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Eigencurves of the $p(\cdot)$ -Biharmonic operator with a Hardy-type term

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ABSTRACT. This paper is devoted to the study of the homogeneous Dirichlet problem for a singular nonlinear equation which involves the $p(\cdot)$ -biharmonic operator and a Hardy-type term that depend on the solution and with a parameter λ . By using a variational approach and min-max argument based on Ljusternik-Schnirelmann theory on C^1 -manifolds [13], we prove that the considered problem admits at least one non-decreasing sequence of positive eigencurves with a characterization of the principal curve $\mu_1(\lambda)$ and also show that, the smallest curve $\mu_1(\lambda)$ is positive for all $0 \leq \lambda < C_H$, with C_H is the optimal constant of Hardy type inequality.

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1. Introduction

Nonlinear singular elliptic problems have been studied intensively in recent years, we mention the work [8]. For instance, nonlinear singular boundary value problems arise in some parts of science such as in boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, chemical catalyst kinetics, and theory of heat conduction in electrically conducting materials.

Hardy inequality in $W_0^{2,p}(\Omega)$ and $W_0^{2,p(\cdot)}(\Omega)$ and the related problems for nonlinear elliptic equation have attracted considerable interest. We refer to papers [6, 7, 9], where further bibliographical references can be found.

Motivated by such interest, we consider the singular $p(\cdot)$ -biharmonic problem

$$(E_\lambda) \quad \begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) - \lambda \frac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} = \mu |u|^{p(x)-2}u, & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\delta(x) := \text{dist}(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$. Throughout this paper the function $p \in C(\overline{\Omega})$ always denotes a variable exponent satisfying the following bounds

$$1 < p^- := \min_{\overline{\Omega}} p \leq p^+ := \max_{\overline{\Omega}} p < \frac{N}{2}, \quad (1.1)$$

here μ plays the role of an eigenvalue depending on λ (referred as eigencurve : $\lambda \mapsto \mu(\lambda)$), and λ is a real parameter such that $0 \leq \lambda < C_H$, where

$$C_H := \min \left(\frac{p^-}{p^+} \gamma_1^{p^-}; \frac{p^-}{p^+} \gamma_2^{p^+} \right),$$

is a positive constant in the $p(\cdot)$ -Hardy inequality, with

$$\gamma_1 = \left\lceil \frac{N(p^- - 1)(N - 2p^-)}{(p^-)^2} \right\rceil \text{ and } \gamma_2 = \left\lceil \frac{N(p^+ - 1)(N - 2p^+)}{(p^+)^2} \right\rceil,$$

and the $p(\cdot)$ -Hardy inequality [6, Lemma 3.1] is given by

$$\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx \geq C_H \int_{\Omega} \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx, \quad (1.2)$$

for all $u \in W_0^{2,p(\cdot)}(\Omega)$.

$\Delta_{p(\cdot)}^2 u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(\cdot)$ -biharmonic operator. It is reduced to the p -biharmonic (for a constant exponent $p > 1$). For variable exponent the $p(\cdot)$ -biharmonic presents a more complicated nonlinearity than the p -biharmonic, we loose homogeneity of any order.

In [6], the case $\mu = 0$ was considered. By the Ljusternik-Schnirelmann theory on C^1 -manifolds [13, 4], the existence of a sequence of positive eigenvalues was proved. Let Λ be the set of all nonnegative eigenvalues. It was shown that $\sup \Lambda = \infty$. Sufficient conditions were also found to guarantee that $\inf \Lambda > 0$.

The eigencurve problems involving the p -Laplacian operator has been the object of study in [2]. Precisely, the authors have studied the problem $-\Delta_p u - \lambda \rho(x)|u|^{p-2}u = \mu|u|^{p-2}u$, on a bounded regular domain.

In [5], without any assumption on the regularity of the bounded domain Ω , the authors have studied the problem $\Delta_p^2 u - \lambda \rho(x)|u|^{p-2}u = \mu|u|^{p-2}u$. Using the variational methods, they obtained the existence of an increasing sequence of positive eigencurves.

The goal of this paper is to show that problem (E_λ) has at least one non-decreasing sequence of eigencurves $(\mu_k(\lambda))_{k \geq 1}$, as we give a direct characterization of the principal curve, by using a variational technique based on Ljusternik- Schnirelmann theory on C^1 -manifolds [13].

The paper is organized as follows. In the second section, we provide a suitable functional framework for problem (E_λ) and prove some preliminary results. In the third section, we prove the compactness of the functional Φ_λ associated with (E_λ) .

Finally, in the fourth Section, we obtain the existence of an increasing sequence of positive eigencurves of the $p(\cdot)$ -biharmonic operator with a Hardy-type term.

2. Functional framework and preliminaries

In this section, we briefly review the definitions and list some results on the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, where Ω is given as in problem (E_λ) . We refer the reader to the book [3] and the references therein, for the basic properties of these spaces.

Let

$$C_1^+(\overline{\Omega}) := \left\{ h \mid h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ on } \overline{\Omega} \right\}.$$

Let $p \in C_1^+(\overline{\Omega})$ be fixed. We define the generalized Lebesgue space by

$$L^{p(\cdot)}(\Omega) := \left\{ u \mid u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \rho_{p(\cdot)}(u) < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \alpha > 0 \mid \rho_{p(\cdot)}\left(\frac{u}{\alpha}\right) \leq 1 \right\},$$

where $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ is the convex modular mapping defined by

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

We define the potential

$$\Psi_{p(\cdot)}(u) = \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx.$$

Proposition 2.1 (see [14]). *Under hypothesis (1.1), the space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is separable, uniformly convex, reflexive and its conjugate dual space is $L^{p'(\cdot)}(\Omega)$ where*

$$p'(x) = \frac{p(x)}{p(x) - 1} \quad \text{for all } x \in \Omega.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ we have the Hölder-type inequality

$$\left| \int_{\Omega} u(x)v(x) dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \leq 2 |u|_{p(\cdot)} |v|_{p'(\cdot)}. \quad (2.1)$$

The Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$, where m is a positive integer, is defined as

$$W^{m,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) \mid D^{\alpha} u \in L^{p(\cdot)}(\Omega), \quad |\alpha| \leq m \right\}.$$

Equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha| \leq m} |D^{\alpha} u|_{p(\cdot)}.$$

$W^{m,p(\cdot)}(\Omega)$ are Banach, separable and reflexive spaces. For more details, we refer the reader to [10, 11, 12] and the references therein. $W_0^{m,p(\cdot)}(\Omega)$ is denoted by the closure of $C_0^{\infty}(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

Proposition 2.2 (see [11, Theorem 1.3]). *Let $u_n, u \in L^{p(\cdot)}(\Omega)$, we have*

1. $|u|_{p(\cdot)} = a \Leftrightarrow \rho_{p(\cdot)}\left(\frac{u}{a}\right) = 1$ for $u \neq 0$ and $a > 0$.
2. $|u|_{p(\cdot)} < (=; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < (=; > 1)$.
3. $|u_n| \rightarrow 0$ (resp $\rightarrow +\infty$) $\Leftrightarrow \rho_{p(\cdot)}(u_n) \rightarrow 0$, (resp $\rightarrow +\infty$).
4. *The following statements are equivalent to one another:*
 - (i) $\lim_{n \rightarrow +\infty} |u_n - u|_{p(\cdot)} = 0$,
 - (ii) $\lim_{n \rightarrow +\infty} \rho_{p(\cdot)}(u_n - u) = 0$,
 - (iii) $u_n \rightarrow u$ in measure in Ω and $\lim_{n \rightarrow +\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

Throughout this paper, we let $X := W_0^{2,p(\cdot)}(\Omega)$, and $X^* := W^{-2,p'(\cdot)}(\Omega)$ its dual. Define a norm $\|\cdot\|$ of X by

$$\|u\| := \inf \left\{ \alpha > 0 \mid \rho_{p(\cdot)}\left(\frac{\Delta u}{\alpha}\right) \leq 1 \right\}.$$

Remark 2.1. *It easy to see that $\|\cdot\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta(\cdot)|_{p(\cdot)}$ in the space X . Consequently, the norms $\|\cdot\|_{2,p(\cdot)}$, $\|\cdot\|$ and $|\Delta(\cdot)|_{p(\cdot)}$ are equivalent. Furthermore in view of Proposition 2.2, for every $u \in X$, we have*

$$\min \left\{ \|u\|^{p^+}, \|u\|^{p^-} \right\} \leq \int_{\Omega} |\Delta u|^{p(x)} dx \leq \max \left\{ \|u\|^{p^+}, \|u\|^{p^-} \right\}. \quad (2.2)$$

Note that a weak solution for problem (E_{λ}) satisfies the following definition.

Definition 2.1. A function $u \in X$ is termed a weak solution of (E_λ) , if for all $v \in X$,

$$\int_{\Omega} |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta v(x) dx = \int_{\Omega} \left(\lambda \frac{|u(x)|^{p(x)-2} u(x)}{\delta(x)^{2p(x)}} + \mu |u(x)|^{p(x)-2} u \right) v(x) dx. \quad (2.3)$$

If u is not identically zero, then we say that u is called an eigenfunction of the $p(\cdot)$ -biharmonic operator associated to the eigenvalue couple (λ, μ) .

Definition 2.2. We call by the principal eigencurve of the $p(\cdot)$ -biharmonic operator, the graph of the function $\lambda \rightarrow \mu_1(\lambda)$ from the interval $[0, C_H[$ into $\mathbb{R}$ defined by:

$$\mu_1(\lambda) = \inf \left\{ \frac{\int_{\Omega} \frac{|\Delta u|^{p(x)}}{p(x)} dx - \lambda \int_{\Omega} \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx}{\int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx} \middle| u \in X \setminus \{0\} \right\}. \quad (2.4)$$

Define the following functionals on X .

$$\begin{aligned} \Phi_\lambda(u) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx \\ &= \Phi(u) - \lambda \varphi(u). \end{aligned}$$

We set

$$\mathcal{M} := \left\{ u \in X \middle| \Psi_{p(\cdot)}(u) = 1 \right\}.$$

Lemma 2.1. The following properties hold true:

- (i) Φ_λ , $\Psi_{p(\cdot)}$ and φ are even, and of class C^1 on X .
- (ii) $\mathcal{M}$ is a closed C^1 -manifold.

Proof.

(i) It is clear that Φ_λ , $\Psi_{p(\cdot)}$ and φ are even and of class C^1 on X .

(ii) $\mathcal{M} = \Psi_{p(\cdot)}^{-1}(\{1\})$, therefore $\mathcal{M}$ is closed. Moreover, $\Psi'_{p(\cdot)}(u)$ is onto for all $u \in \mathcal{M}$. Indeed, let $u \in \mathcal{M}$ and consider the mapping

$$f(t) = \Psi_{p(\cdot)}(tu) = \int_{\Omega} \frac{1}{p(x)} t^{p(x)} |u|^{p(x)} dx, t \geq 0.$$

f is differentiable for every $t > 0$ and $f'(t) = \langle \Psi'_{p(\cdot)}(tu), u \rangle > 0$ since

$$\langle \Psi'_{p(\cdot)}(tu), u \rangle = \int_{\Omega} t^{p(x)-1} |u|^{p(x)} dx \geq \min \{t^{p^- - 1}, t^{p^+ - 1}\} \int_{\Omega} |u|^{p(x)} dx > \min \{t^{p^- - 1}, t^{p^+ - 1}\} p^- > 0$$

Hence, f is strictly increasing, further $f(0) = 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$ since for t large one has

$$f(t) \geq \frac{1}{p^+} t^{p^-} \int_{\Omega} |u|^{p(x)} dx$$

Thus for every $m \in \mathbb{R}$, there exists a unique $t_0 > 0$ such that $f(t_0) = m$, so that $\Psi_{p(\cdot)}(t_0 u) = m$ and we also have

$$\left\langle \Psi'_{p(\cdot)}(u), \frac{t_0 p(x)}{p(x)} u \right\rangle = \int_{\Omega} |u|^{p(x)-2} u \frac{t_0^{p(x)}}{p(x)} u dx = \int_{\Omega} \frac{1}{p(x)} t_0 p(x) |u|^{p(x)} dx = \Psi_{p(\cdot)}(t_0 u) = m$$

This means that $\Psi'_{p(\cdot)}(u)$ is onto for all $u \in \mathcal{M}$. Therefore $\Psi_{p(\cdot)}$ is a submersion. Then $\mathcal{M}$ is a C^1 -manifold of X .

Next, we write Φ'_λ as

$$\Phi'_\lambda(\cdot) = \Phi'(\cdot) - \lambda \varphi'(\cdot),$$

where $\Phi', \varphi' : X \rightarrow X^*$ are defined by

$$\begin{aligned} \langle \Phi'(u), v \rangle &:= \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \\ \langle \varphi'(u), v \rangle &:= \int_{\Omega} \frac{|u|^{p(x)-2}}{\delta(x)^{2p(x)}} u v dx. \end{aligned}$$

Remark 2.2. Problem (E_λ) can be equivalently written as

$$\Phi'_\lambda(u) = \mu \Psi'_{p(\cdot)}(u), \quad u \in \mathcal{M}, \quad (2.5)$$

it is known that (μ, u) solves (2.5) if and only if u is a critical point of Φ_λ with respect to $\mathcal{M}$.

We have the following properties about the operator Φ' .

Lemma 2.2 (see [6, Lemma 3.4]). *The mapping Φ' is continuous, bounded homeomorphism, strictly monotone, and of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ in } X \text{ and } \overline{\lim}_{n \rightarrow \infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0, \text{ imply } u_n \rightarrow u \text{ in } X.$$

Lemma 2.3 (see [6, Lemma 3.5-(i)]). *The functional φ' is sequentially weakly-strongly continuous, namely,*

$$u_n \rightharpoonup u \text{ in } X \implies \varphi'(u_n) \rightarrow \varphi'(u) \text{ in } X^*.$$

Lemma 2.4. *The functional $\Psi'_{p(\cdot)}$ is sequentially weakly-strongly continuous.*

Proof.

Let $u_n \rightharpoonup u$ in X . For any $v \in X$, by Hölder-type inequality (2.1) and continuous embedding $X \hookrightarrow L^{p(\cdot)}(\Omega)$, it follows that

$$\begin{aligned} \left| \langle \Psi'_{p(\cdot)}(u_n) - \Psi'_{p(\cdot)}(u), v \rangle \right| &= \left| \int_{\Omega} (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) v dx \right| \\ &\leq 2 \left| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right|_{\frac{p(\cdot)}{p(\cdot)-1}} \|v\|_{p(\cdot)}, \end{aligned}$$

$$\leq k_1 \left| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right|_{\frac{p(\cdot)}{p(\cdot)-1}} \|v\|, \quad k_1 > 0,$$

On the other hand, using the compact embedding of X into $L^{p(\cdot)}(\Omega)$, we have $u_n \rightarrow u$ in $L^{p(\cdot)}(\Omega)$. Thus,

$$|u_n|^{p(x)-2} u_n \rightarrow |u|^{p(x)-2} u \quad \text{in } L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega).$$

Lemma 2.5. *For any $\lambda \in [0, C_H[$, the functional Φ_λ is bounded from below on $\mathcal{M}$.*

Proof.

Let $u \in \mathcal{M}$. By (1.2), we have

$$\frac{\lambda}{C_H} \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx \geq \lambda \int_{\Omega} \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx.$$

This implies that

$$\Phi_\lambda(u) \geq \left(1 - \frac{\lambda}{C_H}\right) \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx. \quad (2.6)$$

Thus

$$\Phi_\lambda(u) \geq \lambda_1 \left(1 - \frac{\lambda}{C_H}\right) > -\infty, \quad (2.7)$$

where $\mu_1(0) = \lambda_1$ is the first eigenvalue for the $p(\cdot)$ -biharmonic operator, see [1].

Remark 2.3. *We deduce from (2.7), that the eigenvalue μ_1 is positive.*

3. Palais-Smale condition

Proposition 3.1. *The functional Φ_λ satisfies the Palais-Smale condition on $\mathcal{M}$, i.e., for $\{u_n\}_{n \geq 1} \subset \mathcal{M}$, if*

$$\{\Phi_\lambda(u_n)\}_n \text{ is bounded,} \quad (3.1)$$

and

$$\alpha_n = \Phi'_\lambda(u_n) - \beta_n \Psi'_{p(\cdot)}(u_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (3.2)$$

where

$$\beta_n = \frac{\langle \Phi'_\lambda(u_n), u_n \rangle}{\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle},$$

then $\{u_n\}_{n \geq 1}$ has a convergent subsequence in X .

Proof.

We have

$$\begin{aligned} \Phi(u_n) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \geq \frac{1}{p^+} \int_{\Omega} |\Delta u_n|^{p(x)} dx \\ &= \frac{1}{p^+} \rho_{p(\cdot)}(\Delta u_n). \end{aligned}$$

We know from (3.1) that $\{\Phi_\lambda(u_n)\}_n$ being bounded, and by (2.6) we have that $\rho_{p(\cdot)}(\Delta u_n)$ is bounded in $\mathbb{R}$. Thus, without loss of generality, we can assume that

$$u_n \text{ converges weakly in } X \text{ to a some function } u \in X,$$

and

$$\rho_{p(\cdot)}(\Delta u_n) \rightarrow \ell.$$

For the rest we distinguish two cases:

Case 1. If $\ell = 0$, then $u_n \rightarrow 0$ in X .

Case 2. If $\ell \neq 0$, let us prove that

$$\overline{\lim}_{n \rightarrow \infty} \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle \leq 0.$$

Indeed, notice that

$$\langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle = \rho_{p(\cdot)}(\Delta u_n) - \langle \Delta_{p(\cdot)}^2 u_n, u \rangle.$$

Applying α_n of (3.2) to u , we deduce that

$$\theta_n = \langle \Delta_{p(\cdot)}^2 u_n, u \rangle - \lambda \langle \varphi'(u_n), u \rangle - \beta_n \langle \Psi'_{p(\cdot)}(u_n), u \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore

$$\begin{aligned} \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle &= \rho_{p(\cdot)}(\Delta u_n) - \lambda \langle \varphi'(u_n), u \rangle - \theta_n \\ &\quad - \frac{\langle \Phi'_\lambda(u_n), u_n \rangle}{\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle} \langle \Psi'_{p(\cdot)}(u_n), u \rangle. \end{aligned}$$

That is,

$$\begin{aligned} \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle &= \frac{\rho_{p(\cdot)}(\Delta u_n)}{\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle} \left(\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u_n), u \rangle \right) \\ &\quad - \theta_n - \lambda \langle \varphi'(u_n), u \rangle + \lambda \frac{\langle \varphi'(u_n), u_n \rangle}{\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle} \cdot \langle \Psi'_{p(\cdot)}(u_n), u \rangle. \end{aligned}$$

On one hand, according to Lemma 2.4, we deduce that

$$\Psi'_{p(\cdot)}(u_n) \rightarrow \Psi'_{p(\cdot)}(u), \quad \text{and} \quad \langle \Psi'_{p(\cdot)}(u_n), u_n \rangle \rightarrow \langle \Psi'_{p(\cdot)}(u), u \rangle.$$

By Lemma 2.3, we know that

$$\varphi'(u_n) \rightarrow \varphi'(u), \quad \langle \varphi'(u_n), u_n \rangle \rightarrow \langle \varphi'(u), u \rangle \quad \text{and} \quad \langle \varphi'(u_n), u \rangle \rightarrow \langle \varphi'(u), u \rangle.$$

Then

$$\begin{aligned} \left| \langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u_n), u \rangle \right| &\leq \left| \langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u), u \rangle \right| \\ &\quad + \left| \langle \Psi'_{p(\cdot)}(u), u \rangle - \langle \Psi'_{p(\cdot)}(u), u \rangle \right|. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u_n), u \rangle \right| &\leq \left| \langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u), u \rangle \right| \\ &\quad + \left| \|\Psi'_{p(\cdot)}(u_n) - \Psi'_{p(\cdot)}(u)\|_* \|u\| \right|. \end{aligned}$$

This implies that

$$\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u_n), u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

Combining with the above equalities, we obtain

$$\overline{\lim}_{n \rightarrow +\infty} \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle \leq \frac{\ell}{\langle \Psi'_{p(\cdot)}(u), u \rangle} \overline{\lim}_{n \rightarrow \infty} \left(\langle \Psi'_{p(\cdot)}(u_n), u_n \rangle - \langle \Psi'_{p(\cdot)}(u_n), u \rangle \right).$$

We deduce

$$\overline{\lim}_{n \rightarrow \infty} \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle \leq 0. \quad (3.4)$$

On the other hand,

$$\langle \Phi'(u_n), u_n - u \rangle = \langle \Delta_{p(\cdot)}^2 u_n, u_n - u \rangle.$$

Then according to (3.4), we conclude that

$$\overline{\lim}_{n \rightarrow \infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0. \quad (3.5)$$

In view of Lemma 2.2, $u_n \rightarrow u$ strongly in X .

4. Main results

In this section we prove the existence of at least one non-decreasing sequence of eigencurves to problem (E_λ) . Our approach is based on Ljusternik-Schnirelmann theory on C^1 -manifolds [13].

Let us now state our first main result of this paper.

Theorem 4.1. *For any $\lambda \in [0, C_H[$, the problem (E_λ) has at least one non-decreasing sequence of eigencurves $(\mu_j)_{j \geq 1}$ given by*

$$\mu_j(\lambda) := \inf_{K \in \Gamma_j} \max_{u \in K} \Phi_\lambda(u) \rightarrow \infty, \quad \text{as } j \rightarrow \infty,$$

where

$$\Gamma_j := \{K \subset \mathcal{M} \mid K \text{ is compact, } K = -K \text{ and } \gamma(K) \geq j\}, \quad (4.1)$$

here $\gamma(K)$ indicates the genus of K .

We start with two auxiliary results.

Lemma 4.1. *For any $j \in \mathbb{N}^*$, $\Gamma_j \neq \emptyset$.*

Proof.

Since X is separable. Therefore, for any $j \in \mathbb{N}^*$, there exists $(\varphi_i)_{i \geq 1}$ linearly dense in X such that

$$\begin{cases} \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset & \text{if } i \neq j, \\ \text{meas}(\text{supp}(\varphi_i)) > 0 & \text{for } i \in \{1, 2, \dots, j\}. \end{cases}$$

Let $X_j = \text{Span}\{\varphi_1, \varphi_2, \dots, \varphi_j\}$ be the vector subspace of X generated by j vectors $\{\varphi_1, \varphi_2, \dots, \varphi_j\}$, then $\dim X_j = j$.

Note that $X_j \subset L^{p(\cdot)}(\Omega)$ because $X_j \subset X \subset L^{p(\cdot)}(\Omega)$. Thus the norm $\|\cdot\|$ and $|\cdot|_{p(\cdot)}$ are equivalent on X_j because X_j is a finite dimensional space. Consequently the map

$$u \mapsto |u|_{p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

defines a norm on X_j . Denote $S := \{u \in X_j : |u|_{p(\cdot)} = 1\}$ the unit sphere of X_j .

Let us introduce the functional

$$\begin{aligned} g : \mathbb{R}^+ \times X_j &\longrightarrow \mathbb{R} \\ (\xi, u) &\mapsto \Psi_{p(\cdot)}(\xi u). \end{aligned}$$

On one hand, it is clear that

- $g(0, u) = 0$.
- $g(\xi, u)$ is non-decreasing with respect to ξ .

Moreover, for $\xi > 1$ we have

$$g(\xi, u) \geq \xi^{p^-} \Psi_{p(\cdot)}(u),$$

so that $\lim_{\xi \rightarrow +\infty} g(\xi, u) = +\infty$. Therefore, for every $u \in S$ fixed, there is a unique value $\xi = \xi(u) > 0$ such that $g(\xi(u), u) = 1$.

On the other hand, since

$$\frac{\partial g}{\partial \xi}(\xi(u), u) = \int_{\Omega} (\xi(u))^{p(x)-1} |u|^{p(x)} dx \geq \frac{p^-}{\xi(u)} g(\xi(u), u) = \frac{p^-}{\xi(u)} > 0.$$

The implicit function theorem implies that the map $u \mapsto \xi(u)$ is continuous and even by uniqueness. Now, take the compact $H_j := \mathcal{M} \cap X_j$. Since the map $h : S \rightarrow H_j$ defined by $h(u) = \xi(u).u$ is continuous and odd, it follows by the property of genus that $\gamma(H_j) = j$. Therefore $H_j \in \Gamma_j$.

Lemma 4.2.

$$\mu_j(\lambda) \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Proof.

Let $(e_k, e_n^*)_{k,n}$ be a bi-orthogonal system satisfying

- $(e_k)_k$ are linearly dense in X ,
- $(e_n^*)_n$ total for X^* ,
- $\langle e_n^*, e_k \rangle = 0$ if $n \neq k$,
- $\langle e_n^*, e_n \rangle = 1, \forall n \in \mathbb{N}^*$.

Set for every $k \in \mathbb{N}^*$,

$$F_k := \text{span}\{e_1, \dots, e_k\} \quad \text{and} \quad F_k^\perp := \text{span}\{e_{k+1}, e_{k+2}, \dots\}.$$

By the property of genus, we have for any $K \in \Gamma_k$,

$$K \cap F_{k-1}^\perp \neq \emptyset.$$

Let us prove that

$$t_k := \inf_{K \in \Gamma_k} \sup_{u \in K \cap F_{k-1}^\perp} \Phi_\lambda(u) \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Arguing by contradiction. Assume that for k is large enough, there exists $u_k \in F_{k-1}^\perp$ with $\int_\Omega \frac{|u_k|^{p(x)}}{p(x)} dx = 1$ such that

$$t_k \leq \Phi_\lambda(u_k) \leq M,$$

for some constant $M > 0$ independent on k . Thus in view of (2.6), we get

$$\|u_k\| \leq \left(\frac{p^+ C_H M}{C_H - \lambda} \right)^{\frac{1}{p^-}}.$$

This implies that $(u_k)_k$ is bounded in X . We may assume that $u_k \rightharpoonup u$ in X and $u_k \rightarrow u$ in $L^{p(\cdot)}(\Omega)$. By our choice of $F_{k-1}^\perp$, we observe that $u_k \rightarrow 0$ in X because $\langle e_n^*, e_k \rangle = 0$, for any $k > n$. This contradicts the fact that $1 = \int_\Omega \frac{|u_k|^{p(x)}}{p(x)} dx \rightarrow 0$ for every k . Hence $\lambda_k \geq t_k$, for every $k \geq 1$.

Proof of Theorem 4.1. Applying lemmas 4.1, 4.2 and Ljusternik-Schnirelemaan theory to the problem (E_λ) , we have for each $j \in \mathbb{N}^*$, $\mu_j(\lambda)$ is a critical value of (E_λ) on C^1 -manifold $\mathcal{M}$, such that

$$\mu_j(\lambda) \rightarrow \infty, \text{ as } j \rightarrow \infty.$$

Corollary 4.1. *The following statements hold true:*

$$(i) \mu_1(\lambda) = \inf \left\{ \frac{\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx} \mid u \in X \setminus \{0\} \right\}.$$

$$(ii) 0 < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \leq \mu_n(\lambda) \rightarrow +\infty.$$

Proof.

(i) For $u \in \mathcal{M}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, Φ_λ is even and

$$\Phi_\lambda(u) = \max_{K_1} \Phi_\lambda \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi_\lambda(u).$$

Thus

$$\inf_{u \in \mathcal{M}} \Phi_\lambda(u) \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi_\lambda(u) = \mu_1(\lambda).$$

On the other hand, for all $K \in \Gamma_1$ and $u \in K$, we have

$$\max_{u \in K} \Phi_\lambda \geq \Phi_\lambda(u) \geq \inf_{u \in \mathcal{M}} \Phi_\lambda(u).$$

It follows that

$$\inf_{K \in \Gamma_1} \max_K \Phi_\lambda = \mu_1(\lambda) \geq \inf_{u \in \mathcal{M}} \Phi_\lambda(u).$$

Then

$$\mu_1(\lambda) = \inf \left\{ \frac{\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx} \mid u \in X \setminus \{0\} \right\}.$$

(ii) For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of the definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\mu_i(\lambda) \geq \mu_j(\lambda)$. As regards $\mu_n(\lambda) \rightarrow \infty$, it has been proved in Theorem 4.1.

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