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A generalized sequential problem of Lane-Emden type via fractional calculus

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ABSTRACT. In this paper, we combine the Riemann-Liouville integral operator and Caputo derivative to investigate a nonlinear time-singular differential equation of Lane Emden type. The considered problem involves n fractional Caputo derivatives under the conditions that neither commutativity nor semi group property is satisfied for these derivatives. We prove an existence and uniqueness analytic result by application of Banach contraction principle. Then, another result that deals with the existence of at least one solution is delivered and some sufficient conditions related to this result are established by means of the fixed point theorem of Schaefer. We end the paper by presenting to the reader some illustrative examples.

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1. Introduction

Fractional calculus and differential equations of fractional order are of great importance since they can be used in analyzing and modeling real word phenomena [10, 11, 21]. Recently, there has been a very important progress in the study of the theory of differential equations of fractional order. For some related research works with applications, the reader can consult the references [3, 4, 20, 22]. In particular, research on the existence of unique solutions for fractional

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differential equations is of big importance since it help physicians to better understand the behaviour of real word phenomena. For more details, see the papers [2, 15, 20, 22, 23, 24, 27]. Moreover, the singular differential equations are also very important in applied sciences, for more information, see [2, 6, 7, 25]. Among these equations, we cite the standard Lane-Emden equation which is part of the present work. This equation has a considerable importance in astrophysics, for more details, one can consult [19, 26, 27] and the reference therein. Before recalling some other equations and problems that have motivated the present work, we remind the reader that the standard form of Lane Emden equation has the following form:

$$y''(t) + \frac{a}{t}y'(t) + f(t, y(t)) = g(t), \quad t \in (0, 1], a \geq 0, \quad (1.1)$$

with

$$y(0) = a_1, \quad y'(0) = a_2,$$

where a_1, a_2 are constants, f and g are continuous functions (see [28]).

In [23], the authors have proposed the following fractional order Lane Emden model:

$$\begin{cases} D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\beta y(t) + f(t, y(t)) = g(t), \quad t \in (0, 1], \\ k \geq 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \end{cases} \quad (1.2)$$

with

$$y(0) = y_0, \quad y'(0) = y_1,$$

where y_0 and y_1 are constants, f and g are two continuous functions and the derivatives are in the sense of Riemann-Liouville. The authors have used the collocation method to establish some numerical solutions for this model.

In [18, 19], Rabha W. Ibrahim has studied two sequential singular equations having the same form, but different in their initial conditions. The considered equations are given by:

$$\begin{cases} D^\beta \left(D^\alpha + \frac{a}{t} \right) u(t) + f(t, u(t)) = g(t), \\ 0 < \alpha, \beta \leq 1, \quad 0 < t \leq 1, \quad a \geq 0. \end{cases} \quad (1.3)$$

For her first sequential singular equation, Rabha W. Ibrahim has taken the conditions $u(0) = u(1) = u(r) = 0, 0 < r < 1$; for this case, the existence of solutions by Krasnoselskii theorem has been studied in [18]. The second sequential problem that has been considered by Rahba W. Ibrahim has the conditions $u(0) = \mu, u(1) = \nu$; for this case, the existence and the Ulam stability of solutions have been discussed in [19].

Also in [8], A. Bakkouche et al. have studied the existence of solutions and the Δ -Ulam stabilities for the following two dimension non homogeneous Lane-Emden fractional system:

$$\begin{cases} D^{\beta_1} (D^{\alpha_1} + b_1 g_1(t)) x_1(t) + f_1(t, x_1(t), x_2(t)) \\ \quad = \omega_1 S_1(t, x_1(t), x_2(t)), \quad 0 < t < 1, \\ D^{\beta_2} (D^{\alpha_2} + b_2 g_2(t)) x_2(t) + f_2(t, x_1(t), x_2(t)) \\ \quad = \omega_2 S_2(t, x_1(t), x_2(t)), \quad 0 < t < 1, \\ x_k(0) = 0, \quad D^\alpha x_k(1) + b_k g_k(1) x_k(1) = 0, \end{cases} \quad (1.4)$$

under the conditions: $0 < \beta_k < 1, 0 < \alpha_k < 1, b_k \geq 0, 0 < \omega_k < \infty, k = 1, 2$ and the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo. The functions $f_k : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $S_k : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous, $g_k :]0, 1] \rightarrow [0, +\infty)$ is continuous and singular at $t = 0$.

In the paper [9], A. Benzidane and Z. Dahmani have considered the following class of nonlinear equations of Lane Emden type:

$$\left\{ \begin{array}{l} D^{\beta_1} \left(D^{\alpha_1} + g_1(t) \right) x_1(t) + f_1(t, x_1(t), x_2(t), D^{\delta_1} x_1(t), D^{\delta_2} x_2(t)) \\ \quad = h_1(t, x_1(t), x_2(t)), \\ D^{\beta_2} \left(D^{\alpha_2} + g_2(t) \right) x_2(t) + f_2(t, x_1(t), x_2(t), D^{\delta_1} x_1(t), D^{\delta_2} x_2(t)) \\ \quad = h_2(t, x_1(t), x_2(t)), \\ x_k(0) = a_k, x_k(1) = b_k, t \in J, \end{array} \right. \quad (1.5)$$

where $J = [0, 1]$, $0 < \alpha_k, \beta_k < 1, 0 < \delta_k < \alpha_k < 1, k = 1, 2$; the functions $f_k : [0, 1] \times \mathbb{R}^4, k = 1, 2$ are continuous, $g_k : (0, 1] \rightarrow [0, +\infty)$ are continuous functions, singular at $t = 0$, and $\lim_{t \rightarrow 0^+} g_k(t) = \infty$; the operators $D^{\alpha_k}, D^{\beta_k}$ and $D^{\delta_k}, k = 1, 2$ are the derivatives in the sense of Caputo and the constants a_k, b_k are reals. The authors have studied the existence and uniqueness of solutions and the Ulam stability for the considered class.

Recently, Y. Bahous and Z. Dahmani [5] have considered a Lane Emden type problem involving both Caputo derivative and Riemann-Liouville integral in its nonlinearities. The problem is the following:

$$\left\{ \begin{array}{l} D^{\beta} \left(D^{\alpha} + \frac{k}{t^{\lambda}} \right) y(t) + f(t, y(t), D^{\delta} y(t)) + g(t, y(t), I^{\rho} y(t)) \\ \quad = h(t), t \in]0, 1[, \\ y(0) = v, y(1) = b \int_0^{\eta} q(s) y(s) ds, 0 < \eta < 1, \\ 0 < \beta, \alpha < 1, k > 0, \lambda > 0, \end{array} \right. \quad (1.6)$$

where D^{α} is in the sense of Caputo, I^{ρ} is the Riemann-Liouville integral of order ρ , the functions $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous, and h and q are continuous over $[0, 1]$. The authors have investigated the existence and uniqueness of solutions. Then, they have studied the Ulam-Hyers stability.

Very recently, Y. Gouari et al. [16] have proposed the following nonlinear singular integro-differential equation of Lane Emden type with nonlocal multi point integral conditions:

$$\left\{ \begin{array}{l} D^{\beta} \left(D^{\alpha} + \frac{k}{t^{\lambda}} \right) y(t) + \Delta_1 f(t, y(t), D^{\delta} y(t)) + \Delta_2 g(t, y(t), I^{\rho} y(t)) \\ \quad + h(t, y(t)) = l(t) \\ y(0) = 0 \\ y(1) = b \int_0^{\eta} y(s) ds, 0 < \eta < 1 \\ I^q y(u) = y(1), 0 < u < 1 \\ k > 0, 0 < \lambda \leq 1, 1 \leq \beta \leq 2, 0 \leq \alpha, \delta \leq 1, \end{array} \right. \quad (1.7)$$

where $\Delta_1 > 0, \Delta_2 > 0$ are positive real numbers, I^ρ is the Riemann-Liouville integral of order ρ , and f, g are two given functions defined on $[0, 1] \times \mathbb{R}^2$, and h and l are two given functions defined over $[0, 1]$. The authors have proved the existence and the uniqueness of solutions by application of Banach contraction principle, then, by means of Schaefer fixed point theorem, they have studied the existence of at least one solution for the problem.

In this paper, we are concerned with the following sequential time-singular fractional problem of Lane Emden type:

$$\left\{ \begin{array}{l} D^{\alpha_1}(D^{\alpha_2} \dots (D^{\alpha_n}(D^\beta + \frac{k}{t^\lambda})) \dots) u(t) + f(t, u(t), D^\delta u(t)) + g(t, u(t), I^\rho u(t)) \\ + h(t, u(t)) = l(t), \quad t \in]0, 1[, \\ u(0) = 0, \\ u(1) = \theta, \\ D^{\alpha_n}(D^\beta u(0)) = 0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^\beta u(0))) = 0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4} \dots (D^{\alpha_n}(D^\beta u(0))) \dots) = 0, \\ D^{\alpha_2}(D^{\alpha_3} \dots (D^{\alpha_n}(D^\beta u(1) + \phi_{k,\lambda}(1)u(1))) \dots) = 0, \\ k > 0. \end{array} \right. \quad (1.8)$$

For (1.8), we take $J := [0, 1]$, $0 \leq \beta \leq 1, 0 \leq \alpha_i < 1; i = 1, 2, \dots, n, \delta < \min(\beta, \alpha_i), \phi_{k,\lambda}(t) = \frac{k}{t^\lambda}$, the sequential derivatives are in the sense of Caputo, I^ρ denotes the Riemann-Liouville integral of order ρ , and $f, g : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are two given functions, also $h : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function and l is a function which is defined on J .

We need also to shed light on the fact that for the above problem, we take into account the following points:

1. We introduce the Caputo derivative in both sides of equation of the problem.
2. Also, we introduce the Riemann Liouville integral in one nonlinearity of the right hand side of the equation.
3. Another particularity of the problem, is that it includes n sequential Caputo derivations which allow us to introduce a new sequential problem with absence of commutativity and semi group properties between the introduced derivatives; this absence is not true in the classical case of differentiation. (It is also absent when dealing with, at least, two parameters $\alpha_1, \alpha_2 \in [0, 1]$, that satisfy $\alpha_1 + \alpha_2 \notin [0, 1]$). So, for our case, we have to find new arguments to overcome this type of problem.
4. Another important point in this paper is the time singularity at the origin for the above problem.

So based on the above four conditions, we think that in this paper, we are concerned with a more general sequential problem of Lane Emden type; it is "more general" in the sense that it can be used to describe many problems that arise in mathematical physics, since it includes several particular types of equations with important applications. For instance, it is easy to verify that our problem includes the standard Lane Emden equation as a special case. Also,

it includes the Emden Fowler equation that has been introduced to model several phenomena in mathematical physics and astrophysics, such as the theory of stellar structure. Also, the fractional problem proposed by the authors of [16] can be derived from the above problem under some special values on the parameters and the functions of our problem.

To the best of our knowledge, this is the first time in the literature where such problem is investigated.

The paper is organised as follows: in section 2, we recall some definitions and lemmas that are used for the proof of our main results. In section 3, we prove the main theorems of this paper, and we discuss some illustrative examples. In the last section, a conclusion follows.

2. Preliminaries

In this paper, we work with the following definitions and lemmas. We refer the reader to [24] for more information.

Definition 2.1. *The Riemann-Liouville integral of order $\alpha > 0$ for all continuous function on $[0, 1]$ is given by:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (2.1)$$

with $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2.2. *If $f \in C^n([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, then, the Caputo derivative is given by:*

$$\begin{aligned} D^\alpha f(t) &= I^{n-\alpha} \frac{d^n}{dt^n} (f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned} \quad (2.2)$$

Lemma 2.1. *Taking $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$, then the solutions of $D^\alpha y(t) = 0$ are:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i; \quad (2.3)$$

$c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n - 1$.

Lemma 2.2. *Taking $n \in \mathbb{N}^*, n - 1 < \alpha < n$, then,*

$$I^\alpha D^\alpha y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i, \quad (2.4)$$

and $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n - 1$.

The following result establishes a relation between the integral representation and the differential problem given in (1.8).

Lemma 2.3. *Let $G \in C([0, 1])$. Then, we can state that*

$$\left\{ \begin{array}{l} D^{\alpha_1}(D^{\alpha_2} \dots (D^{\alpha_n}(D^\beta + \frac{k}{t^\lambda})) \dots) u(t) = G(t), \quad t \in]0, 1[, \\ u(0) = 0, \\ u(1) = \theta, \\ D^{\alpha_n}(D^\beta u(0)) = 0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^\beta u(0))) = 0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4} \dots (D^{\alpha_n}(D^\beta u(0))) \dots) = 0, \\ D^{\alpha_2}(D^{\alpha_3} \dots (D^{\alpha_n}(D^\beta u(1) + \phi_{k,\lambda}(1)u(1))) \dots) = 0, \\ k > 0, 0 \leq \alpha_i < 1, 0 \leq \beta \leq 1; i = 1, 2, \dots, n \end{array} \right. \quad (2.5)$$

admits as integral solution the following expression

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\frac{1}{\Gamma\left(\sum_{i=1}^n \alpha_i\right)} \int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} G(\tau) d\tau - \frac{k}{s^\lambda} u(s) \right) ds \\ & - \frac{\int_0^1 (1-s)^{\alpha_1-1} G(s) ds}{\Gamma(\alpha_1) \Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right)} t^{\sum_{i=2}^n \alpha_i + \beta} + \left[\frac{\int_0^1 (1-s)^{\alpha_1-1} G(s) ds}{\Gamma(\alpha_1) \Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right)} \right] t^\beta \\ & - \left[\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left(\frac{1}{\Gamma\left(\sum_{i=1}^n \alpha_i\right)} \int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} G(\tau) d\tau - \frac{k}{s^\lambda} u(s) \right) ds + \theta \right] t^\beta. \end{aligned} \quad (2.6)$$

Proof. We use the property established in Lemma 2.2 to (2.5), we find that

$$\begin{aligned} u(t) = & I^\beta \left(\sum_{i=1}^n \alpha_i G(s) - \frac{k}{s^\lambda} \right) (t) + \frac{c_0}{\Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right)} t^{\sum_{i=2}^n \alpha_i + \beta} + \frac{c_1}{\Gamma\left(\sum_{i=3}^n \alpha_i + \beta + 1\right)} t^{\sum_{i=3}^n \alpha_i + \beta} \\ & + \frac{c_2}{\Gamma\left(\sum_{i=4}^n \alpha_i + \beta + 1\right)} t^{\sum_{i=4}^n \alpha_i + \beta} + \dots + \frac{c_{n-2}}{\Gamma(\alpha_n + \beta + 1)} t^{\alpha_n + \beta} + \frac{c_{n-1}}{\Gamma(\beta + 1)} t^\beta + c_n. \end{aligned} \quad (2.7)$$

Some of the initial conditions allow us to write:

$$\begin{aligned} u(0) = 0 & \Rightarrow c_n = 0 \\ D^{\alpha_2}(D^{\alpha_3} \dots (D^{\alpha_n}(D^\beta u(1) + \phi_{k,\lambda}(1)u(1))) \dots) = 0 & \Rightarrow c_0 = -I^{\alpha_1} G(1) \\ D^{\alpha_n}(D^\beta u(0)) = 0 & \Rightarrow c_{n-2} = 0 \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^\beta u(0))) = 0 & \Rightarrow c_{n-3} = 0 \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& D^{\alpha_4}(D^{\alpha_5} \dots (D^{\alpha_n}(D^\beta u(0))) \dots) = 0 \Rightarrow c_2 = 0 \\
& D^{\alpha_3}(D^{\alpha_4} \dots (D^{\alpha_n}(D^\beta u(0))) \dots) = 0 \Rightarrow c_1 = 0 \\
& u(1) = \theta \Rightarrow \\
& \frac{c_{n-1}}{\Gamma(\beta+1)} = \frac{\int_0^1 (1-s)^{\alpha_1-1} G(s) ds}{\Gamma(\alpha_1) \Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)} \\
& - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)} \int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} G(\tau) d\tau - \frac{k}{s^\lambda} u(s) \right) ds + \theta.
\end{aligned} \tag{2.8}$$

Replacing $c_0, c_1, c_2, \dots, c_n$ in (2.7), we end the proof.

Let us now transforming the above problem to a fixed point one.
We begin by considering the following notions related to Banach spaces:

$$X := \{x \in C(J, \mathbb{R}), D^\delta x \in C(J, \mathbb{R})\}; \tag{2.9}$$

$$\|x\|_X = \text{Max}\{\|x\|_\infty, \|D^\delta x\|_\infty\}, \tag{2.10}$$

and

$$\|x\|_\infty = \sup_{t \in J} |x(t)|, \|D^\delta x\|_\infty = \sup_{t \in J} |D^\delta x(t)|. \tag{2.11}$$

Then, we consider the nonlinear operator $H : X \rightarrow X$ that takes the form:

$$\begin{aligned}
Hu(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)} \int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} \left(l(\tau) - h(\tau, u(\tau)) - f(\tau, u(\tau), D^\delta u(\tau)) \right. \right. \\
& \left. \left. - g(\tau, u(\tau), I^\rho u(\tau)) \right) d\tau - \frac{k}{s^\lambda} u(s) \right] ds - \frac{\sum_{i=2}^n \alpha_i + \beta}{\Gamma(\alpha_1) \Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)} \int_0^1 (1-s)^{\alpha_1-1} \left(l(s) \right. \\
& \left. - h(s, u(s)) - f(s, u(s), D^\delta u(s)) - g(s, u(s), I^\rho u(s)) \right) ds - \left[\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)} \right. \right. \\
& \left. \left. \times \int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} \left(l(\tau) - h(\tau, u(\tau)) - f(\tau, u(\tau), D^\delta u(\tau)) - g(\tau, u(\tau), I^\rho u(\tau)) \right) d\tau \right. \right. \\
& \left. \left. - \frac{k}{s^\lambda} u(s) \right) ds + \theta \right] t^\beta,
\end{aligned} \tag{2.12}$$

such that $\lambda < 1$.

At the end of this second section, we are able to prove the results.

3. Main Results

First, we note that we need to work with the following hypotheses:

(A1) : The functions f and g are defined and continuous on $J \times \mathbb{R}^2$ and h is defined and continuous on $J \times \mathbb{R}$.

(A2) : There are four nonnegative constants $L_{f1}, L_{f2}, L_{g1}, L_{g2}$; for any $t \in J, x_i, x_i^* \in \mathbb{R}$,

$$|f(t, x_1, x_2) - f(t, x_1^*, x_2^*)| \leq \sum_{i=1}^2 L_{fi} |x_i - x_i^*|, \quad (3.1)$$

$$|g(t, x_1, x_2) - g(t, x_1^*, x_2^*)| \leq \sum_{i=1}^2 L_{gi} |x_i - x_i^*|, \quad (3.2)$$

and there is one positive number r_0 ; for any $t \in J, x, y \in \mathbb{R}$,

$$|h(t, x) - h(t, y)| \leq r_0 |x - y|. \quad (3.3)$$

We take

$$L'_f := \text{Max}(L_{f1}, L_{f2}), \quad L'_g := \text{Max}(L_{g1}, L_{g2}). \quad (3.4)$$

(A3) : There are three positive constants M_f, M_g, M_h ; for any $t \in J, x \in \mathbb{R}^2, y \in \mathbb{R}$, we have

$$|f(t, x)| \leq M_f, \quad |g(t, x)| \leq M_g, \quad |h(t, y)| \leq M_h. \quad (3.5)$$

(A4) : We take: $\|l\|_\infty = M_l$.

Also, we consider the new expressions:

$$D1 = 2 \left[\left(r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho + 1)} \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)} \right) + \frac{k\Gamma(1 - \lambda)}{\Gamma(\beta - \lambda + 1)} \right], \quad (3.6)$$

$$D2 = \left(r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho + 1)} \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta - \delta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) + k\Gamma(1 - \lambda) \left(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)} \right). \quad (3.7)$$

The first main result deals with the existence of one and only one solution for (1.8). It is based on the application of Banach fixed point theorem for contraction mappings.

Theorem 3.1. *If the conditions $(A_i)_{i=2,3,4}$ are satisfied and $D < 1, D := \max \{D_1, D_2\}$, then, the problem (1.8) has a unique solution on J .*

Proof. It is sufficient for us to prove that H is a contraction mapping.

Let $(x, y) \in X^2$. Then, it is clear that

$$\begin{aligned} \|Hy - Hx\|_\infty \leq & 2 \left[\left(r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho + 1)} \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)} \right) \right. \\ & \left. + \frac{k\Gamma(1 - \lambda)}{\Gamma(\beta - \lambda + 1)} \right] \|y - x\|_X. \end{aligned} \quad (3.8)$$

On the other hand, since

$$\begin{aligned} D^\delta Hu(t) = & \frac{1}{\Gamma(\beta - \delta)} \int_0^t (t - s)^{\beta - \delta - 1} \left[\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)} \int_0^s (s - \tau)^{\sum_{i=1}^n \alpha_i - 1} \left(l(\tau) - h(\tau, u(\tau)) \right. \right. \\ & \left. \left. - f(\tau, u(\tau), D^\delta u(\tau)) - g(\tau, u(\tau), I^\rho u(\tau)) \right) d\tau - \frac{k}{s^\lambda} u(s) \right] ds - \frac{\sum_{i=2}^n \alpha_i + \beta - \delta}{\Gamma(\alpha_1) \Gamma(\sum_{i=2}^n \alpha_i + \beta - \delta + 1)} \\ & \times \int_0^1 (1 - s)^{\alpha_1 - 1} \left(l(s) - h(s, u(s)) - f(s, u(s), D^\delta u(s)) - g(s, u(s), I^\rho u(s)) \right) ds \\ & + \frac{\Gamma(\beta + 1)t^{\beta - \delta}}{\Gamma(\alpha_1) \Gamma(\sum_{i=2}^n \alpha_i + \beta + 1) \Gamma(\beta - \delta + 1)} \int_0^1 (1 - s)^{\alpha_1 - 1} \left(l(s) - h(s, u(s)) \right. \\ & \left. - f(s, u(s), D^\delta u(s)) - g(s, u(s), I^\rho u(s)) \right) ds - \left[\frac{1}{\Gamma(\beta)} \int_0^1 (1 - s)^{\beta - 1} \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)} \right. \right. \\ & \left. \left. \times \int_0^s (s - \tau)^{\sum_{i=1}^n \alpha_i - 1} \left(l(\tau) - h(\tau, u(\tau)) - f(\tau, u(\tau), D^\delta u(\tau)) - g(\tau, u(\tau), I^\rho u(\tau)) \right) d\tau \right. \right. \\ & \left. \left. - \frac{k}{s^\lambda} u(s) \right) ds + \theta \right] \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \delta + 1)} t^{\beta - \delta}, \end{aligned} \quad (3.9)$$

then, with the same arguments as before, we have

$$\begin{aligned}
 \|D^\delta Hy - D^\delta Hx\|_\infty \leq & \left(r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho+1)} \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta - \delta + 1)} \right. \\
 & + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
 & \left. + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) \|y - x\|_X \\
 & + k\Gamma(1 - \lambda) \left(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)} \right) \|y - x\|_X.
 \end{aligned} \tag{3.10}$$

Thanks to (3.8) and (3.10), we obtain

$$\|Hy - Hx\|_X \leq D \|x - y\|_X.$$

The proof is thus achieved.

The following main result deals with the existence of at least one solution of the studied problem.

Theorem 3.2. *Under the hypotheses (A1), (A3) and (A4), the problem (1.8) has at least one solution $u(t), t \in J$.*

Proof. Let us prove the result by considering the following steps:

Continuous of H : The proof is trivial and then it is omitted. (We just apply the fact that the functions are continuous).

Unifrom Boundness of H : Let us take $r > 0$ and consider the (bounded) ball $B_r := \{x \in X; \|x\|_X \leq r\}$. For $y \in B_r$, in virtue of (A3) and (A4), we can write

$$\begin{aligned}
 \|Hy\|_\infty \leq & 2 \left[\left(M_l + M_h + M_f + M_g \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)} \right) \right. \\
 & \left. + \frac{k\Gamma(1 - \lambda)}{\Gamma(\beta - \lambda + 1)} \right] + |\theta| < +\infty
 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\|D^\delta Hy\|_\infty \leq & \left(M_l + M_h + M_f + M_g \right) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta - \delta + 1)} \right. \\
& + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
& \left. + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) + k\Gamma(1 - \lambda) \left(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)} \right) \\
& + |\theta| < +\infty.
\end{aligned} \tag{3.12}$$

The above two inequalities show that $\|Hy\|_X < +\infty$.

Consequently H is uniformly bounded.

Equicontinuity of H : We prove that for any bounded set (B_r for instance), we obtain that $H(B_r)$ is an equicontinuous set of X .

Taking $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and consider the above (bounded) ball B_r of X . So by considering $y \in B_r$, we can state that

$$\begin{aligned}
|Hy(t_1) - Hy(t_2)| \leq & \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)} \left| t_1^{\sum_{i=1}^n \alpha_i + \beta} - t_2^{\sum_{i=1}^n \alpha_i + \beta} \right| \\
& + kr \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda + \beta)} |t_1^{\beta - \lambda} - t_2^{\beta - \lambda}| \\
& + \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(1 + \alpha_1)} \left(\left| t_1^{\sum_{i=2}^n \alpha_i + \beta} - t_2^{\sum_{i=2}^n \alpha_i + \beta} \right| + |t_1^\beta - t_2^\beta| \right) \\
& + \left(\frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)} + kr \frac{\Gamma(1 - \lambda)}{\Gamma(1 - \lambda + \beta)} + \theta \right) |t_1^\beta - t_2^\beta|
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
 |D^\delta Hy(t_1) - D^\delta Hy(t_2)| \leq & \frac{M_l + M_h + M_f + M_g}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta - \delta + 1\right)} \left| \sum_{i=1}^n \alpha_i - \delta + \beta - \sum_{i=1}^n \alpha_i - \delta + \beta \right| \\
 & + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta-\delta)} |t_1^{\beta-\delta-\lambda} - t_2^{\beta-\delta-\lambda}| \\
 & + \frac{M_l + M_h + M_f + M_g}{\Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right) \Gamma(1+\alpha_1)} \left| \sum_{i=2}^n \alpha_i + \beta - \delta - \sum_{i=2}^n \alpha_i + \beta - \delta \right| \\
 & + \frac{\left(M_l + M_h + M_f + M_g\right) \Gamma(\beta+1)}{\Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right) \Gamma(\beta-\delta+1) \Gamma(\alpha_1+1)} |t_1^{\beta-\delta} - t_2^{\beta-\delta}| \\
 & + \left(\frac{M_l + M_h + M_f + M_g}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta + 1\right)} + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta)} + \theta \right) \frac{\Gamma(\beta+1)}{\Gamma(\beta-\delta+1)} \\
 & \times |t_1^{\beta-\delta} - t_2^{\beta-\delta}|.
 \end{aligned} \tag{3.14}$$

For (3.13) and (3.14), their right hand sides tend to zero for $t_1 \rightarrow t_2$. As a consequence the Ascoli-Arzelà theorem, we conclude that H is completely continuous.

Boundness of A_γ : The set $A_\gamma := \{x \in X : x = \gamma Hx, \gamma \in]0, 1[\}$ is bounded. Let $y \in A_\gamma$. Then we have $y = \gamma Hy$ for some $0 < \gamma < 1$. Hence,

$$\begin{aligned}
 \|y\|_\infty \leq & \gamma \left(2 \left[\left(M_l + M_h + M_f + M_g \right) \left(\frac{1}{\Gamma\left(\sum_{i=1}^n \alpha_i + \beta + 1\right)} + \frac{1}{\Gamma\left(\sum_{i=2}^n \alpha_i + \beta + 1\right) \Gamma(\alpha_1 + 1)} \right) \right. \right. \\
 & \left. \left. + \frac{k\Gamma(1-\lambda)}{\Gamma(\beta-\lambda+1)} \right] + |\theta| \right).
 \end{aligned} \tag{3.15}$$

Do not forget that we have

$$\begin{aligned}
\|D^\delta y\|_\infty \leq & \gamma \left((M_l + M_h + M_f + M_g) \left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i + \beta - \delta + 1)} \right. \right. \\
& + \frac{1}{\Gamma(\sum_{i=2}^n \alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=2}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
& \left. \left. + \frac{\Gamma(\beta + 1)}{\Gamma(\sum_{i=1}^n \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) + k\Gamma(1 - \lambda) \left(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)} \right) \right. \\
& \left. + |\theta| \right). \tag{3.16}
\end{aligned}$$

Using (3.11) and (3.12), we state that $\|y\|_X < \infty$. The set is thus bounded.

Consequently, thanks to Schaefer fixed point theorem, we deduce that H has at least one fixed point. Thus, the problem (1.8) has a solution.

At the end of this section, let us give the following two examples.

Example 3.1.

$$\begin{cases}
D^{0.1}(D^{0.4}(D^{0.3}(D^{0.2}(D^{0.9} + \frac{3}{10t^{0.1}})))y(t) + \frac{|y(t)|}{(44\pi + t)(1 + |y(t)|)} + \frac{|D^\delta y(t)|}{(200 + t)(1 + |D^\delta y(t)|)} \\
+ \frac{\cos(y(t) + I^\rho y(t))}{400(t^2 + 1)} + \frac{|y(t)|}{(72e^{t+2})(1 + |y(t)|)} = 3t, \quad t \in]0, 1[, \\
y(0) = 0, \\
y(1) = 1, \\
D^{0.2}(D^{0.9}y(0)) = 0, \\
D^{0.3}(D^{0.2}(D^{0.9}y(0))) = 0, \\
D^{0.4}(D^{0.3}(D^{0.2}(D^{0.9}y(1) + \phi_{\frac{3}{10}, 0.1}(1)y(1)))) = 0, \\
\phi_{\frac{3}{10}, 0.1}(t) = \frac{3}{10t^{0.1}},
\end{cases} \tag{3.17}$$

$$\text{where: } f(t, x_1, x_2) = \frac{|x_1|}{(44\pi + t)(1 + |x_1|)} + \frac{|x_2|}{(200 + t)(1 + |x_2|)}, \quad g(t, x_1, x_2) = \frac{\cos(x_1 + x_2)}{400(t^2 + 1)},$$

$$h(t, x) = \frac{|x|}{(72e^{t+2})(1 + |x|)}, \quad l(t) = 3t, \quad \delta = 0.1, \quad \rho = 0.5, \quad D1 = 0.9613, \quad D2 = 0.9924, \quad \text{and} \\
D = \max \{D1, D2\} = 0.9924.$$

The conditions of our first main result hold. Therefore, the example has a unique solution on $[0, 1]$.

Example 3.2.

$$\left\{ \begin{array}{l} D^{0.2}(D^{0.4}(D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}} + \frac{2}{5t^{0.2}}))))y(t) + \frac{\sin y(t) + \sin D^{\delta}y(t)}{100e^{t^2+1}} + \\ + \frac{1}{48e^t} \left(\cos y(t) + \frac{|I^{\rho}y(t)|}{1 + |I^{\rho}y(t)|} \right) + \frac{|y(t)|}{(144e^{t^2})(1 + |y(t)|)} = 4t, \quad t \in]0, 1[, \\ y(0) = 0, \\ y(1) = 1, \\ D^{0.2}(D^{\frac{1}{2}}y(0)) = 0, \\ D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(0))) = 0, \\ D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(0)))) = 0, \\ D^{0.4}(D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(1) + \phi_{\frac{2}{5}, 0.2}(1)y(1))))) = 0, \\ \phi_{\frac{2}{5}, 0.2}(t) = \frac{4}{10t^{0.2}}. \end{array} \right. \quad (3.18)$$

We have:

$$\begin{aligned} f(t, x_1, x_2) &= \frac{\sin x_1 + \sin x_2}{100e^{t^2+1}}, \quad g(t, x_1, x_2) = \frac{1}{48e^t} \left(\cos x_1 + \frac{|x_2|}{1 + |x_2|} \right), \\ h(t, x) &= \frac{|x|}{(144e^{t^2})(1 + |x|)}, \quad l(t) = 4t, \end{aligned}$$

and

$$\delta = \frac{1}{2}, \quad \rho = \frac{1}{7}.$$

Also, we have

$$|f(t, x_1, x_2)| \leq \frac{1}{50}, \quad |g(t, x_1, x_2)| \leq \frac{1}{24}, \quad |h(t, x)| \leq \frac{1}{144}, \quad \|l\|_{\infty} = 4,$$

Since f, g, h are continuous, then by Theorem 3.2, the problem presented in this example has at least one solution on $[0, 1]$.

4. Conclusion

The aim of this paper was to introduce the more general singular fractional problem of Lane Emden type given in (8), by means of n sequential Caputo derivatives. Using the well known Banach fixed point theorem, an existence and uniqueness result has been proved, then by application of Scafefer fixed point theorem, the existence of at least one solution for (8) has also been discussed. To justify the applicability of our main results, two examples have been studied in details.

Since in this field of interest it is important to increase ability of researchers for investigating differential equations and trying to find applicability of the studied problems in real word phenomena, in the paper we have discussed a Lane Emden type problem with a time singularity. However, for future developments, we think it will be more suitable to discuss the

above problem by considering new nonlinearities with an arbitrary time singularity or with singularities in space variables. Another important physical point when dealing with the type of space singularities is to be concerned with positive solutions for the considered problem in order better to satisfy the needs of physicist researchers and to contribute with them in physical applications.

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