Moroccan J. of Pure and Appl. Anal. (MJPAA)

Volume 6(2), 2020, Pages 281–302

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2020-0022](https://doi.org/10.2478/mjpaa-2020-0022)

Some class of nonlinear inequalities with gradient constraints in Orlicz spaces

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ABSTRACT. In the present paper, we show the existence of solutions of some nonlinear inequalities of the form $\langle Au + g(x, u, \nabla u), v - u \rangle \geq \langle f, v - u \rangle$ with gradient constraint that depend on the solution itself, where A is a Leray-Lions operator defined on Orlicz spaces, g is a nonlinearity and $f \in L^1$.

Mathematics Subject Classification (2020). 35J87, 35J60, 35J92, 46E30.

Key words and phrases. Nonlinear variational inequalities, Nonlinear elliptic equations, p -Laplacian, Orlicz spaces.

1. Introduction

Problems with gradient constraint have recently been the subject of many mathematical and numerical studies, also these kind of inequalities arising in different physical models, such that sandpile growth [12], the superconductivity see [13] and also in image processing [2]. In this paper we deal with some nonlinear inequalities with obstacles depending on the gradients of the solutions in the setting of Orlicz-Sobolev spaces.

Let Ω be a bounded subset of $\mathbb{R}^N$ ($N \geq 2$) with the segment property.

Let A be a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega) \rightarrow W^{-1} L_{\overline{M}}(\Omega)$, with M an

Received : 15 June 2020 - Accepted: 30 September 2020.

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N -function and consider the non-linearity g which satisfies the following "natural" growth condition:

$$|g(x, s, \xi)| \leq b(|s|) (c(x) + M(|\xi|))$$

and the classical sign condition $g(x, s, \xi)s \geq 0$.

Our goal is to prove the existence result of the solution of the following problem

$$(\mathcal{P}) \begin{cases} \langle Au + g(x, u, \nabla u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K_u \\ u \in K_u, \end{cases}$$

where K_u is the convex defined as follows

$$K_u = \left\{ v \in W_0^{1,\infty}(\Omega) : |\nabla v| \leq h(u) \text{ a.e. in } \Omega \right\}$$

where h is a nonnegative continuous function such that $\int_0^{+\infty} \frac{ds}{h(s)} = +\infty$ and $h \geq \delta$ with $\delta > 0$ is a real number and $f \in L^1(\Omega), f \geq 0$.

In order to get our aim, we deal with the asymptotic behaviour of the entropy solution u_p ($p > 1$) of the following problem, when p goes to ∞

$$\begin{cases} -\operatorname{div} \left(\frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \right) - \operatorname{div} (a(x, u_p, \nabla u_p)) + g(x, u_p, \nabla u_p) = f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where $f \in L^1(\Omega)$ and h is the continuous function defined above and which can be for example $h(s) = 1 + |s|$ or $h(s) = 1 + |s| \log^\alpha(e + |s|)$ with $\alpha \leq 1$.

The existence of the entropy solution of this problem has been proved by A. Elmahi and D. Meskine in [4], they also solved it in the variational case (see [5]).

Firstly, we will study the limit when p goes to infinity of u_p and we will show that, up to a subsequence, we have

$$u_p \rightarrow u \quad \text{a.e. in } \Omega.$$

Moreover, we show that u satisfies

$$|\nabla u| \leq h(u) \quad \text{a.e. in } \Omega.$$

Secondly, we will prove that this function limit u is solution of the obstacle problem $(\mathcal{P})$.

The plan of this paper is as follows: in section 2 we recall some preliminaries concerning Orlicz-Sobolev spaces and some lemmas applied on the truncation operators while in section 3 we prove the existence of solutions for problems of the forme $(\mathcal{P})$.

2. Notations and Preliminaries

In this section we recall some notations and preliminaries that will be used later.

First, we start with recalling the definition of N -Functions, Orlicz spaces and Orlicz-Sobolev spaces.

N-Functions.

Let $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ and $\frac{M(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \rightarrow \infty$.

The N -function $\bar{M}$ conjugate to M is defined by $\bar{M}(t) = \int_0^t \bar{a}(s)ds$, where $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $\bar{a}(t) = \sup \{s : a(s) \leq t\}$ (see [1], [10],[9]).

While the N -function M satisfy for some $k > 0$,

$$M(2t) \leq kM(t) \quad \forall t \geq 0, \quad (2.1)$$

we said that satisfy the Δ_2 -condition, when (2.1) holds only for $t \geq$ some $t_0 > 0$, then M is said to satisfy the Δ_2 -condition near infinity.

Let P and H two N -functions, $P \ll H$ means that P grows less rapid to H , i.e. for all $\eta > 0$

$$\frac{P(t)}{H(\eta t)} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Orlicz spaces.

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_{\Omega} M(u(x)) dx < +\infty \text{ (resp. } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 -condition for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x)dx$, and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\bar{M}}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\bar{M}$ satisfy the Δ_2 -condition, for all t or for t large, according to whether Ω has infinite measure or not.

Orlicz-Sobolev spaces.

The space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$) is denoted by $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). It is a Banach space under

the norm

$$\|u\|_{1,M} = \sum_{\alpha \leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1 E_M(\Omega)$ is defined as the (norm) closure of the Schwarz space $D(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1 L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$

$$\int_{\Omega} M \left(\frac{D^\alpha u_n - D^\alpha u}{\lambda} \right) dx \rightarrow 0 \quad \text{for all } \alpha \leq 1.$$

This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$.

If M satisfies the Δ_2 -condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1} L_{\overline{M}}(\Omega)$ (resp. $W^{-1} E_{\overline{M}}(\Omega)$) denote the space of distribution on Ω which can written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $D(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (see [6], [7]). Consequently, the action of a distribution S in $W^{-1} L_{\overline{M}}(\Omega)$ on an element u of $W_0^1 L_M(\Omega)$ is well defined. It will be denoted by $\langle S, u \rangle$.

Now, we quote some lemmas applied on to the truncation operators.

Lemma 1. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N -function and let $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N -function, then the mapping: $F : W^1 L_M(\Omega) \rightarrow W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

Lemma 3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N -functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 s),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from

$$\mathcal{P} \left(E_M(\Omega), \frac{1}{K_2} \right) = \left\{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2} \right\}$$

into $E_Q(\Omega)$.

Lemma 4. Let $(f_n), f \in L^1(\Omega)$ such that

- (1) $f_n \geq 0$ a.e in Ω ;
- (2) $f_n \rightarrow f$ a.e in Ω ;
- (3) $\int_{\Omega} f_n(x) dx \rightarrow \int_{\Omega} f(x) dx$;

then $f_n \rightarrow f$ strongly in $L^1(\Omega)$.

3. The existence result

Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property.

Let M and P be two N -functions such that $P \ll M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \rightarrow W^{-1} L_{\bar{M}}(\Omega)$ be a mapping (non-everywhere defined) given by

$$A(u_p) = -\operatorname{div} (a(x, u_p, \nabla u_p)),$$

for all $p > 1$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $x \in \Omega$, and for all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$|a(x, s, \xi)| \leq c(x) + k_1 \bar{P}^{-1} M(k_2 s) + k_3 \bar{M}^{-1} M(k_4 \xi), \quad (3.1)$$

$$[a(x, s, \xi) - a(x, s, \xi^*)] [\xi - \xi^*] > 0, \quad (3.2)$$

$$\alpha M(|\xi|) \leq a(x, s, \xi) \xi, \quad (3.3)$$

where $c(x)$ belongs to $E_{\bar{M}}(\Omega)$, $c \geq 0$, k_1, k_2, k_3, k_4 to $\mathbb{R}^+$ and $\alpha \in \mathbb{R}_*^+$.

Let h be a nonnegative continuous function such that

$$\int_0^{+\infty} \frac{ds}{h(s)} = +\infty \quad \text{and} \quad h \geq \delta \quad (3.4)$$

with $\delta > 0$ is a real number.

Let $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$,

$$g(x, s, \xi) s \geq 0, \quad (3.5)$$

$$|g(x, s, \xi)| \leq b(|s|) (c'(x) + M(|\xi|)), \quad (3.6)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nondecreasing function and $c'(x)$ is a given non-negative function in $L^1(\Omega)$.

Finally, we suppose that

$$f \in L^1(\Omega), f \geq 0. \quad (3.7)$$

Consider the following elliptic problem with Dirichlet boundary condition:

$$A(u_p) - \operatorname{div} \left(\frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \right) + g(x, u_p, \nabla u_p) = f \quad \text{in } \Omega. \quad (3.8)$$

Define $T_0^{1,M_p}(\Omega)$ to be the set of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $T_k(u_p) \in W_0^1 L_{M_p}(\Omega)$, where $M_p(t) = M(t) + t^p$, $p > 1$, and for all $k \geq 0$,

$$T_k(s) = \max(-k, \min(k, s)) \quad \text{for all } s \in \mathbb{R}.$$

The existence of the solution of the problem (3.8) has been proved in [4] within the meaning of the following definition.

Definition 3.1. (Entropy solution) A function u_p is an entropy solution of the equation (3.8) if:

$$\left\{ \begin{array}{l} u_p \in T_0^{1,M_p}(\Omega), g(x, u_p, \nabla u_p) \in L^1(\Omega), \\ \langle A(u_p), T_k(u_p - v) \rangle + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_k(u_p - v) dx + \int_{\Omega} g(x, u_p, \nabla u_p) T_k(u_p - v) dx \\ \leq \int_{\Omega} f T_k(u_p - v) dx \\ \text{for all } v \in W_0^1 L_{M_p}(\Omega) \cap L^\infty(\Omega) \text{ and for all } k \geq 0. \end{array} \right. \quad (3.9)$$

The main result in this work is given by the following theorem:

Theorem 3.1. Assume that (3.1), (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) hold true. Let u_p be an entropy solution of the problem (3.8), then there exists a measurable function u such that, up to a subsequence, as p goes to ∞ ,

$$u_p \rightarrow u \quad \text{a.e in } \Omega,$$

in addition, we have

$$|\nabla u| \leq h(u) \quad \text{a.e in } \Omega.$$

Moreover, u satisfies the following inequality:

$$\int_{\Omega} a(x, u, \nabla u) [\nabla u - \nabla v] dx + \int_{\Omega} g(x, u, \nabla u)(u - v) dx \leq \langle f, u - v \rangle,$$

for all $v \in W_0^{1,\infty}(\Omega) \cap L^\infty(\Omega)$ such that $|\nabla v| \leq h(u)$.

Remark 3.1. Our result seems to be new even in the classical Sobolev spaces (see [11]–[13]).

Proof. In order to get this result, we start at first by establishing some a priori estimates. For every fixed $k > 0$, taking $v = 0$ in (3.9), so we have

$$\begin{aligned} \int_{\Omega} a(x, u_p, \nabla u_p) \nabla T_k(u_p) dx + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_k(u_p) dx \\ + \int_{\Omega} g(x, u_p, \nabla u_p) T_k(u_p) dx \leq \int_{\Omega} f T_k(u_p) dx, \end{aligned} \quad (3.10)$$

since

$$\int_{\{|u_p| \leq k\}} g(x, u_p, \nabla u_p) T_k(u_p) dx \geq 0,$$

and

$$\int_{\{|u_p| \leq k\}} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_k(u_p) dx \geq 0,$$

hence

$$\int_{\{|u_p| \leq k\}} a(x, u_p, \nabla u_p) \nabla u_p dx \leq C_1 k,$$

where C_1 is a constant such that $\|f\|_{L^1(\Omega)} \leq C_1$. Therefore, by (3.3), we get

$$\alpha \int_{\Omega} M(|\nabla T_k(u_p)|) dx \leq C_1 k. \quad (3.11)$$

Thanks to Lemma 5.7 of [6], there exists two positive constants γ and λ such that

$$\int_{\Omega} M(v) dx \leq \gamma \int_{\Omega} M(\lambda |\nabla v|) dx \quad \text{for all } v \in W_0^1 L_M(\Omega).$$

Taking $v = \frac{1}{\lambda} T_k(u_p)$, so we obtain

$$\alpha \int_{\Omega} M\left(\frac{1}{\lambda} T_k(u_p)\right) dx \leq C_2 k,$$

which implies that

$$\text{meas}\{|u_p| > k\} \leq \frac{C_3 k}{M\left(\frac{k}{\lambda}\right)}, \quad \forall p > 1, \forall k > 0.$$

For any $\delta > 0$, we have

$$\begin{aligned} \text{meas}\{|u_p - u_q| > \delta\} &\leq \text{meas}\{|u_p| > k\} + \text{meas}\{|u_q| > k\} \\ &\quad + \text{meas}\{|T_k(u_p) - T_k(u_q)| > \delta\} \end{aligned}$$

and so that

$$\text{meas}\{|u_p - u_q| > \delta\} \leq \frac{2C_3 k}{M\left(\frac{k}{\lambda}\right)} + \text{meas}\{|T_k(u_p) - T_k(u_q)| > \delta\}.$$

By using (3.11), we deduce that $(T_k(u_p))_p$ is bounded in $W_0^1 L_M(\Omega)$, and then there exists some $w_k \in W_0^1 L_M(\Omega)$ such that

$$\begin{aligned} T_k(u_p) &\rightharpoonup w_k \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \\ &\quad \text{strongly in } E_M(\Omega) \text{ and a.e in } \Omega. \end{aligned}$$

Consequently, we can assume that $(T_k(u_p))$ is a Cauchy sequence in measure in Ω .

Let $\epsilon > 0$, there exists some $k(\epsilon) > 0$ such that

$$\text{meas}\{|u_p - u_q| > \delta\} \leq \epsilon, \quad \text{for all } p, q \geq p_0(k(\epsilon), \delta).$$

This proves that (u_p) is a Cauchy sequence in measure, thus, u_p converges almost everywhere to some measurable function u .

By Lemma 4.4 [6], we obtain for all $k > 0$,

$T_k(u_p) \rightharpoonup T_k(u)$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ strongly in $E_M(\Omega)$ and a.e in Ω .

On the other hand, thanks to (3.10) we get the following estimate

$$\int_{\{|u_p| \leq k\}} \frac{|\nabla u_p|^p}{h(u_p)^{p-1}} dx \leq C_1 k,$$

then

$$\int_{\{|u_p| \leq k\}} \frac{|\nabla u_p|^p}{h(u_p)^p} h(u_p) dx \leq C_1 k,$$

so

$$\int_{\{|u_p| \leq k\}} \frac{|\nabla u_p|^p}{h(u_p)^p} dx \leq Ck.$$

For every $q < p$, we have

$$\int_{\Omega} \frac{|\nabla T_k(u_p)|^q}{h(u_p)^q} dx \leq \left(\int_{\Omega} \frac{|\nabla T_k(u_p)|^p}{h(u_p)^p} dx \right)^{\frac{q}{p}} |\Omega|^{1-\frac{q}{p}} \leq (Ck)^{\frac{q}{p}} |\Omega|^{1-\frac{q}{p}},$$

then

$$\left(\int_{\Omega} \frac{|\nabla T_k(u_p)|^q}{h(u_p)^q} dx \right)^{\frac{1}{q}} \leq (Ck)^{\frac{1}{p}} |\Omega|^{\frac{1}{q}-\frac{1}{p}},$$

hence

$$\left(\int_{\Omega} \left| \nabla \left(\int_0^{T_k(u_p)} \frac{ds}{h(s)} \right) \right|^q dx \right)^{\frac{1}{q}} \leq (Ck)^{\frac{1}{p}} |\Omega|^{\frac{1}{q}-\frac{1}{p}}.$$

Since $(Ck)^{\frac{1}{p}} |\Omega|^{\frac{1}{q}-\frac{1}{p}} \rightarrow |\Omega|^{\frac{1}{q}}$ as $p \rightarrow +\infty$, then $\int_0^{T_k(u_p)} \frac{ds}{h(s)}$ is bounded in $W_0^{1,q}(\Omega)$ and since $T_k(u_p) \rightarrow T_k(u)$ a.e. in Ω . So

$$\int_0^{T_k(u_p)} \frac{ds}{h(s)} \rightharpoonup \int_0^{T_k(u)} \frac{ds}{h(s)} \quad \text{in } W_0^{1,q}(\Omega).$$

Consequently, we have

$$\left(\int_{\Omega} \left(\frac{|\nabla T_k(u)|}{h(u)} \right)^q dx \right)^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}}.$$

As the above inequality holds for every q , taking the limit as $q \rightarrow +\infty$, we get that

$$|\nabla T_k(u)| \leq h(T_k(u)) \quad \text{a.e. } x \in \Omega, \quad \text{for all } k > 0. \quad (3.12)$$

Now we will show that u is bounded, indeed we have for $q_0 > N$,

$$\int_{\Omega} \left| \nabla \left(\int_0^{T_k(u)} \frac{ds}{h(s)} \right) \right|^{q_0} dx \leq |\Omega|.$$

Then by Sobolev injection $W_0^{1,q_0}(\Omega)$ in $L^\infty(\Omega)$ we obtain

$$\left| \int_0^{T_k(u)} \frac{ds}{h(s)} \right| \leq C.$$

By letting $k \rightarrow +\infty$, we get

$$\left| \int_0^u \frac{ds}{h(s)} \right| \leq C$$

and since $\int_0^{+\infty} \frac{ds}{h(s)} = +\infty$, we conclude that

$$|u| \leq M_0 \quad \text{a.e. } x \in \Omega \quad (M_0 \text{ is a constant}). \quad (3.13)$$

Go back to equation (3.12) and letting $k \rightarrow +\infty$, we obtain that

$$|\nabla u| \leq h(u) \quad \text{a.e. } x \in \Omega. \quad (3.14)$$

Finally, since $\|u\|_\infty \leq k$, we get that

$$T_k(u_p) \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \text{ and also weakly in } W_0^{1,q}(\Omega) (q > N) \quad (3.15)$$

strongly in $E_M(\Omega)$ and a.e in Ω .

Now we will show that $(a(x, T_k(u_p), \nabla T_k(u_p)))_p$ is bounded in $(L_{\overline{M}}(\Omega))^N$ for all $k > 0$, by using the dual norm of $(E_{\overline{M}}(\Omega))^N$.

Let $\varphi \in (E_{\overline{M}}(\Omega))^N$, such that $\|\varphi\|_{M,\Omega} = 1$. In view of (3.2), we have

$$\int_{\Omega} \left[a(x, T_k(u_p), \nabla T_k(u_p)) - a\left(x, T_k(u_p), \frac{\varphi}{k_4}\right) \right] \left[\nabla T_k(u_p) - \frac{\varphi}{k_4} \right] dx \geq 0,$$

where k_i ($i = 1, \dots, 4$) are the constants in (3.1).

This implies that

$$\begin{aligned} & \int_{\Omega} \frac{1}{k_4} a(x, T_k(u_p), \nabla T_k(u_p)) \varphi dx \\ & \leq \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx - \int_{\Omega} a\left(x, T_k(u_p), \frac{\varphi}{k_4}\right) \left[\nabla T_k(u_p) - \frac{\varphi}{k_4} \right] dx \\ & \leq C_1 k - \int_{\Omega} a\left(x, T_k(u_p), \frac{\varphi}{k_4}\right) \nabla T_k(u_p) dx + \frac{1}{k_4} \int_{\Omega} a\left(x, T_k(u_p), \frac{\varphi}{k_4}\right) \varphi dx, \end{aligned}$$

so, by using Young's inequality in the last two terms of last side of the above inequality, we deduce that

$$\begin{aligned} & \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \varphi dx \\ & \leq 3k_3(1 + k_4) \int_{\Omega} \overline{M} \left[\frac{a\left(x, T_k(u_p), \frac{\varphi}{k_4}\right)}{3k_3} \right] dx \\ & \quad + C_1 k_4 k + 3k_3 k_4 \int_{\Omega} M[|\nabla T_k(u_p)|] dx + 3k_3 \int_{\Omega} M[\varphi] dx \end{aligned}$$

$$\begin{aligned} &\leq 3k_3(1+k_4) \int_{\Omega} \overline{M} \left[\frac{\left| a \left(x, T_k(u_p), \frac{\varphi}{k_4} \right) \right|}{3k_3} \right] dx \\ &\quad + C_1 k_4 k + \frac{3C_1 k_3 k_4 k}{\alpha} + 3k_3. \end{aligned}$$

Moreover, thanks to (3.1) and the convexity of the N -function $\overline{M}$, we have

$$\overline{M} \left[\frac{a \left(x, T_k(u_p), \frac{\varphi}{k_4} \right)}{3k_3} \right] \leq \frac{1}{3} \overline{M} \left(\frac{c(x)}{k_3} \right) + \frac{k_1}{3k_3} \overline{M} P^{-1} M(k_2 k) + \frac{1}{3} M(\varphi).$$

Integrating this over Ω , we get

$$\int_{\Omega} \overline{M} \left[\frac{a \left(x, T_k(u_p), \frac{\varphi}{k_4} \right)}{3k_3} \right] dx \leq C_k, \quad (3.16)$$

where C_k is a constant depending on k .

We deduce that

$$\int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \varphi dx \leq C_k \quad \forall \varphi \in (E_M(\Omega))^N,$$

such that $\|\varphi\|_{M,\Omega} = 1$, and so

$$\|a(x, T_k(u_p), \nabla T_k(u_p))\|_{\overline{M},\Omega} \leq C_k,$$

which proves that $[a(x, T_k(u_p), \nabla T_k(u_p))]_p$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

In a second step, we will demonstrate the almost everywhere convergence of the gradients, for that we consider the function $\varphi(t) = te^{\sigma t^2}$, $\sigma > 0$. It is known that when $\sigma > \left(\frac{b(k)}{2\alpha}\right)^2$ one has

$$\varphi'(t) - \frac{b(k)}{\alpha} |\varphi(t)| \geq \frac{1}{2} \quad \forall t \in \mathbb{R}, \quad (3.17)$$

where $k > 0$ is a fixed real number which will be used as a level of the truncation, and we define the function ρ_m where $m > k$, by

$$\begin{cases} 1 & \text{if } |s| \leq m, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \\ 0 & \text{if } |s| > m+1. \end{cases} \quad (3.18)$$

For every fixed $k' > 0$, we have

$$\begin{aligned} &\int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_{k'}(u_p - v) dx + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla T_{k'}(u_p - v) dx \\ &\quad + \int_{\Omega} g(x, u_p, \nabla u_p) T_{k'}(u_p - v) dx \leq \int_{\Omega} f T_{k'}(u_p - v) dx, \end{aligned}$$

choosing for all $\eta > 0$, $v = T_\eta(u_p) - \varphi(T_k(u_p) - \theta u)\rho_m(u_p)$ in this inequality, where θ is a real number such that $0 < \theta \leq 1$ then by letting k' and η go to $+\infty$, we obtain

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla \cdot [\varphi(T_k(u_p) - \theta u)\rho_m(u_p)] dx \\ & + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla \cdot [\varphi(T_k(u_p) - \theta u)\rho_m(u_p)] dx \\ & + \int_{\Omega} g(x, u_p, \nabla u_p) \varphi(T_k(u_p) - \theta u)\rho_m(u_p) dx \leq \int_{\Omega} f \varphi(T_k(u_p) - \theta u) \rho_m(u_p) dx. \end{aligned}$$

We denote by $\epsilon_i(p, \theta)$ ($i=0,1,2,\dots$) various sequences of real numbers which tend to 0 when p tend to $+\infty$ and θ tend to 1 i.e.

$$\lim_{\theta \rightarrow 1} \lim_{p \rightarrow +\infty} \epsilon_i(p, \theta) = 0.$$

Since, we have $T_k(u_p) \rightarrow u$, as $p \rightarrow \infty$, because $k \geq |u|$, hence

$$\varphi(T_k(u_p) - \theta u)\rho_m(u_p) \rightarrow \varphi(u - \theta u)\rho_m(u) \text{ weakly* in } L^\infty(\Omega),$$

as $p \rightarrow +\infty$ and thus

$$\int_{\Omega} f \varphi(T_k(u_p) - \theta u)\rho_m(u_p) dx \rightarrow \int_{\Omega} f \varphi(u - \theta u)\rho_m(u) dx,$$

letting $\theta \rightarrow 1$, we obtain that

$$\int_{\Omega} f \varphi(u - \theta u)\rho_m(u) dx \rightarrow 0,$$

which means that

$$\int_{\Omega} f \varphi(T_k(u_p) - \theta u)\rho_m(u_p) dx = \epsilon_0(p, \theta). \quad (3.19)$$

Denoting $\varphi(T_k(u_p) - \theta u)\rho_m(u_p)$ by $z_{p,m}^\theta$ and since $g(x, u_p, \nabla u_p)z_{p,m}^\theta \geq 0$ on the set $\{x \in \Omega : |u_p| > k\}$, so

$$\left\langle A(u_p), z_{p,m}^\theta \right\rangle + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla z_{p,m}^\theta dx + \int_{\{|u_p| \leq k\}} g(x, u_p, \nabla u_p) z_{p,m}^\theta dx \leq \epsilon_0(p, \theta). \quad (3.20)$$

About the first term in this last inequality, we have

$$\begin{aligned} \left\langle A(u_p), z_{p,m}^\theta \right\rangle &= \int_{\Omega} a(x, u_p, \nabla u_p) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &+ \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \\ &= \int_{\{|u_p| \leq k\}} a(x, u_p, \nabla u_p) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &- \theta \int_{\{|u_p| > k\}} a(x, u_p, \nabla u_p) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &+ \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx. \end{aligned}$$

Since $m > k$, we have $\rho_m(u_p) = 1$ on the subset $\{x \in \Omega : |u_p| \leq k\}$, so

$$\begin{aligned} \langle A(u_p), z_{p,m}^\theta \rangle &= \int_{\{|u_p| \leq k\}} a(x, T_k(u_p), \nabla T_k(u_p)) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx \\ &\quad - \theta \int_{\{|u_p| > k\}} a(x, u_p, \nabla u_p) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx, \end{aligned}$$

and then

$$\begin{aligned} \langle A(u_p), z_{p,m}^\theta \rangle &= \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\ &\quad \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx \\ &\quad + \int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx \\ &\quad - \theta \int_{\{|u_p| > k\}} a(x, u_p, \nabla u_p) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx. \end{aligned}$$

Thanks to Lemma 3, we have

$$a(x, T_k(u_p), \theta \nabla u) \varphi'(T_k(u_p) - \theta u) \longrightarrow a(x, u, \theta \nabla u) \varphi'(u - \theta u),$$

strongly in $(E_{\overline{M}}(\Omega))^N$ as $p \rightarrow +\infty$, and from (3.15) we have

$$\nabla T_k(u_p) \rightharpoonup \nabla u \text{ weakly in } (L_M(\Omega))^N, \quad (3.21)$$

we deduce that

$$\begin{aligned} &\int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx \\ &\quad \rightarrow \int_{\Omega} a(x, u, \theta \nabla u) [\nabla u - \theta \nabla u] \varphi'(u - \theta u) dx \text{ as } p \rightarrow +\infty, \end{aligned}$$

we pass to the limit when θ tends to 1, we see that

$$\int_{\Omega} a(x, u, \theta \nabla u) [\nabla u - \theta \nabla u] \varphi'(u - \theta u) dx \rightarrow 0 \text{ as } \theta \rightarrow 1,$$

in other words, it means that

$$\int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx = \epsilon_1(p, \theta).$$

Since $\rho_m(u_p) = 0$ on the subset $\{|u_p| > m + 1\}$, it follows therefore that

$$\begin{aligned} &- \theta \int_{\{|u_p| > k\}} a(x, u_p, \nabla u_p) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &= - \theta \int_{\{|u_p| > k\}} a(x, T_{m+1}(u_p), \nabla T_{m+1}(u_p)) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx. \end{aligned}$$

Furthermore, since $[a(x, T_k(u_p), \nabla T_k(u_p))]_p$ is bounded in $(L_{\overline{M}}(\Omega))^N$, so for a subsequence, we have

$$a(x, T_k(u_p), \nabla T_k(u_p)) \rightharpoonup h_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N$$

for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ for some h_k in $(L_{\overline{M}}(\Omega))^N$, and since

$$\nabla u \in (E_M(\Omega))^N,$$

by letting $p \rightarrow +\infty$ and $\theta \rightarrow 1$, we obtain that

$$\begin{aligned} & -\theta \int_{\{|u_p|>k\}} a(x, T_{m+1}(u_p), \nabla T_{m+1}(u_p)) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ & \rightarrow - \int_{\{|u|>k\}} h_{m+1} \nabla u \varphi'(u - \theta u) \rho_m(u) dx. \end{aligned}$$

For $k > M_0$ (see (3.14)), we have

$$\text{meas}\{|u| > k\} = 0.$$

Consequently,

$$\int_{\{|u|>k\}} h_{m+1} \nabla u \varphi'(u - \theta u) \rho_m(u) dx = 0,$$

which gives that

$$-\theta \int_{\{|u_p|>k\}} a(x, u_p, \nabla u_p) \nabla u \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx = \epsilon_2(p, \theta).$$

On the other hand,

$$\begin{aligned} & \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \\ & = \int_{\{m \leq |u_p| \leq m+1\}} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx, \end{aligned}$$

hence for $k > M_0$, we get

$$\begin{aligned} & \left| \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \right| \\ & \leq \varphi(2k) \int_{\{m \leq |u_p| \leq m+1\}} a(x, u_p, \nabla u_p) \nabla u_p dx. \end{aligned}$$

Taking $v = u_p - T_1(u_p - T_m(u_p))$ in (3.9) and then letting $k \rightarrow \infty$, thus we have

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_1(u_p - T_m(u_p)) dx + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla T_1(u_p - T_m(u_p)) dx \\ & + \int_{\Omega} g(x, u_p, \nabla u_p) T_1(u_p - T_m(u_p)) dx \leq \int_{\Omega} f T_1(u_p - T_m(u_p)) dx, \end{aligned}$$

then

$$\int_{\{m \leq |u_p| \leq m+1\}} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \cdot \nabla u_p dx + \int_{\{m \leq |u_p| \leq m+1\}} a(x, u_p, \nabla u_p) \cdot \nabla u_p dx$$

$$+ \int_{\Omega} g(x, u_p, \nabla u_p) T_1(u_p - T_m(u_p)) dx \leq \int_{\{|u_p| \geq m\}} f dx. \quad (3.22)$$

Since

$$\int_{\{m \leq |u_p| \leq m+1\}} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \cdot \nabla u_p dx \geq 0,$$

and

$$g(x, u_p, \nabla u_p) T_1(u_p - T_m(u_p)) \geq 0,$$

we obtain

$$\int_{\{m \leq |u_p| \leq m+1\}} a(x, u_p, \nabla u_p) \nabla u_p dx \leq \int_{\{|u_p| > m\}} f dx,$$

we deduce then that

$$\left| \int_{\Omega} a(x, u_p, \nabla u_p) \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \right| \leq \varphi(2k) \int_{\{|u_p| > m\}} f dx.$$

Consequently,

$$\begin{aligned} \langle A(u_p), z_{p,m}^\theta \rangle &\geq -\varphi(2k) \int_{\{|u_p| > m\}} f dx \\ &\quad + \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\ &\quad \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx + \epsilon_3(p, \theta). \end{aligned} \quad (3.23)$$

Concerning the second term of (3.20), we can write it as follows

$$\begin{aligned} &\int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \cdot [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \cdot \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx, \end{aligned}$$

which is equal to

$$\begin{aligned} I &:= \int_{\Omega} \left[\frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p - \frac{|\theta \nabla u|^{p-2}}{h(u_p)^{p-1}} \theta \nabla u \right] [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} \theta^{p-1} \frac{|\nabla u|^{p-2}}{h(u_p)^{p-1}} \nabla u [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \cdot \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx, \end{aligned}$$

by the definition of monotonicity, we have

$$\int_{\Omega} \left[\frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p - \frac{|\theta \nabla u|^{p-2}}{h(u_p)^{p-1}} \theta \nabla u \right] [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \geq 0,$$

then

$$\begin{aligned} I &\geq \int_{\Omega} \theta^{p-1} \frac{|\nabla u|^{p-2}}{h(u_p)^{p-1}} \nabla u \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \\ &\quad + \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx. \end{aligned}$$

Since we have

$$\begin{aligned} &\left| \int_{\Omega} \theta^{p-1} \frac{|\nabla u|^{p-2}}{h(u_p)^{p-1}} \nabla u \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx \right| \\ &\leq \int_{\Omega} \left(\theta \frac{|\nabla u|}{h(u_p)} \right)^{p-1} |\nabla T_k(u_p) - \theta \nabla u| |\varphi'(T_k(u_p) - \theta u)| |\rho_m(u_p)| dx \end{aligned}$$

and then since $|\nabla u| \leq h(u)$, so

$$\begin{aligned} &\int_{\Omega} \left(\theta \frac{|\nabla u|}{h(u_p)} \right)^{p-1} \times |\nabla T_k(u_p) - \theta \nabla u| |\varphi'(T_k(u_p) - \theta u)| |\rho_m(u_p)| dx \\ &\leq \int_{\Omega} \left(\theta \frac{h(u)}{h(u_p)} \right)^{p-1} |\nabla T_k(u_p) - \theta \nabla u| |\varphi'(T_k(u_p) - \theta u)| |\rho_m(u_p)| dx \end{aligned}$$

and moreover since $\theta < 1$ hence by passing the limit as $p \rightarrow +\infty$ and using the fact that $h(T_{m+1}(u_p)) \rightarrow h(T_{m+1}(u))$ in $L^\infty(\Omega)$, we obtain that

$$\int_{\Omega} \theta^{p-1} \frac{|\nabla u|^{p-2}}{h(u_p)^{p-1}} \nabla u \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) \rho_m(u_p) dx = \epsilon_4(p, \theta). \quad (3.24)$$

On the other hand, we have for $k > M_0$

$$\begin{aligned} &\left| \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \right| \\ &= \left| \int_{\{m \leq |u_p| \leq m+1\}} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \right| \\ &\leq \varphi(2k) \int_{\{m \leq |u_p| \leq m+1\}} \frac{|\nabla u_p|^p}{h(u_p)^{p-1}} dx. \end{aligned}$$

Going back to (3.22) and since

$$g(x, u_p, \nabla u_p) T_1(u_p - T_m(u_p)) \geq 0,$$

and

$$\int_{\{m \leq |u_p| \leq m+1\}} a(x, u_p, \nabla u_p) \nabla u_p dx \geq \alpha \int_{\{m \leq |u_p| \leq m+1\}} M(|\nabla u_p|) dx \geq 0,$$

it follows that

$$\int_{\{m \leq |u_p| \leq m+1\}} \frac{|\nabla u_p|^p}{h(u_p)^{p-1}} dx \leq \int_{\{|u_p| \geq m\}} f dx,$$

which implies that

$$\left| \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla u_p \varphi(T_k(u_p) - \theta u) \rho'_m(u_p) dx \right| \leq \varphi(2k) \int_{\{|u_p| \geq m\}} f dx. \quad (3.25)$$

For what concern the third term of (3.20), we have

$$\begin{aligned} \left| \int_{\{|u_p| \leq k\}} g(x, u_p, \nabla u_p) z_{p,m}^{\theta} dx \right| &= \left| \int_{\{|u_p| \leq k\}} g(x, T_k(u_p), \nabla T_k(u_p)) \varphi(T_k(u_p) - \theta u) dx \right| \\ &\leq \int_{\Omega} b(k) c'(x) |\varphi(T_k(u_p) - \theta u)| dx + b(k) \int_{\Omega} M(|\nabla T_k(u_p)|) |\varphi(T_k(u_p) - \theta u)| dx \\ &\leq \epsilon_5(p, \theta) + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) |\nabla T_k(u_p)| |\varphi(T_k(u_p) - \theta u)| dx. \end{aligned}$$

Moreover, since

$$\begin{aligned} &\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) |\nabla T_k(u_p)| |\varphi(T_k(u_p) - \theta u)| dx \\ &= \frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\ &\quad \times [\nabla T_k(u_p) - \theta \nabla u] |\varphi(T_k(u_p) - \theta u)| dx \\ &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] |\varphi(T_k(u_p) - \theta u)| dx \\ &\quad - \frac{b(k)}{\alpha} \theta \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla u |\varphi(T_k(u_p) - \theta u)| dx, \end{aligned}$$

and in the same way as the terms above, we remark that

$$\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] |\varphi(T_k(u_p) - \theta u)| dx = \epsilon_6(p, \theta)$$

and that

$$-\frac{b(k)}{\alpha} \theta \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla u |\varphi(T_k(u_p) - \theta u)| dx = \epsilon_7(p, \theta),$$

then we conclude that

$$\begin{aligned} \left| \int_{\{|u_p| \leq k\}} g(x, u_p, \nabla u_p) z_{p,m}^{\theta} dx \right| &\leq \frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\ &\quad \times [\nabla T_k(u_p) - \theta \nabla u] |\varphi(T_k(u_p) - \theta u)| dx + \epsilon_8(p, \theta). \end{aligned} \quad (3.26)$$

Combining the estimates (3.23), (3.24), (3.25) and (3.26), we obtain

$$-2\varphi(2k) \int_{\{|u_p| > m\}} f dx$$

$$\begin{aligned}
& + \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\
& \quad \times [\nabla T_k(u_p) - \theta \nabla u] \varphi'(T_k(u_p) - \theta u) dx \\
& \leq \frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\
& \quad \times [\nabla T_k(u_p) - \theta \nabla u] |\varphi(T_k(u_p) - \theta u)| dx + \epsilon_9(p, \theta),
\end{aligned}$$

so

$$\begin{aligned}
& \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \\
& \quad \times [\nabla T_k(u_p) - \theta \nabla u] \left[\varphi'(T_k(u_p) - \theta u) - \frac{b(k)}{\alpha} |\varphi(T_k(u_p) - \theta u)| \right] dx \\
& \leq 2\varphi(2k) \int_{\{|u_p| \geq m\}} f dx + \epsilon_9(p, \theta).
\end{aligned}$$

Consequently, from (3.17), we have

$$\begin{aligned}
& \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] [\nabla T_k(u_p) - \theta \nabla u] dx \\
& \leq 4\varphi(2k) \int_{\{|u_p| \geq m\}} f dx + 2\epsilon_9(p, \theta).
\end{aligned} \tag{3.27}$$

On the other hand,

$$\begin{aligned}
& \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \nabla u)] \times [\nabla T_k(u_p) - \nabla u] dx \\
& = \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] \times [\nabla T_k(u_p) - \theta \nabla u] dx \\
& \quad + \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) [\theta \nabla u - \nabla u] dx \\
& \quad - \int_{\Omega} a(x, T_k(u_p), \nabla u) [\nabla T_k(u_p) - \nabla u] dx \\
& \quad + \int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] dx.
\end{aligned}$$

Letting $p \rightarrow +\infty$ and $\theta \rightarrow 1$ in the last three terms of the last side of the above equality, we obtain that

$$\int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) [\theta \nabla u - \nabla u] dx = \epsilon_{10}(p, \theta)$$

and

$$\int_{\Omega} a(x, T_k(u_p), \nabla u) [\nabla T_k(u_p) - \nabla u] dx = \epsilon_{11}(p, \theta)$$

and

$$\int_{\Omega} a(x, T_k(u_p), \theta \nabla u) [\nabla T_k(u_p) - \theta \nabla u] dx = \epsilon_{12}(p, \theta), \tag{3.28}$$

which implies that

$$\begin{aligned} & \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \nabla u)] [\nabla T_k(u_p) - \nabla u] dx \\ &= \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \theta \nabla u)] [\nabla T_k(u_p) - \theta \nabla u] dx + \epsilon_{13}(p, \theta) \\ &\leq \epsilon_{14}(p, \theta) + 4\varphi(2k) \int_{\{|u_p| \geq m\}} f dx, \end{aligned}$$

we passing to the limit sup over p and then over θ , we get

$$\begin{aligned} & \limsup_{p \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \nabla u)] [\nabla T_k(u_p) - \nabla u] dx \\ &\leq 4\varphi(2k) \int_{\{|u| \geq m\}} f dx, \end{aligned}$$

by letting $m \rightarrow \infty$ and due to the fact that $|\{|u| \geq m\}| \rightarrow 0$, we obtain that

$$\int_{\Omega} [a(x, T_k(u_p), \nabla T_k(u_p)) - a(x, T_k(u_p), \nabla u)] \times [\nabla T_k(u_p) - \nabla u] dx \rightarrow 0$$

as $p \rightarrow \infty$.

We deduce then as in [3] that there exists a subsequence still denoted by u_p such that

$$\nabla u_p \rightarrow \nabla u \text{ a.e. in } \Omega, \quad (3.29)$$

which gives that, for all $k > 0$,

$$\begin{aligned} a(x, T_k(u_p), \nabla T_k(u_p)) &\rightharpoonup a(x, T_k(u), \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \\ &\text{for } \sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega)). \end{aligned} \quad (3.30)$$

After showing the almost everywhere convergence of the gradients, in the third step we will prove the modular convergence of the truncations.

Returning to (3.27), we can write

$$\begin{aligned} & \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \\ &\leq \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \theta \nabla u dx \\ &+ \int_{\Omega} a(x, T_k(u_p), \theta \nabla u) \times [\nabla T_k(u_p) - \theta \nabla u] dx \\ &+ 2\epsilon_9(p, \theta) + 4\varphi(2k) \int_{\{|u_p| \geq m\}} f dx, \end{aligned}$$

by using (3.28), this implies that

$$\begin{aligned} & \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \\ &\leq \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \theta \nabla u dx + \epsilon_{15}(p, \theta) + 4\varphi(2k) \int_{\{|u_p| \geq m\}} f dx, \end{aligned}$$

passing to the limit sup over p , we have

$$\begin{aligned} & \limsup_p \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \\ & \leq \int_{\Omega} a(x, u, \nabla u) \theta \nabla u dx + \lim_{p \rightarrow +\infty} \epsilon_{15}(p, \theta) + 4\varphi(2k) \int_{\{|u| \geq m\}} f dx, \end{aligned}$$

letting $\theta \rightarrow 1$, we get

$$\begin{aligned} & \limsup_p \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \\ & \leq \int_{\Omega} a(x, u, \nabla u) \nabla u dx + 4\varphi(2k) \int_{\{|u| \geq m\}} f dx, \end{aligned}$$

letting $m \rightarrow +\infty$, we obtain

$$\limsup_p \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \leq \int_{\Omega} a(x, u, \nabla u) \nabla u dx.$$

On the other hand, by using the Lemma's Fatou

$$\int_{\Omega} a(x, u, \nabla u) \nabla u dx \leq \liminf_{p \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx,$$

so

$$\int_{\Omega} a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) dx \rightarrow \int_{\Omega} a(x, u, \nabla u) \nabla u dx, \text{ as } p \rightarrow +\infty$$

and, by using Lemma 4, we conclude that

$$a(x, T_k(u_p), \nabla T_k(u_p)) \nabla T_k(u_p) \rightarrow a(x, u, \nabla u) \nabla u, \text{ in } L^1(\Omega). \quad (3.31)$$

This implies, by using (3.3), that

$$T_k(u_p) \rightarrow u \text{ in } W_0^1 L_M(\Omega)$$

for the modular convergence for all $k > \|u\|_{\infty}$.

Now we will show the equi-integrability of the non-linearities.

In fact, what we want to demonstrate is that

$$g(x, u_p, \nabla u_p) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega)$$

by using Vitali's theorem. Since

$$\nabla u_p \rightarrow \nabla u \text{ a.e in } \Omega,$$

then

$$g(x, u_p, \nabla u_p) \rightarrow g(x, u, \nabla u) \text{ a.e in } \Omega,$$

it suffices to prove that $g(x, u_p, \nabla u_p)$ are uniformly equi-integrable in Ω .

Let $E \subset \Omega$ be a measurable subset of Ω . For any $m > 0$, we have

$$\int_E g(x, u_p, \nabla u_p) dx = \int_{E \cap \{|u_p| \leq m\}} g(x, u_p, \nabla u_p) dx + \int_{\{|u_p| > m\}} g(x, u_p, \nabla u_p) dx$$

Taking $v = T_m(u_p)$ in (3.9), then we get

$$\int_{\{|u_p|>m\}} g(x, u_p, \nabla u_p) dx \leq \int_{\{|u_p|\geq m\}} f dx.$$

Let $\epsilon > 0$. So there exists $m = m(\epsilon) > 0$ such that

$$\int_{\{|u_p|>m\}} g(x, u_p, \nabla u_p) dx \leq \frac{\epsilon}{2}.$$

On the other hand, we have

$$\begin{aligned} \int_{E \cap \{|u_p|\leq m\}} g(x, u_p, \nabla u_p) dx &\leq \int_E g(x, T_m(u_p), \nabla T_m(u_p)) dx \\ &\leq b(m) \int_E [c'(x) + M(|\nabla T_m(u_p)|)] dx \\ &\leq b(m) \int_E a(x, T_m(u_p), \nabla T_m(u_p)) \nabla T_m(u_p) dx \\ &\quad + b(m) \int_E c'(x) dx. \end{aligned}$$

From (3.31) and since $c'(x) \in L^1(\Omega)$, there exists $\mu > 0$ such that

$$E < \mu \implies \int_{E \cap \{|u_p|\leq m\}} g(x, u_p, \nabla u_p) dx \leq \frac{\epsilon}{2}.$$

Consequently,

$$E < \mu \implies \int_E g(x, u_p, \nabla u_p) dx \leq \epsilon,$$

which proves that $g(x, u_p, \nabla u_p)$ are uniformly equi-integrable in Ω as required.

Finally, we finish our proof by the passage to the limit.

Let φ be a function such that $|\nabla \varphi| \leq h(u)$, choosing $v = \theta \varphi$ in (3.9), where $0 < \theta \leq 1$ is a real number, we have for every fixed $k > 0$ that

$$\begin{aligned} &\int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p \nabla T_k(u_p - \theta \varphi) dx + \int_{\Omega} a(x, u_p, \nabla u_p) \nabla T_k(u_p - \theta \varphi) dx \\ &\quad + \int_{\Omega} g(x, u_p, \nabla u_p) T_k(u_p - \theta \varphi) dx \leq \int_{\Omega} f T_k(u_p - \theta \varphi) dx. \end{aligned} \tag{3.32}$$

On the other hand, we have

$$\begin{aligned} &\int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p [\nabla T_k(u_p) - \theta \nabla \varphi] dx \\ &= \int_{\Omega} \left[\frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p - \theta^{p-1} \frac{|\nabla \varphi|^{p-2}}{h(u_p)^{p-1}} \nabla \varphi \right] [\nabla T_k(u_p) - \theta \nabla \varphi] dx \\ &\quad + \int_{\Omega} \theta^{p-1} \frac{|\nabla \varphi|^{p-2}}{h(u_p)^{p-1}} \nabla \varphi [\nabla T_k(u_p) - \theta \nabla \varphi] dx \end{aligned}$$

$$\geq \int_{\Omega} \theta^{p-1} \frac{|\nabla \varphi|^{p-2}}{h(u_p)^{p-1}} \nabla \varphi [\nabla T_k(u_p) - \theta \nabla \varphi] dx.$$

Since $|\nabla \varphi| \leq h(u)$, so

$$\left| \int_{\Omega} \theta^{p-1} \frac{|\nabla \varphi|^{p-2}}{h(u_p)^{p-1}} \nabla \varphi [\nabla T_k(u_p) - \theta \nabla \varphi] dx \right| \leq \int_{\Omega} \left(\theta \frac{h(u)}{h(u_p)} \right)^{p-1} |\nabla T_k(u_p) - \theta \nabla \varphi| dx,$$

and as $p \rightarrow +\infty$ and then $\theta \rightarrow 1$, we have

$$\int_{\Omega} \left(\theta \frac{h(u)}{h(u_p)} \right)^{p-1} |\nabla T_k(u_p) - \theta \nabla \varphi| dx \rightarrow 0,$$

hence

$$\lim_{\theta \rightarrow 1} \lim_{p \rightarrow +\infty} \int_{\Omega} \frac{|\nabla u_p|^{p-2}}{h(u_p)^{p-1}} \nabla u_p [\nabla T_k(u_p) - \theta \nabla \varphi] dx \geq 0.$$

We pass to the limit over p on the other integrals in (3.32) and then over θ and lastly over k , consequently we have

$$\int_{\Omega} a(x, u, \nabla u) [\nabla u - \nabla \varphi] dx + \int_{\Omega} g(x, u, \nabla u)(u - \varphi) dx \leq \int_{\Omega} f(u - \varphi) dx. \quad (3.33)$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$ such that $|\nabla \varphi| \leq h(u)$.

References

- [1] R. ADAMS, *Sobolev spaces*, Pure and Applied Mathematics, **65**. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.
- [2] A. ATLAS, F. KARAMI AND D. MESKINE, *The Perona-Malik inequality and application to image denoising*, Nonlinear Anal. Real World Appl. **18** (2014), 57–68.
- [3] A. BENKIRANE AND A. ELMAHI, *Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Anal. **28** 11 (1997), 1769–1784.
- [4] A. ELMAHI AND D. MESKINE, *Non-linear elliptic problems having natural growth and L^1 data in Orlicz spaces*, Ann. Mat. Pura Appl. **184** 2 (2005), 161–184.
- [5] A. ELMAHI AND D. MESKINE, *Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces*, Abstr. Appl. Anal. **12** (2004), 1031–1045.
- [6] J.-P. GOSSEZ, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974), 163–205.
- [7] J.-P. GOSSEZ, *Some approximation properties in Orlicz-Sobolev spaces*, Studia Math. **74** 1 (1982), 17–24.
- [8] J.-P. GOSSEZ, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983), Proc. Sympos. Pure Math., **45**, Part 1, Amer. Math. Soc., Providence, RI, (1986), 455–462.
- [9] A. KUFNER, O. JOHN AND S. FUCIK, *Function spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.
- [10] M. KRASNOSELSKII AND YA. RUTICKII, *Convex functions and Orlicz spaces*, Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen 1961.
- [11] MIRANDA, FERNANDO; RODRIGUES, JOS FRANCISCO; SANTOS, LISA., *Evolutionary quasi-variational and variational inequalities with constraints on the derivatives*, Adv. Nonlinear Anal. **9** (2020), no. 1, 250–277.
- [12] L. PRIGOZHIN, *Variational model of sandpile growth*, European J. Appl. Math. **7** 3 (1996), 225–235.

- [13] J.-F. RODRIGUES AND L. SANTOS, *A parabolic quasi-variational inequality arising in a superconductivity model*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** 1 (2000), 153–169.