a considerable new generalization of two refinements of the Young inequality due to Kittaneh and Manasrah, and Hirzallah and Kittaneh, which correspond to the cases $m = 1$ and $m = 2$, respectively. As applications we give some refined Young type inequalities for generalized euclidean operator radius and the numerical radius of some well-know f -connection of operators and refined some Young type inequalities for the traces, determinants, and norms of positive definite matrices.

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1. Introduction

The arithmetic-geometric mean inequality expressed as follows: For a set of nonnegative real numbers $x_1, x_2, \dots, x_n$, then

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$$\sqrt[n]{\prod_{k=1}^n x_k} \leq \sum_{k=1}^n \frac{x_k}{n}. \quad (1.1)$$

The weighted form of the arithmetic-geometric mean inequality (AM-GM) states as follows:

Theorem 1.1. *Let n be a positive integer. For $k = 1, 2, \dots, n$, let $x_k > 0$, and let $\alpha_k \geq 0$ satisfy $\sum_{k=1}^n \alpha_k = 1$. Then, we have*

$$\prod_{k=1}^n x_k^{\alpha_k} \leq \sum_{k=1}^n \alpha_k x_k. \quad (1.2)$$

The special case of the weighted AM-GM inequality ($n = 2$) is the well-known Young's inequality, for positive real numbers a, b and $0 \leq \alpha \leq 1$, we have

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha)b. \quad (1.3)$$

The first refinements of Young inequality is the squared version proved in [3] as follows

$$(a^\alpha b^{1-\alpha})^2 + r_0^2(a - b)^2 \leq (\alpha a + (1 - \alpha)b)^2, \quad (1.4)$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$.

Later, Kittaneh and Manasrah [8], obtained the other interesting refinement of Young inequality as follows

$$a^\alpha b^{1-\alpha} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \alpha a + (1 - \alpha)b. \quad (1.5)$$

S. Furuichi [4] was refined (1.2) as follows:

$$\prod_{k=1}^n x_k^{\alpha_k} + r \left(\sum_{k=1}^n x_k - n \sqrt[n]{\prod_{k=1}^n x_k} \right) \leq \sum_{k=1}^n \alpha_k x_k, \quad (1.6)$$

where $r = \min\{\alpha_k : k = 1, \dots, n\}$. This inequality generalizes the inequality (1.5).

Recently, Manasrah and Kittaneh [7] gave a generalization of the two refined Young inequalities (1.4) and (1.5) as follows

Theorem 1.2. *Let m be a positive integer and let α a positive number such that $0 \leq \alpha \leq 1$. Then we have*

$$\left(a^\alpha b^{1-\alpha}\right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leq \left(\alpha a + (1 - \alpha)b\right)^m, \quad (1.7)$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$.

This paper is organised as follows: In section 2, by using inequality (1.6) we give a new generalization of the two refined Young inequalities (1.4) and (1.5). In section 3, we apply our main result to establish some inequalities for generalized euclidean operator radius. In section 4, we show some numerical radius of some well-know f -connection of operators. In section 5, we refined some Young type inequalities for the traces, determinants, and norms of positive definite matrices.

2. A New generalizations of two refined Young's inequalities

Before giving the main result, we need the following simple lemma

Lemma 2.1. *Let m be a positive integer, and let α a positive number such that $0 \leq \alpha \leq 1$. Then we have*

$$\sum_{k=1}^m \binom{m}{k} k \alpha^k (1-\alpha)^{m-k} = m\alpha$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k) \alpha^k (1-\alpha)^{m-k} = m(1-\alpha),$$

where $\binom{m}{k}$ is the binomial coefficient.

For a proof of Lemma 2.1, one can see [1].

The first main result of this setion is the following theorem.

Theorem 2.1. *Let a and b be two positive numbers and $0 \leq \alpha \leq 1$. Then for $m = 1, 2, 3, \dots$, we have*

$$\left(a^\alpha b^{1-\alpha}\right)^m + r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right) \leq \left(\alpha a + (1-\alpha)b\right)^m, \quad (2.1)$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Proof. We have,

$$\begin{aligned} \left(\alpha a + (1-\alpha)b\right)^m &= \sum_{k=0}^m \binom{m}{k} \alpha^k (1-\alpha)^{m-k} a^k b^{m-k} \\ &= \sum_{k=0}^m \mu_k x_k. \end{aligned}$$

where x_k is given by: For $0 \leq k \leq m$,

$$x_k := a^k b^{m-k}, \quad \text{with} \quad \mu_k := \binom{m}{k} \alpha^k (1-\alpha)^{m-k}.$$

We have

- (1) $x_k > 0$ for all $k \in \{0, 1, \dots, m\}$,
- (2) $\mu_k \geq 0$ for all $k \in \{0, 1, \dots, m\}$, with $\sum_{k=0}^m \mu_k = 1$.

Hence by (1.6),

$$\begin{aligned} \sum_{k=0}^m \mu_k x_k &\geq \prod_{k=0}^m x_k^{\mu_k} + r \left(\sum_{k=0}^m x_k - (m+1) \left(\prod_{k=0}^m x_k \right)^{\frac{1}{m+1}} \right) \\ &= a^{\alpha(m)} b^{\beta(m)} + r \left(\sum_{k=0}^m a^k b^{m-k} - (m+1) \left(\prod_{k=0}^m a^k b^{m-k} \right)^{\frac{1}{m+1}} \right) \end{aligned}$$

$$= a^{\alpha(m)} b^{\beta(m)} + r \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}} \right),$$

where

$$r = \min\{\mu_k, k = 0, \dots, m\} = r_0^m,$$

$$\alpha(m) = \sum_{k=1}^m \binom{m}{k} k \alpha^k (1 - \alpha)^{m-k} = m\alpha \quad (\text{by Lemma 2.1}),$$

and

$$\beta(m) = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) \alpha^k (1 - \alpha)^{m-k} = m(1 - \alpha) \quad (\text{by Lemma 2.1}).$$

Therefore,

$$\left(a^\alpha b^{1-\alpha} \right)^m + r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}} \right) \leq \left(\alpha a + (1 - \alpha)b \right)^m.$$

Remark 2.1. The Theorem 2.1 is a common generalization of the inequalities (1.4) and (1.5).

The second main result of this section is the following theorem.

Theorem 2.2. Let a and b be two positive numbers and $0 \leq \alpha \leq 1$. Then for $m = 1, 2, 3, \dots$, we have

$$r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \leq r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}} \right) \leq \left(\alpha a + (1 - \alpha)b \right)^m - \left(a^\alpha b^{1-\alpha} \right)^m,$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$.

Proof. The second inequality in Theorem 2.2, are special cases of Theorem 2.1. So, it is enough to prove the first inequality. We have,

$$\begin{aligned} & \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}} \right) - \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \\ &= \sum_{k=0}^m a^k b^{m-k} - (m+1)(ab)^{\frac{m}{2}} - \left(a^m + b^m - 2(ab)^{\frac{m}{2}} \right) \\ &= \sum_{k=1}^{m-1} a^k b^{m-k} - (m-1)(ab)^{\frac{m}{2}} \\ &\geq (m-1) \left(\prod_{k=1}^{m-1} a^k b^{m-k} \right)^{\frac{1}{m-1}} - (m-1)(ab)^{\frac{m}{2}} \\ &\quad (\text{by inequality (1.1)}) \\ &= (m-1)(ab)^{\frac{m}{2}} - (m-1)(ab)^{\frac{m}{2}} = 0. \end{aligned}$$

It is known from [9] that for $1 \leq \alpha \leq 1$ and $r \geq 1$,

$$\alpha a + (1 - \alpha)b \leq (\alpha a^r + (1 - \alpha)b^r)^{1/r}. \quad (2.2)$$

It follows from (2.2) and Theorem 2.1 that

$$\left(a^\alpha b^{1-\alpha}\right)^m + r_0^m \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}}\right) \leq \left(\alpha a^r + (1 - \alpha)b^r\right)^{\frac{m}{r}}, \quad (2.3)$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$. In particular, for $\alpha = \frac{1}{2}$, we get

$$\left(a^{1/2}b^{1/2}\right)^m \leq \frac{1}{2^{\frac{m}{r}}} \left(a^r + b^r\right)^{\frac{m}{r}} - \frac{1}{2^m} \left(\frac{b^{m+1} - a^{m+1}}{b - a} - (m+1)(ab)^{\frac{m}{2}}\right). \quad (2.4)$$

3. Applications to refined some inequalities for generalized euclidean operator radius

In this section we provide some improvements to some inequalities due to A. Sheikhhosseini, M Sal Moslehian and K. Shebrawi [13].

Let $B(\mathcal{H})$, denote the C^* -Algebra of all bounded linear operators acting on a Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$, the numerical radius of T is defined by

$$w(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

The generalized euclidean operator radius ω_p of operators $T_1, \dots, T_n \in B(\mathcal{H})$ is defined for $p \geq 1$ as follows (see, [12]):

$$\omega_p(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^n |\langle T_i x, x \rangle|^p \right)^{\frac{1}{p}}.$$

Before giving our results, we need the following lemmas. The first lemma is known as the mixed Schwarz inequality, this lemma has been proven by Kato [10].

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in (0, 1)$. Then*

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle, \quad (3.1)$$

for all $x, y \in \mathcal{H}$.

The second lemma is a generalization of the mixed Schwarz inequality, this lemma has been proven by F. Kittaneh [6].

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$|\langle Tx, y \rangle|^2 \leq \|f(|T|)\| \|g(|T^*|)\|, \quad (3.2)$$

for all $x, y \in \mathcal{H}$.

The third Lemma follows from spectral theorem for positive operators and Jensen's inequality, this lemma has been proven in [11].

Lemma 3.3. (McCarthy inequality) Let $T \in \mathcal{B}(\mathcal{H})$ $T \geq 0$ and let $x \in \mathcal{H}$ be any unit vector. Then

$$(a) \langle Tx, x \rangle^r \leq \langle T^r x, x \rangle \quad \text{for } r \geq 1,$$

$$(b) \langle T^r x, x \rangle \leq \langle Tx, x \rangle^r \quad \text{for } 0 < r \leq 1.$$

The following Theorem is well-known.

Theorem 3.1. [12] Let $A_i, B_i, T_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3, \dots$, and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$, for all $t \in [0, +\infty)$. Then for $r, p \geq 1$, we have

$$\omega_p^{rp}(A_1^* T_1 B_1, \dots, A_n^* T_n B_n) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^n \left(([B_i^* f^2(|T_i|) B_i]^{rp} + [A_i^* g^2(|T_i^*|) A_i]^{rp}) \right) \right\|.$$

The first main result in this section is the following Theorem.

Theorem 3.2. Let $A_i, B_i, T_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3, \dots$, and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$, for all $t \in [0, +\infty)$. Then for $m = 1, 2, 3, \dots$ and $r, p \geq m$, we have

$$\begin{aligned} \omega_p^p(A_1^* T_1 B_1, \dots, A_n^* T_n B_n) &\leq \frac{n^{1-m/r}}{2^{m/r}} \left\| \sum_{i=1}^n \left(([B_i^* f^2(|T_i|) B_i]^{\frac{pr}{m}} + [A_i^* g^2(|T_i^*|) A_i]^{\frac{pr}{m}}) \right) \right\|^{\frac{m}{r}} \\ &\quad - \inf_{\|x\|=1} \zeta(x), \end{aligned}$$

where

$$\begin{aligned} \zeta(x) &= \frac{1}{2^m} \sum_{i=1}^n \left(\frac{\langle [B_i^* f^2(|T_i^*|) B_i]^{\frac{p}{m}} x, x \rangle^{m+1} - \langle [A_i^* g^2(|T_i|) A_i]^{\frac{p}{m}} x, x \rangle^{m+1}}{\langle [B_i^* f^2(|T_i^*|) B_i]^{\frac{p}{m}} x, x \rangle - \langle [A_i^* g^2(|T_i|) A_i]^{\frac{p}{m}} x, x \rangle} \right. \\ &\quad \left. - (m+1) [\langle [A_i^* g^2(|T_i|) A_i]^{\frac{p}{m}} x, x \rangle \langle [B_i^* f^2(|T_i^*|) B_i]^{\frac{p}{m}} x, x \rangle]^{\frac{m}{2}} \right). \end{aligned}$$

Proof. We have,

$$\begin{aligned} \sum_{i=1}^n |\langle A_i^* T_i B_i x, x \rangle|^p &= \sum_{i=1}^n |\langle T_i B_i x, A_i x \rangle|^p \\ &\leq \sum_{i=1}^n \|f(|T_i|) B_i x\|^p \|g(|T_i^*|) A_i x\|^p \quad (\text{by Lemma 3.2}) \\ &\leq \sum_{i=1}^n \langle f(|T_i|) B_i x, f(|T_i|) B_i x \rangle^{\frac{p}{2}} \langle g(|T_i^*|) A_i x, g(|T_i^*|) A_i x \rangle^{\frac{p}{2}} \\ &\leq \sum_{i=1}^n \left(\langle B_i^* f^2(|T_i|) B_i x, x \rangle^{\frac{p}{2m}} \langle A_i^* g^2(|T_i^*|) A_i x, x \rangle^{\frac{p}{2m}} \right)^m \\ &\leq \sum_{i=1}^n \left(\langle [B_i^* f^2(|T_i|) B_i]^{\frac{p}{m}} x, x \rangle^{\frac{1}{2}} \langle [A_i^* g^2(|T_i^*|) A_i]^{\frac{p}{m}} x, x \rangle^{\frac{1}{2}} \right)^m \end{aligned}$$

$$\begin{aligned}
& \text{(by Lemma 3.3)} \\
& \leq \sum_{i=1}^n \left[\frac{1}{2} \langle [B_i^* f^2(|T_i|) B_i] \frac{p}{m} x, x \rangle^r + \langle [A_i^* g^2(|T_i^*|) A_i] \frac{p}{m} x, x \rangle^r \right]^{\frac{m}{r}} - \zeta(x) \\
& \text{(by inequality (2.4))} \\
& \leq \frac{n^{1-m/r}}{2^{m/r}} \left\langle \sum_{i=1}^n \left(([B_i^* f^2(|T_i^*|) B_i] \frac{pr}{m} + [A_i^* g^2(|T_i^*|) A_i] \frac{pr}{m}) \right) x, x \right\rangle^{\frac{m}{r}} - \zeta(x),
\end{aligned}$$

by lemma 3.3 and the concavity of the function $t \rightarrow t^{\frac{m}{r}}$. Taking the supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we deduce the result. This completes the proof.

Choosing $f(t) = g(t) = \sqrt{t}$, and $T_i = I$ for $i = 1, 2, \dots, n$, in Theorem 3.2 we obtain the following simpler form.

Corollary 3.1. *Let $A_i, B_i \in B(\mathcal{H})$ for $(i = 1, 2, 3, \dots)$. Then for $m = 1, 2, 3, \dots$ and $r, p \geq m$, we have*

$$\omega_p^p(A_1^* B_1, \dots, A_n^* B_n) \leq \frac{n^{1-m/r}}{2^{m/r}} \left\| \sum_{i=1}^n \left((|B_i|^{\frac{2pr}{m}} + |A_i|^{\frac{2pr}{m}}) \right) \right\|^{\frac{m}{r}} - \inf_{\|x\|=1} \zeta(x),$$

where

$$\begin{aligned}
\zeta(x) = \frac{1}{2^m} \sum_{i=1}^n & \left(\frac{\langle |B_i|^{\frac{2p}{m}} x, x \rangle^{m+1} - \langle |A_i|^{\frac{2p}{m}} x, x \rangle^{m+1}}{\langle |B_i|^{\frac{2p}{m}} x, x \rangle - \langle |A_i|^{\frac{2p}{m}} x, x \rangle} \right. \\
& \left. - (m+1) [\langle |A_i|^{\frac{2p}{m}} x, x \rangle \langle |B_i|^{\frac{2p}{m}} x, x \rangle]^{\frac{m}{2}} \right).
\end{aligned}$$

The second result in this section is the following Theorem.

Theorem 3.3. *Let $T_i \in B(\mathcal{H})$ for $i = 1, 2, 3, \dots$, and $p \geq 2m$ for some $m = 1, 2, 3, \dots$. Then for $0 \leq \alpha \leq 1$, we have*

$$\omega_p^p(T_1, \dots, T_n) \leq \left\| \sum_{i=1}^n \left(\alpha |T_i|^{\frac{p}{m}} + (1-\alpha) |T_i^*|^{\frac{p}{m}} \right)^m \right\| - \inf_{\|x\|=1} \zeta(x),$$

where

$$\zeta(x) = r_0^m \sum_{i=1}^n \left(\frac{\langle |T_i|^{\frac{p}{m}} x, x \rangle^{m+1} - \langle |T_i^*|^{\frac{p}{m}} x, x \rangle^{m+1}}{\langle |T_i|^{\frac{p}{m}} x, x \rangle - \langle |T_i^*|^{\frac{p}{m}} x, x \rangle} - (m+1) [\langle |T_i|^{\frac{p}{m}} x, x \rangle \langle |T_i^*|^{\frac{p}{m}} x, x \rangle]^{\frac{m}{2}} \right).$$

Proof. We have,

$$\begin{aligned}
\sum_{i=1}^n |\langle T_i x, x \rangle|^p &= \sum_{i=1}^n \left(|\langle T_i x, x \rangle|^2 \right)^{\frac{p}{2}} \\
&\leq \sum_{i=1}^n \left(\langle |T_i|^{2\alpha} x, x \rangle \langle |T_i^*|^{2(1-\alpha)} x, x \rangle \right)^{\frac{p}{2}} \text{ (by Lemma 3.1)}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^n \left(\langle |T_i|^{\frac{p\alpha}{m}} x, x \rangle \langle |T_i^*|^{\frac{p(1-\alpha)}{m}} x, x \rangle \right)^m \text{ (by Lemma 3.3)} \\
&\leq \sum_{i=1}^n \left(\langle |T_i|^{\frac{p}{m}} x, x \rangle^\alpha \langle |T_i^*|^{\frac{p}{m}} x, x \rangle^{(1-\alpha)} \right)^m \text{ (by Lemma 3.2)} \\
&\leq \sum_{i=1}^n \left(\alpha \langle |T_i|^{\frac{p}{m}} x, x \rangle + (1-\alpha) \langle |T_i^*|^{\frac{p}{m}} x, x \rangle \right)^m - \zeta(x) \\
&\quad \text{(by Theorem 2.1)} \\
&\leq \sum_{i=1}^n \left(\langle (\alpha |T_i|^{\frac{p}{m}} + (1-\alpha) |T_i^*|^{\frac{p}{m}}) x, x \rangle \right)^m - \inf_{||x||=1} \zeta(x) \\
&\leq \sum_{i=1}^n \langle (\alpha |T_i|^{\frac{p}{m}} + (1-\alpha) |T_i^*|^{\frac{p}{m}})^m x, x \rangle - \inf_{||x||=1} \zeta(x) \\
&\quad \text{(by Lemma 3.3)} \\
&\leq \langle \sum_{i=1}^n (\alpha |T_i|^{\frac{p}{m}} + (1-\alpha) |T_i^*|^{\frac{p}{m}})^m x, x \rangle - \inf_{||x||=1} \zeta(x).
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$, $||x|| = 1$, we deduce the result. This completes the proof.

4. Some results for f -connections of operators

For positive definite operators $A, B \in B(\mathcal{H})$, the operator geometric mean is defined by

$$A \sharp B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

Let f be a continuous function defined on the real interval J containing the spectrum of the operator $A^{-1/2} B A^{-1/2}$, where B is a self-adjoint operators and A is a positive invertible operator. By using the continuous functional calculus, Tafazoli et al. [14] defined f -connection σ_f as follows:

$$A \sigma_f B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

The aim of this section is to give some refinements of numerical radius inequalities for the f -connection of operators [14].

The first main result of this section which can be reads as follows.

Theorem 4.1. *Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integer m , and $r \geq 1$, we have*

$$w^m((A \sigma_f B) X) \leq 2^{\frac{-m}{r}} w^{\frac{m}{r}} \left((X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X)^r + A^r \right) - \inf_{||x||=1} \zeta(x),$$

where

$$\zeta(x) = \frac{1}{2^m} \left(\frac{\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle^{m+1} - \langle A x, x \rangle^{m+1}}{\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle - \langle A x, x \rangle} \right)$$

$$-(m+1)(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle \langle A x, x \rangle)^{\frac{m}{2}}).$$

Proof. We have,

$$\begin{aligned} |\langle A \sigma_f B X x, x \rangle|^m &= |\langle A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle|^m \\ &\leq \|f(A^{-1/2} B A^{-1/2}) A^{1/2} X x\|^m \|A^{1/2} x\|^m \\ &\leq \left(\langle f(A^{-1/2} B A^{-1/2}) A^{1/2} X x, f(A^{-1/2} B A^{-1/2}) A^{1/2} X x \rangle^{\frac{1}{2}} \langle A^{1/2} x, A^{1/2} x \rangle^{\frac{1}{2}} \right)^m \\ &\leq \left(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle^{\frac{1}{2}} \langle A x, x \rangle^{\frac{1}{2}} \right)^m \\ &\leq 2^{\frac{-m}{r}} \left(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle^r + \langle A x, x \rangle^r \right)^{\frac{m}{r}} - \xi(x) \\ &\quad \text{(by inequality (2.4))} \\ &\leq 2^{\frac{-m}{r}} \left(\langle (X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X)^r + A^r x, x \rangle \right)^{\frac{m}{r}} - \xi(x) \\ &\quad \text{(by Lemma 3.3).} \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we deduce the result.

Taking $X = I$, in Theorem 4.1, then we have the following corollary

Corollary 4.1. *Let $A, B \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integers m , and $r \geq 1$, we have*

$$w(A \sigma_f B) \leq 2^{\frac{-m}{r}} w_r^m (A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2})^r + A^r - \inf_{\|x\|=1} \xi(x),$$

where

$$\begin{aligned} \xi(x) &= \frac{1}{2^m} \left(\frac{\langle A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} x, x \rangle^{m+1} - \langle A x, x \rangle^{m+1}}{\langle A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} x, x \rangle - \langle A x, x \rangle} \right. \\ &\quad \left. - (m+1)(\langle A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} x, x \rangle \langle A x, x \rangle)^{\frac{m}{2}} \right). \end{aligned}$$

Taking $r = 1$, $X = I$, and $f(t) = \sqrt{t}$, in Theorem 4.1, we have the following simplified form.

Corollary 4.2. *Let $A, B \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integers m , we have*

$$w^m(A \sharp B) \leq 2^{-m} w^m(A + B) - \inf_{\|x\|=1} \xi(x),$$

where

$$\xi(x) = \frac{1}{2^m} \left(\frac{\langle B x, x \rangle^{m+1} - \langle A x, x \rangle^{m+1}}{\langle B x, x \rangle - \langle A x, x \rangle} - (m+1)(\langle A x, x \rangle \langle B x, x \rangle)^{\frac{m}{2}} \right).$$

The second main result of this section which can be reads as follows.

Theorem 4.2. Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integers m , and $r > 1$, we have

$$\|(A\sigma_f B)X\|^m \leq 2^{\frac{-m}{r}} \left(\|(X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X)\|^r + \|A\|^r \right)^{\frac{m}{r}} - \inf_{\|x\|=1, \|y\|=1} \xi(x, y),$$

where

$$\begin{aligned} \xi(x, y) = \frac{1}{2^m} & \left(\frac{\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle^{m+1} - \langle A y, y \rangle^{m+1}}{\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle - \langle A y, y \rangle} \right. \\ & \left. - (m+1)(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} X x, x \rangle \langle A y, y \rangle)^{\frac{m}{2}} \right). \end{aligned}$$

Proof. We have,

$$\begin{aligned} |\langle A\sigma_f B)Xx, y \rangle|^m &= |\langle A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} Xx, y \rangle|^m \\ &\leq |\langle f(A^{-1/2} B A^{-1/2}) A^{1/2} Xx, A^{1/2} y \rangle|^m \\ &\leq \left(\langle f(A^{-1/2} B A^{-1/2}) A^{1/2} Xx, f(A^{-1/2} B A^{-1/2}) A^{1/2} Xx \rangle^{\frac{1}{2}} \langle A^{1/2} y, A^{1/2} y \rangle^{\frac{1}{2}} \right)^m \\ &\leq \left(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} Xx, x \rangle^{\frac{1}{2}} \langle A y, y \rangle^{\frac{1}{2}} \right)^m \\ &\leq 2^{\frac{-m}{r}} \left(\langle X^* A^{1/2} f^2(A^{-1/2} B A^{-1/2}) A^{1/2} Xx, x \rangle^r + \langle A y, y \rangle^r \right)^{\frac{m}{r}} - \xi(x, y) \\ &\quad \text{(by inequality (2.4)).} \end{aligned}$$

Taking the supremum over $x, y \in \mathcal{H}$, $\|x\| = \|y\| = 1$, we deduce the result.

Taking $f(t) = \sqrt{t}$, in Theorem 4.2, we have the following simplified form.

Corollary 4.3. Let $A, B, X \in B(\mathcal{H})$ be such that A, B are positive definite. Then for all integers m , and $r > 1$, we have

$$\|(A\sharp B)X\|^m \leq 2^{-m} (\|X^* B X\|^r + \|A\|^r)^{\frac{m}{r}} - \inf_{\|x\|=1, \|y\|=1} \xi(x, y),$$

where

$$\xi(x, y) = \frac{1}{2^m} \left(\frac{\langle X^* B X x, x \rangle^{m+1} - \langle A y, y \rangle^{m+1}}{\langle X^* B X x, x \rangle - \langle A y, y \rangle} - (m+1)(\langle A y, y \rangle \langle X^* B X x, x \rangle)^{\frac{m}{2}} \right).$$

5. Applications to refined Young type inequalities for traces, determinants, and norms of positive definite matrices

In this section, we give some refined Young type inequalities for traces, determinants, and norms of positive definite matrices.

Let $\mathbf{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. A matrix $A \in \mathbf{M}_n(\mathbb{C})$ is called positive semidefinite, denoted as $A \geq 0$ if $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$, and it is called positive definite denoted as $A > 0$ if $x^* A x > 0$ for all nonzero $x \in \mathbb{C}^n$. The singular values of a matrix

$A \in \mathbf{M}_n(\mathbb{C})$ are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, denoted by $s_i(A)$ for $i = 1, 2, 3, \dots, n$. A norm $|||\cdot|||$, on $\mathbf{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbf{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$.

The trace norm $||\cdot||$ is given by $||A||_1 = \text{tr}|A| = \sum_{k=1}^n s_k(A)$, where tr is the usual trace. This norm is unitarily invariant.

A matrix Young's inequality due to Ando [2] asserts that

$$s_j(A^\alpha B^{1-\alpha}) \leq s_j(\alpha A + (1-\alpha)B),$$

the above singular value inequality entails the following unitarily invariant norm inequality

$$|||A^\alpha B^{1-\alpha}||| \leq |||\alpha A + (1-\alpha)B|||.$$

A determinant version of Young's inequalities is also known [11; p. 467]: For positive semidefinite matrices A, B and $0 \leq \alpha \leq 1$,

$$\det(A^\alpha B^{1-\alpha}) \leq \det(\alpha A + (1-\alpha)B). \quad (5.1)$$

To prove the result of this section, we need the following two lemmas, the first lemma (see, e.g., [11; p. 482,]) is the Minkowski inequality for determinants. the second lemma [5] is a Heinz-Kato type inequality for unitarily invariant norms.

Lemma 5.1. *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices. Then we have*

$$\det(A+B)^{\frac{1}{n}} \geq \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}. \quad (5.2)$$

Lemma 5.2. *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices. Then we have*

$$|||A^\alpha XB^{1-\alpha}||| \leq |||AX|||^\alpha |||XB|||^{1-\alpha}. \quad (5.3)$$

In particular

$$\text{tr}|A^\alpha XB^{1-\alpha}| \leq (\text{tr}A)^\alpha (\text{tr}B)^{1-\alpha}. \quad (5.4)$$

The first result of this section concerns the determinat of positive definite matrices which can be reads as follows:

Theorem 5.1. *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \leq \alpha \leq 1$. Then for $m = 1, 2, 3, \dots$, we have*

$$\begin{aligned} \left(\det(A^\alpha B^{1-\alpha}) \right)^m &+ r_0^{nm} \left(\frac{\det(B)^{m+\frac{1}{n}} - \det(A)^{m+\frac{1}{n}}}{\det(B)^{\frac{1}{n}} - \det(A)^{\frac{1}{n}}} - (m+1)(\det(AB))^{\frac{m}{2}} \right) \\ &\leq \det(\alpha A + (1-\alpha)B)^m, \end{aligned}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Proof. We have

$$\begin{aligned} \det(\alpha A + (1-\alpha)B)^m &= \left[\det(\alpha A + (1-\alpha)B)^{\frac{1}{n}} \right]^{nm} \\ &\geq \left[\det(\alpha A)^{\frac{1}{n}} + \det((1-\alpha)B)^{\frac{1}{n}} \right]^{nm} \quad (\text{by Lemma 5.1}) \end{aligned}$$

$$\begin{aligned}
&= \left[\alpha \det(A)^{\frac{1}{n}} + (1 - \alpha) \det(B)^{\frac{1}{n}} \right]^{nm} \\
&\geq \left[\left(\det(A)^{\frac{1}{n}} \right)^{\alpha} \left(\det(B)^{\frac{1}{n}} \right)^{1-\alpha} \right]^{nm} \\
&+ r_0^{nm} \left(\frac{\det(B)^{m+\frac{1}{n}} - \det(A)^{m+\frac{1}{n}}}{\det(B)^{\frac{1}{n}} - \det(A)^{\frac{1}{n}}} - (m+1)(\det(AB))^{\frac{m}{2}} \right) \\
&\quad (\text{by Theorem 2.1}) \\
&= \left(\det(A^{\alpha} B^{1-\alpha}) \right)^m \\
&+ r_0^{nm} \left(\frac{\det(B)^{m+\frac{1}{n}} - \det(A)^{m+\frac{1}{n}}}{\det(B)^{\frac{1}{n}} - \det(A)^{\frac{1}{n}}} - (m+1)(\det(AB))^{\frac{m}{2}} \right).
\end{aligned}$$

The second result of this section concerns the traces of positive definite matrices which can be reads as follows:

Theorem 5.2. *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \leq \alpha \leq 1$. Then for $m = 1, 2, 3, \dots$, we have*

$$\begin{aligned}
&\left(\text{tr}(|A^{\alpha} B^{1-\alpha}|) \right)^m + r_0^m \left(\frac{(\text{tr} B)^{m+1} - (\text{tr} A)^{m+1}}{(\text{tr} B) - (\text{tr} A)} - (m+1)((\text{tr} A)(\text{tr} B))^{\frac{m}{2}} \right) \\
&\leq \left[\text{tr}(\alpha A + (1 - \alpha) B) \right]^m,
\end{aligned}$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$.

Proof. We have

$$\begin{aligned}
&\left(\text{tr}(|A^{\alpha} B^{1-\alpha}|) \right)^m + r_0^m \left(\frac{(\text{tr} B)^{m+1} - (\text{tr} A)^{m+1}}{(\text{tr} B) - (\text{tr} A)} - (m+1)((\text{tr} A)(\text{tr} B))^{\frac{m}{2}} \right) \\
&\leq \left[(\text{tr} A)^{\alpha} (\text{tr} B)^{1-\alpha} \right]^m + r_0^m \left(\frac{(\text{tr} B)^{m+1} - (\text{tr} A)^{m+1}}{(\text{tr} B) - (\text{tr} A)} - (m+1)((\text{tr} A)(\text{tr} B))^{\frac{m}{2}} \right) \\
&\quad (\text{by inequality (5.4)}) \\
&\leq \left[\text{tr}(\alpha A + (1 - \alpha) B) \right]^m \quad (\text{by Theorem 2.1}).
\end{aligned}$$

The third result of this section concerns the norms of positive semi-definite matrices which can be reads as follows:

Theorem 5.3. *Let $A, X, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices and $0 \leq \alpha \leq 1$. Then for $m = 1, 2, 3, \dots$*

$$\left\| |A^{\alpha} X B^{1-\alpha}| \right\|^m + r_0^m \left(\frac{(\|XB\|)^{m+1} - (\|AX\|)^{m+1}}{(\|XB\|) - (\|AX\|)} - (m+1)(\|AX\|)(\|XB\|)^{\frac{m}{2}} \right)$$

$$\leq \left[\alpha |||AX||| + (1 - \alpha) |||XB||| \right]^m,$$

where $r_0 = \min\{\alpha, 1 - \alpha\}$.

Proof. We have

$$\begin{aligned} & |||A^\alpha XB^{1-\alpha}|||^m + r_0^m \left(\frac{(|||XB|||)^{m+1} - (|||AX|||)^{m+1}}{(|||XB|||) - (|||AX|||)} - (m+1)(|||AX|||)(|||XB|||)^{\frac{m}{2}} \right) \\ & \leq \left[|||AX|||^\alpha |||XB|||^{1-\alpha} \right]^m \\ & \quad + r_0^m \left(\frac{(|||XB|||)^{m+1} - (|||AX|||)^{m+1}}{(|||XB|||) - (|||AX|||)} - (m+1)(|||AX|||)(|||XB|||)^{\frac{m}{2}} \right) \\ & \quad (\text{by inequality (5.2)}) \\ & \leq \left[\alpha |||AX||| + (1 - \alpha) |||XB||| \right]^m \quad (\text{by Theorem 2.1}). \end{aligned}$$

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