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Multivariate Caputo left fractional Landau inequalities

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ABSTRACT. Relied on author's first ever found multivariate Caputo fractional Taylor's formula (2009, [1], Chapter 13), we develop and prove several multivariate left side Caputo fractional uniform Landau type inequalities.

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1. Introduction

We are motivated by the following two results. First from [3], we have the following nice result:

If $\Delta f = \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} \right)^2 f$ is the Laplacian of a function defined on $\mathbb{R}^d$, then for $0 < k < 2n$,

$$\left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \right\| \leq C(n, k) \|f\|^{1 - \left(\frac{k}{2n}\right)} \|\Delta^n f\|^{\frac{k}{2n}}, \quad (1)$$

where $\|\cdot\|$ denotes the $L_\infty(\mathbb{R}^d)$ norm. $\Delta^n f$ needs to be defined only in the distributional sense and the result is valid in many other Banach spaces such as L^p spaces. $C(n, k)$ is an existential constant and $\Delta^k = \Delta(\Delta^{k-1})$.

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From [4], we have the following interesting result which is the sharp inequality

$$\|\Delta f\|_\infty \leq 2\sqrt{\frac{d}{(d+2)}}\sqrt{\|f\|_\infty \|\Delta^2 f\|_\infty}, \quad (2)$$

in $\mathbb{R}^d$, $d \geq 2$, where Δf is the Laplacian and $\Delta^2 = \Delta(\Delta)$.

This result is an analogue of the famous Landau inequality ([5]) in $\mathbb{R}$:

$$\|f'\|_\infty \leq \sqrt{2}\sqrt{\|f\|_\infty \|f''\|_\infty}, \quad (3)$$

where $\|\cdot\|_\infty$ means the L_∞ norm on $\mathbb{R}^d$ (respectively on $\mathbb{R}$).

In this article we present left Caputo fractional multivariate Landau type inequalities where all the constants are exactly calculated and we have best upper bounds. We need to mention the following background coming from [1], Chapter 13, pp. 269-278.

We need

Definition 1.1. Let $\nu \geq 0$; the operator J_a^ν , defined on $L_1(a, b)$ by

$$J_a^\nu f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \quad (4)$$

for $a \leq x \leq b$, is called the Riemann-Liouville left fractional integral operator of order ν . For $\nu = 0$, we set $J_a^0 := I$, the identity operator. Here Γ stands for the gamma function.

By [2], p. 13, $J_a^\nu f(x)$, $\nu > 0$, exists for almost all $x \in [a, b]$ and $J_a^\nu f \in L_1(a, b)$, where $f \in L_1(a, b)$. Here $AC^n([a, b])$ is the space of functions with absolutely continuous $(n-1)$ -st derivative.

We need to mention

Definition 1.2. ([2], p. 49, [1], p. 394). Let $\nu \geq 0$; $n := \lceil \nu \rceil$, $\lceil \cdot \rceil$ is the ceiling of the number, $f \in AC^n([a, b])$. We call the left Caputo fractional derivative

$$D_{*a}^\nu f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) dt, \quad (5)$$

$\forall x \in [a, b]$.

The above function $D_{*a}^\nu f(x)$ exists almost everywhere for $x \in [a, b]$.

If $\nu \in \mathbb{N}$, then $D_{*a}^\nu f = f^{(\nu)}$ the ordinary derivative; it is also $D_{*a}^0 f = f$.

We also mention

Theorem 1.1. (Taylor expansion for left Caputo fractional derivatives, [2, p. 54]). Assume $\nu \geq 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a, b])$. Then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^\nu f(t) dt, \quad (6)$$

$\forall x \in [a, b]$.

Remark 1.1. Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \dots, z_k)$, $x_0 := (x_{01}, \dots, x_{0k}) \in Q$. Let $f \in C^n(Q)$, $n \in \mathbb{N}$.

Set

$$\begin{aligned} g_z(t) &:= f(x_0 + t(z - x_0)), \quad 0 \leq t \leq 1; \\ g_z(0) &= f(x_0), \quad g_z(1) = f(z). \end{aligned} \quad (7)$$

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_0 + t(z - x_0)), \quad (8)$$

$j = 0, 1, 2, \dots, n$,
and

$$g_z^{(n)}(0) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0). \quad (9)$$

We mention the following basic multivariate Caputo left fractional fundamental theorem.

Theorem 1.2. ([1], p. 272). Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \dots, z_k)$, $x_0 := (x_{01}, \dots, x_{0k}) \in Q$, $f \in C^1(Q)$, $0 < \nu \leq 1$. Then

$$\begin{aligned} f(z) &= f(x_0) + \\ &\sum_{i=1}^k (z_i - x_{0i}) \left[\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} J_0^{1-\nu} \left(\frac{\partial}{\partial x_i} (x_0 + t(z - x_0)) \right) dt \right]. \end{aligned} \quad (10)$$

We also mention the following Caputo left fractional bivariate Taylor formula.

Theorem 1.3. ([1], p. 274). Let $f \in C^2(Q)$, $Q \subseteq \mathbb{R}^2$ compact and convex, $z := (z_1, z_2)$, $x_0 := (x_{01}, x_{02}) \in Q$, and $1 < \nu \leq 2$. Then
(1)

$$\begin{aligned} f(z_1, z_2) &= f(x_{01}, x_{02}) + (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_{01}, x_{02}) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_{01}, x_{02}) + \\ &(z_1 - x_{01})^2 \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_1^2} (x_0 + t(z - x_0)) \right) \right) dt \right) + \\ &2(z_1 - x_{01})(z_2 - x_{02}) \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} (x_0 + t(z - x_0)) \right) \right) dt \right) \\ &+ (z_2 - x_{02})^2 \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_2^2} (x_0 + t(z - x_0)) \right) \right) dt \right). \end{aligned} \quad (11)$$

Additionally assume that

$$f(x_0) = \frac{\partial f}{\partial x_1}(x_0) = \frac{\partial f}{\partial x_2}(x_0) = 0,$$

then

(2)

$$f(z_1, z_2) =$$

$$\begin{aligned}
& (z_1 - x_{01})^2 \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_1^2} (x_0 + t(z - x_0)) \right) \right) dt \right) + \\
& 2(z_1 - x_{01})(z_2 - x_{02}) \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} (x_0 + t(z - x_0)) \right) \right) dt \right) \\
& + (z_2 - x_{02})^2 \left(\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left(J_0^{2-\nu} \left(\frac{\partial^2 f}{\partial x_2^2} (x_0 + t(z - x_0)) \right) \right) dt \right). \quad (12)
\end{aligned}$$

We finally mention the following general multivariate Caputo left fractional Taylor formula.

Theorem 1.4. ([1], p. 276). Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in C^n(Q)$, where Q is a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \dots, z_k)$, $x_0 := (x_{01}, \dots, x_{0k}) \in Q$. Then

1)

$$\begin{aligned}
f(z) &= f(x_0) + \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f(x_0)}{\partial x_i} + \\
& \sum_{l=2}^{n-1} \frac{\left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^l f \right](x_0)}{l!} + \\
& \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left[J_0^{n-\nu} \left\{ \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\} \right] dt. \quad (13)
\end{aligned}$$

Additionally assume that $f_\alpha(x_0) = 0$, $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$, $|\alpha| := \sum_{i=1}^k \alpha_i =: r$, $r = 0, \dots, n-1$; then

2)

$$\begin{aligned}
f(z) &= \\
& \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left[J_0^{n-\nu} \left\{ \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\} \right] dt. \quad (14)
\end{aligned}$$

2. Main Results

Here we present several left Caputo fractional multivariate Landau inequalities.

Theorem 2.1. Let $f \in C^2(\mathbb{R}_+^k)$, $k \geq 2$, with $\|f\|_{\infty, \mathbb{R}_+^k} < \infty$, and $1 < \nu < 2$. Assume further that

$$K_\nu := \max \left\{ \sup_{\substack{x_0, z \in \mathbb{R}_+^k, t \in [0,1], \\ i=1, \dots, k}} |J_0^{2-\nu} (f_{x_i x_i} (x_0 + t(z - x_0)))|, \right.$$

$$\left. \sup_{\substack{x_0, z \in \mathbb{R}_+^k, t \in [0,1], \\ \left\{ \begin{array}{l} l = l(i, j) = 1, \dots, \frac{k(k-1)}{2}; \\ i < j \end{array} \right\}}} \left| J_0^{2-\nu} \left(f_{x_i x_j} (x_0 + t(z - x_0)) \right) \right| \right\} < +\infty. \quad (15)$$

Then we derive the best upper bound

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} \leq 2k \sqrt{\frac{2}{\Gamma(\nu+1)}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{1}{2}} K_{\nu}^{\frac{1}{2}}. \quad (16)$$

That is

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} < \infty.$$

Proof. For any $x_0, z \in \mathbb{R}_+^k$ by Theorem 1.4 (see also Theorem 1.3) we have that

$$\begin{aligned} f(z) &= f(x_0) + \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f(x_0)}{\partial x_i} + \\ &\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left[J_0^{2-\nu} \left\{ \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^2 f \right] (x_0 + t(z - x_0)) \right\} \right] dt \\ &= f(x_0) + \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f(x_0)}{\partial x_i} + \\ &\sum_{i=1}^k (z_i - x_{0i})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_i} (x_0 + t(z - x_0))) dt + \\ &2 \sum_{\substack{l = l(i, j) = 1 \\ i < j}}^{\frac{k(k-1)}{2}} (z_i - x_{0i}) (z_j - x_{0j}) \frac{1}{\Gamma(\nu)} \\ &\int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_j} (x_0 + t(z - x_0))) dt. \end{aligned} \quad (17)$$

We choose $z_i - x_{0i} = h > 0$, for all $i = 1, \dots, k$.

Then

$$f(z) = f(x_0) + h \sum_{i=1}^k \frac{\partial f}{\partial x_i} (x_0) +$$

$$\begin{aligned} & \frac{h^2}{\Gamma(\nu)} \left[\sum_{i=1}^k \int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_i} (x_0 + t(z-x_0))) dt + \right. \\ & \left. 2 \sum_{\substack{l=1 \\ : i < j}}^{\frac{k(k-1)}{2}} \int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_j} (x_0 + t(z-x_0))) dt \right]. \end{aligned} \quad (18)$$

Hence

$$\begin{aligned} & h \sum_{i=1}^k \frac{\partial f(x_0)}{\partial x_i} = f(z) - f(x_0) - \\ & \frac{h^2}{\Gamma(\nu)} \left[\sum_{i=1}^k \int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_i} (x_0 + t(z-x_0))) dt + \right. \\ & \left. 2 \sum_{\substack{l=1 \\ : i < j}}^{\frac{k(k-1)}{2}} \int_0^1 (1-t)^{\nu-1} J_0^{2-\nu} (f_{x_i x_j} (x_0 + t(z-x_0))) dt \right]. \end{aligned} \quad (19)$$

Therefore

$$\begin{aligned} & h \left| \sum_{i=1}^k \frac{\partial f}{\partial x_i} (x_0) \right| \leq 2 \|f\|_{\infty, \mathbb{R}_+^k} + \\ & \frac{h^2}{\Gamma(\nu+1)} \left\{ \sum_{i=1}^k \left\| J_0^{2-\nu} (f_{x_i x_i} (x_0 + t(z-x_0))) \right\|_{\infty, [0,1]} + \right. \\ & \left. 2 \sum_{\substack{l=1 \\ : i < j}}^{\frac{k(k-1)}{2}} \left\| J_0^{2-\nu} (f_{x_i x_j} (x_0 + t(z-x_0))) \right\|_{\infty, [0,1]} \right\} \leq \\ & 2 \|f\|_{\infty, \mathbb{R}_+^k} + \frac{K_\nu h^2}{\Gamma(\nu+1)} \left\{ \sum_{i=1}^k 1 + 2 \sum_{l=1}^{\frac{k(k-1)}{2}} 1 \right\} = \\ & 2 \|f\|_{\infty, \mathbb{R}_+^k} + \frac{k^2 K_\nu h^2}{\Gamma(\nu+1)}. \end{aligned} \quad (20)$$

Thus

$$\left| \sum_{i=1}^k \frac{\partial f}{\partial x_i} (x_0) \right| \leq \frac{2 \|f\|_{\infty, \mathbb{R}_+^k}}{h} + \frac{k^2 K_\nu}{\Gamma(\nu+1)} h, \quad (21)$$

$\forall h > 0$, for any $x_0 \in \mathbb{R}_+^k$.

Consequently we find that

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} \leq \frac{2 \|f\|_{\infty, \mathbb{R}_+^k}}{h} + \frac{k^2 K_\nu}{\Gamma(\nu+1)} h, \quad (22)$$

$\forall h > 0$.

Set $A := 2 \|f\|_{\infty, \mathbb{R}_+^k}$ and $B := \frac{k^2 K_\nu}{\Gamma(\nu+1)}$, both are positive.

We consider the function

$$y(h) = \frac{A}{h} + Bh, \quad \forall h > 0. \quad (23)$$

We have that

$$y'(h) = -Ah^{-2} + B = 0,$$

with only one critical number

$$h_0 = \sqrt{\frac{A}{B}}. \quad (24)$$

Notice that $y''(h) = 2Ah^{-3}$, and $y''(h_0) = 2A^{-\frac{1}{2}}B^{\frac{3}{2}} > 0$.

Therefore y has a global minimum which is

$$y(h_0) = 2\sqrt{AB}. \quad (25)$$

Hence

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} \leq \frac{2\sqrt{2}k}{\sqrt{\Gamma(\nu+1)}} \sqrt{\|f\|_{\infty, \mathbb{R}_+^k} K_\nu}. \quad (26)$$

The theorem is proved.

We continue with

Theorem 2.2. Let $f \in C^3(\mathbb{R}_+^k)$, $k \geq 2$, with $\|f\|_{\infty, \mathbb{R}_+^k} < \infty$, and $2 < \nu < 3$. Assume further that

$$M_\nu :=$$

$$\sup_{x_0, z \in \mathbb{R}_+^k, t \in [0,1]} \left| \sum_{\left\{ \begin{array}{l} \lambda_1 + \dots + \lambda_k = 3 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k \end{array} \right\}} \frac{1}{\lambda_1! \dots \lambda_k!} J_0^{3-\nu} \left(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0)) \right) \right| < +\infty. \quad (27)$$

Then we derive the best upper bounds:

1)

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} \leq 3 \left(\frac{3}{2} \right)^{\frac{2}{3}} \left(\frac{12}{\Gamma(\nu+1)} \right)^{\frac{1}{3}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{2}{3}} M_\nu^{\frac{1}{3}} < +\infty, \quad (28)$$

and

2)

$$\left\| \frac{1}{2} \left(\sum_{i=1}^k \frac{\partial^2 f}{\partial x_i^2} \right) + \sum_{\substack{l=1 \\ i < j}}^{\frac{k(k-1)}{2}} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\| \leq 3 \left(\frac{9}{\Gamma(\nu+1)} \right)^{\frac{2}{3}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{1}{3}} M_{\nu}^{\frac{2}{3}} < +\infty. \quad (29)$$

Proof. We set

$$\Delta_1(x_0) := \sum_{i=1}^k \frac{\partial f}{\partial x_i}(x_0),$$

and

$$\Delta_2(x_0) := \frac{1}{2} \left(\sum_{i=1}^k \frac{\partial^2 f}{\partial x_i^2}(x_0) \right) + \sum_{\substack{l=1 \\ i < j}}^{\frac{k(k-1)}{2}} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \quad (30)$$

By Theorem 1.4 we have, we choose also $z_i - x_{0i} = h > 0$, all $i = 1, \dots, k$:

$$\begin{aligned} f(z) &= f(x_0) + h\Delta_1(x_0) + h^2\Delta_2(x_0) + \\ &\quad \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left\{ J_0^{3-\nu} \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^3 f \right] (x_0 + t(z - x_0)) \right\} dt \\ &= f(x_0) + \Delta_1(x_0)h + \Delta_2(x_0)h^2 + h^3 \frac{6}{\Gamma(\nu)} \\ &\quad \left[\sum_{\substack{(\lambda_1 + \dots + \lambda_k = 3 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k)} \frac{1}{\lambda_1! \dots \lambda_k!} \int_0^1 (1-t)^{\nu-1} J_0^{3-\nu} \left(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0)) \right) dt \right]. \quad (31) \end{aligned}$$

We set

$$R_3 := \frac{6}{\Gamma(\nu)}$$

$$\left[\sum_{\substack{(\lambda_1 + \dots + \lambda_k = 3 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k)} \frac{1}{\lambda_1! \dots \lambda_k!} \int_0^1 (1-t)^{\nu-1} J_0^{3-\nu} \left(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0)) \right) dt \right]. \quad (32)$$

That is ($h > 0$)

$$f(z) = f(x_0) + \Delta_1(x_0)h + \Delta_2(x_0)h^2 + R_3h^3. \quad (33)$$

So that

$$\Delta_1(x_0)h + \Delta_2(x_0)h^2 = f(z) - f(x_0) - R_3h^3 =: A_1, \quad (34)$$

and

$$2\Delta_1(x_0)h + 4\Delta_2(x_0)h^2 = f(z) - f(x_0) - 8R_3h^3 =: A_2. \quad (35)$$

Solving the system (34), (35) we find

$$\begin{aligned} \Delta_1(x_0) &= \frac{4A_1 - A_2}{2h}, \\ \text{and} \\ \Delta_2(x_0) &= \frac{A_2 - 2A_1}{2h^2}. \end{aligned} \quad (36)$$

We notice that

$$\begin{aligned} |4A_1 - A_2| &= \left| 3(f(z) - f(x_0)) + 4R_3h^3 \right| \leq 6\|f\|_{\infty, \mathbb{R}_+^k} + \\ &4h^3 \frac{6}{\Gamma(\nu+1)} \left\| \sum_{\substack{(\lambda_1 + \dots + \lambda_k = 3) \\ \lambda_i \in \mathbb{Z}^+, i=1, \dots, k}} \frac{1}{\lambda_1! \dots \lambda_k!} J_0^{3-\nu}(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0))) \right\|_{\infty, [0,1]} \\ &\leq 6\|f\|_{\infty, \mathbb{R}_+^k} + \frac{24}{\Gamma(\nu+1)} M_\nu h^3. \end{aligned} \quad (37)$$

Hence

$$|\Delta_1(x_0)| \leq \frac{3\|f\|_{\infty, \mathbb{R}_+^k}}{h} + \frac{12M_\nu}{\Gamma(\nu+1)} h^2, \quad (38)$$

$\forall h > 0$, for all $x_0 \in \mathbb{R}_+^k$.

Consequently it holds

$$\|\Delta_1\|_{\infty, \mathbb{R}_+^k} \leq \frac{3\|f\|_{\infty, \mathbb{R}_+^k}}{h} + \frac{12M_\nu}{\Gamma(\nu+1)} h^2, \quad (39)$$

$\forall h > 0$.

Next we have

$$A_2 - 2A_1 = - \left((f(z) - f(x_0)) + 6R_3h^3 \right). \quad (40)$$

So that

$$|A_2 - 2A_1| = \left| f(z) - f(x_0) + 6R_3h^3 \right| \leq 2\|f\|_{\infty, \mathbb{R}_+^k} + 6h^3 \frac{6}{\Gamma(\nu+1)} M_\nu. \quad (41)$$

Hence

$$|\Delta_2(x_0)| \leq \frac{\|f\|_{\infty, \mathbb{R}_+^k}}{h^2} + \frac{18M_\nu}{\Gamma(\nu+1)} h, \quad (42)$$

$\forall h > 0, \forall x_0 \in \mathbb{R}_+^k$.

So that

$$\|\Delta_2\|_{\infty, \mathbb{R}_+^k} \leq \frac{\|f\|_{\infty, \mathbb{R}_+^k}}{h^2} + \left(\frac{18M_\nu}{\Gamma(\nu+1)} \right) h, \quad (43)$$

$\forall h > 0$.

Next we work on (39).

Call $\mu := 3\|f\|_{\infty, \mathbb{R}_+^k}$, $\theta := \frac{12M_\nu}{\Gamma(\nu+1)}$ that are both positive.

We study the function

$$y(h) = \frac{\mu}{h} + \theta h^2, \quad \forall h > 0. \quad (44)$$

We have

$$y'(h) = -\mu h^{-2} + 2\theta h = 0.$$

Hence

$$h_0 = h_{crit.no.} = \left(\frac{\mu}{2\theta}\right)^{\frac{1}{3}} > 0. \quad (45)$$

We have $y''(h) = 2\mu h^{-3} + 2\theta$, and

$$y''(h_0) = 2\mu h_0^{-3} + 2\theta > 0. \quad (46)$$

Therefore y has a global minimum which is

$$y(h_0) = \mu h_0^{-1} + \theta h_0^2 = h_0^2 (\mu h_0^{-3} + \theta) = \frac{3}{2^{\frac{2}{3}}} \theta^{\frac{1}{3}} \mu^{\frac{2}{3}}. \quad (47)$$

Hence

$$y(h_0) = \frac{3}{2^{\frac{2}{3}}} \theta^{\frac{1}{3}} \mu^{\frac{2}{3}} = 3 \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{12}{\Gamma(\nu+1)}\right)^{\frac{1}{3}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{2}{3}} M_{\nu}^{\frac{1}{3}}. \quad (48)$$

We have proved that

$$\|\Delta_1\|_{\infty, \mathbb{R}_+^k} \leq 3 \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{12}{\Gamma(\nu+1)}\right)^{\frac{1}{3}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{2}{3}} M_{\nu}^{\frac{1}{3}}. \quad (49)$$

Next we work on (43).

We call $\varphi := \|f\|_{\infty, \mathbb{R}_+^k}$, $\psi := \frac{18M_{\nu}}{\Gamma(\nu+1)}$, that are both positive.

We study

$$y(h) := \frac{\varphi}{h^2} + \psi h, \quad \forall h > 0. \quad (50)$$

We have

$$y'(h) = -2\varphi h^{-3} + \psi = 0,$$

so that

$$h_0 := h_{crit.no.} = \left(\frac{2\varphi}{\psi}\right)^{\frac{1}{3}} > 0. \quad (51)$$

We have

$$y''(h) = 6\varphi h^{-4},$$

and

$$y''(h_0) = 6\varphi h_0^{-4} > 0. \quad (52)$$

Therefore y has a global minimum which is

$$y(h_0) = \varphi h_0^{-2} + \psi h_0 = h_0 (\varphi h_0^{-3} + \psi) = h_0 \left(\varphi \frac{\psi}{2\varphi} + \psi\right) = \frac{3}{2} 2^{\frac{1}{3}} \psi^{\frac{2}{3}} \varphi^{\frac{1}{3}}. \quad (53)$$

That is

$$y(h_0) = \frac{3}{2} \sqrt[3]{2} \psi^{\frac{2}{3}} \varphi^{\frac{1}{3}}. \quad (54)$$

Consequently it holds

$$\|\Delta_2\|_{\infty, \mathbb{R}_+^k} \leq 3 \left(\frac{9}{\Gamma(\nu+1)} \right)^{\frac{2}{3}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{1}{3}} M_{\nu}^{\frac{2}{3}}. \quad (55)$$

The theorem is proved.

We also present

Theorem 2.3. *Let $f \in C^4(\mathbb{R}_+^k)$, $k \geq 2$, with $\|f\|_{\infty, \mathbb{R}_+^k} < \infty$, and $3 < \nu < 4$. Assume further that*

$$H_{\nu} :=$$

$$\sup_{x_0, z \in \mathbb{R}_+^k, t \in [0,1]} \left| \sum_{\left\{ \begin{array}{l} \lambda_1 + \dots + \lambda_k = 4 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k \end{array} \right\}} \frac{1}{\lambda_1! \dots \lambda_k!} J_0^{4-\nu} \left(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0)) \right) \right| < +\infty. \quad (56)$$

Then we derive the best upper bounds:

1)

$$\left\| \sum_{i=1}^k \frac{\partial f}{\partial x_i} \right\|_{\infty, \mathbb{R}_+^k} \leq 4 \left(\frac{11}{9} \right)^{\frac{3}{4}} \left(\frac{144}{\Gamma(\nu+1)} \right)^{\frac{1}{4}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{3}{4}} H_{\nu}^{\frac{1}{4}} < +\infty, \quad (57)$$

2)

$$\left\| \frac{1}{2} \left(\sum_{i=1}^k \frac{\partial^2 f}{\partial x_i^2} \right) + \sum_{\left\{ \begin{array}{l} l = l(i, j) = 1, \\ i < j \end{array} \right\}}^{\frac{k(k-1)}{2}} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{\infty, \mathbb{R}_+^k} \leq$$

$$2 \sqrt{\frac{528}{\Gamma(\nu+1)}} \sqrt{\|f\|_{\infty, \mathbb{R}_+^k}} H_{\nu} < +\infty, \quad (58)$$

and

3)

$$\left\| \sum_{\left\{ \begin{array}{l} \lambda_1 + \dots + \lambda_k = 3 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k \end{array} \right\}} \frac{1}{\lambda_1! \dots \lambda_k!} f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}} \right\|_{\infty, \mathbb{R}_+^k} \leq$$

$$\frac{4}{3} \left(\frac{144}{\Gamma(\nu+1)} \right)^{\frac{3}{4}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{1}{4}} H_{\nu}^{\frac{3}{4}} < +\infty. \quad (59)$$

Proof. We set

$$\begin{aligned}\Delta_1(x_0) &:= \sum_{i=1}^k \frac{\partial f}{\partial x_i}(x_0), \\ \Delta_2(x_0) &:= \frac{1}{2} \left(\sum_{i=1}^k \frac{\partial^2 f}{\partial x_i^2}(x_0) \right) + \sum_{\substack{l=1 \\ i < j}}^{\frac{k(k-1)}{2}} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j},\end{aligned}\quad (60)$$

and

$$\Delta_3(x_0) := \sum_{\left\{ \begin{array}{l} \lambda_1 + \dots + \lambda_k = 3 \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k \end{array} \right\}} \frac{1}{\lambda_1! \dots \lambda_k!} f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0).$$

By Theorem 1.4 we have, we choose also $z_i - x_{0i} = h > 0$, all $i = 1, \dots, k$:

$$f(z) = f(x_0) + \Delta_1(x_0)h + \Delta_2(x_0)h^2 + \Delta_3(x_0)h^3 + R_4h^4, \quad (61)$$

where

$$\begin{aligned}R_4 &:= \frac{1}{\Gamma(\nu)} \left[\sum_{\left\{ \begin{array}{l} \lambda_1 + \dots + \lambda_k = 4, \\ \lambda_i \in \mathbb{Z}^+, i = 1, \dots, k \end{array} \right\}} \left(\frac{4!}{\lambda_1! \dots \lambda_k!} \right) \right. \\ &\quad \left. \int_0^1 (1-t)^{\nu-1} J_0^{4-\nu} \left(f_{x_1^{\lambda_1} \dots x_k^{\lambda_k}}(x_0 + t(z - x_0)) \right) dt \right].\end{aligned}\quad (62)$$

Therefore, $\forall h > 0$, we get

$$\begin{aligned}\Delta_1(x_0)h + \Delta_2(x_0)h^2 + \Delta_3(x_0)h^3 &= f(z) - f(x_0) - R_4h^4 =: A_1, \\ \Delta_1(x_0)2h + \Delta_2(x_0)4h^2 + \Delta_3(x_0)8h^3 &= f(z) - f(x_0) - R_416h^4 =: A_2, \\ \text{and} \\ \Delta_1(x_0)3h + \Delta_2(x_0)9h^2 + \Delta_3(x_0)27h^3 &= f(z) - f(x_0) - R_481h^4 =: A_3.\end{aligned}\quad (63)$$

We solve the system of three equations (63) and we find:

$$\begin{aligned}\Delta_1(x_0) &= \frac{18A_1 - 9A_2 + 2A_3}{6h}, \\ \Delta_2(x_0) &= \frac{4A_2 - 5A_1 - A_3}{2h^2}, \\ \text{and} \\ \Delta_3(x_0) &= \frac{3A_1 - 3A_2 + A_3}{6h^3}.\end{aligned}\quad (64)$$

We have also that

$$|R_4| \leq \frac{24H_\nu}{\Gamma(\nu+1)}. \quad (65)$$

It is

$$18A_1 - 9A_2 + 2A_3 = 11(f(z) - f(x_0)) - 36R_4h^4. \quad (66)$$

Therefore it holds

$$|\Delta_1(x_0)| \leq \frac{11\|f\|_{\infty, \mathbb{R}_+^k}}{3h} + \frac{144H_\nu}{\Gamma(\nu+1)}h^3, \quad (67)$$

$\forall h > 0, \forall x_0 \in \mathbb{R}_+^k$.

We get that

$$\|\Delta_1\|_{\infty, \mathbb{R}_+^k} \leq \frac{11 \|f\|_{\infty, \mathbb{R}_+^k}}{3h} + \left(\frac{144H_\nu}{\Gamma(\nu+1)} \right) h^3, \quad (68)$$

$\forall h > 0$.

It is

$$4A_2 - 5A_1 - A_3 = -2(f(z) - f(x_0)) + 22R_4h^4. \quad (69)$$

Hence

$$|\Delta_2(x_0)| \leq \frac{2 \|f\|_{\infty, \mathbb{R}_+^k}}{h^2} + \left(\frac{264H_\nu}{\Gamma(\nu+1)} \right) h^2, \quad (70)$$

$\forall h > 0, \forall x_0 \in \mathbb{R}_+^k$.

The last implies

$$\|\Delta_2\|_{\infty, \mathbb{R}_+^k} \leq \frac{2 \|f\|_{\infty, \mathbb{R}_+^k}}{h^2} + \left(\frac{264H_\nu}{\Gamma(\nu+1)} \right) h^2, \quad (71)$$

$\forall h > 0$.

It is

$$3A_1 - 3A_2 + A_3 = f(z) - f(x_0) - 36R_4h^4. \quad (72)$$

Therefore

$$|\Delta_3(x_0)| \leq \frac{\|f\|_{\infty, \mathbb{R}_+^k}}{3h^3} + \left(\frac{144H_\nu}{\Gamma(\nu+1)} \right) h, \quad (73)$$

$\forall h > 0, \forall x_0 \in \mathbb{R}_+^k$.

The last implies

$$\|\Delta_3\|_{\infty, \mathbb{R}_+^k} \leq \frac{\|f\|_{\infty, \mathbb{R}_+^k}}{3h^3} + \left(\frac{144H_\nu}{\Gamma(\nu+1)} \right) h, \quad (74)$$

$\forall h > 0$.

We will work on (68).

Call $\mu := \frac{11\|f\|_{\infty, \mathbb{R}_+^k}}{3}$, $\theta := \left(\frac{144H_\nu}{\Gamma(\nu+1)} \right)$, both are positive.

We study

$$y(h) = \frac{\mu}{h} + \theta h^3, \quad \forall h > 0. \quad (75)$$

We have

$$y'(h) = -\mu h^{-2} + 3\theta h^2 = 0,$$

with one critical number

$$h_{crit.no.} = h_0 = \left(\frac{\mu}{3\theta} \right)^{\frac{1}{4}} > 0. \quad (76)$$

We have that

$$y''(h_0) = 2\mu h_0^{-3} + 6\theta h_0 > 0, \quad (77)$$

implying that y has a global minimum. We see that

$$y(h_0) = \mu h_0^{-1} + \theta h_0^3 = h_0^3 \left(\mu h_0^{-4} + \theta \right) = \frac{4}{3^{\frac{3}{4}}} \theta^{\frac{1}{4}} \mu^{\frac{3}{4}}. \quad (78)$$

I.e.

$$y(h_0) = \frac{4}{3^{\frac{3}{4}}} \theta^{\frac{1}{4}} \mu^{\frac{3}{4}}, \quad (79)$$

is the global minimum.

Consequently

$$\|\Delta_1\|_{\infty, \mathbb{R}_+^k} \leq 4 \left(\frac{11}{9} \right)^{\frac{3}{4}} \left(\frac{144}{\Gamma(\nu+1)} \right)^{\frac{1}{4}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{3}{4}} H_{\nu}^{\frac{1}{4}}. \quad (80)$$

Next we work on (71).

Call $\varphi := 2 \|f\|_{\infty, \mathbb{R}_+^k}$, $\psi := \frac{264H_{\nu}}{\Gamma(\nu+1)} > 0$.

We study

$$y(h) := \frac{\varphi}{h^2} + \psi h^2, \quad \forall h > 0. \quad (81)$$

We have

$$y'(h) = -2\varphi h^{-3} + 2\psi h = 0, \quad (82)$$

with

$$h_{crit.no.} = h_0 = \left(\frac{\varphi}{\psi} \right)^{\frac{1}{4}} > 0.$$

Notice that

$$y''(h_0) = 6\varphi h_0^{-4} + 2\psi > 0. \quad (83)$$

Thus we have that y has global minimum which is

$$y(h_0) = \varphi h_0^{-2} + \psi h_0^2 = h_0^2 (\varphi h_0^{-4} + \psi) = 2\psi^{\frac{1}{2}} \varphi^{\frac{1}{2}},$$

that is

$$y(h_0) = 2\psi^{\frac{1}{2}} \varphi^{\frac{1}{2}}. \quad (84)$$

Therefore we derive

$$\|\Delta_2\|_{\infty, \mathbb{R}_+^k} \leq 2 \sqrt{\frac{528}{\Gamma(\nu+1)}} \sqrt{\|f\|_{\infty, \mathbb{R}_+^k} H_{\nu}}. \quad (85)$$

Finally we work on (74).

Let $A := \frac{\|f\|_{\infty, \mathbb{R}_+^k}}{3}$, $B := \frac{144H_{\nu}}{\Gamma(\nu+1)}$, both are positive.

We study

$$y(h) := \frac{A}{h^3} + Bh, \quad \forall h > 0. \quad (86)$$

Then

$$y'(h) = -3Ah^{-4} + B = 0,$$

so that

$$h_{crit.no.} = h_0 = \left(\frac{3A}{B} \right)^{\frac{1}{4}} > 0. \quad (87)$$

Here

$$y''(h_0) = 12Ah_0^{-5} > 0. \quad (88)$$

Hence y has global minimum which is

$$y(h_0) = Ah_0^{-3} + Bh_0 = h_0 \left(Ah_0^{-4} + B \right) = \frac{4}{3^{\frac{3}{4}}} A^{\frac{1}{4}} B^{\frac{3}{4}}. \quad (89)$$

That is

$$y(h_0) = \frac{4}{3^{\frac{3}{4}}} A^{\frac{1}{4}} B^{\frac{3}{4}}, \quad (90)$$

the global minimum of y .

Consequently it holds

$$\|\Delta_3\|_{\infty, \mathbb{R}_+^k} \leq \frac{4}{3} \left(\frac{144}{\Gamma(\nu+1)} \right)^{\frac{3}{4}} \|f\|_{\infty, \mathbb{R}_+^k}^{\frac{1}{4}} H_{\nu}^{\frac{3}{4}}. \quad (91)$$

The theorem is proved.

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