Moroccan J. of Pure and Appl. Anal. (MJPAA)

Volume 7(1), 2021, Pages 43–49

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2021-0005](https://doi.org/10.2478/mjpaa-2021-0005)

On a chain of reproducing kernel Cartan subalgebras

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ABSTRACT. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{j}^*$, the dual of $\mathfrak{j}$, $\mathfrak{j}^\vee$ the bidual of $\mathfrak{j}$ and $B(.,.)$ the restriction to $\mathfrak{j}$ of the Killing form of $\mathfrak{g}$. In this work, we will construct a chain of reproducing kernel Cartan subalgebras ordered by inclusion.

Mathematics Subject Classification (2020). 17B20, 46E22, 17B22.

Key words and phrases. Cartan subalgebra, Root systems, Reproducing kernel.

1. Introduction

Let E be a set. A reproducing kernel Hilbert space H on E is a Hilbert space of functions from E with values in $\mathbb{C}$, for which the evaluation function in each point of E is continuous (see [1, 2, 5, 7, 8]). Thanks to *Riesz – Fréchet* theorem, one can deduce the existence of a kernel $K : E \times E \rightarrow \mathbb{C}$ such that for all $x \in E$,

$$f(x) = (f, K(., x))_H, \text{ for all } f \in H.$$

The kernel verifies

$$K(x, y) = K_y(x) = (K(., y), K(., x))_H \text{ for all } x, y \in E.$$

Received : 23 March 2020 - Accepted: 30 October 2020.

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We have another approach of the reproducing kernel Hilbert space in the case of linear transformations which is given by the theorem of Aronszajn.

Furthermore, the theorem of Aronszajn (see [10, 13]) ensures that for every positive definite kernel K , there exists a unique Hilbert space V of complex functions on E where K is the reproducing kernel. That is, for all $x \in E$,

$$f(x) = (f, K(., x))_V, \text{ for all } f \in V.$$

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{j}^*$ the dual of $\mathfrak{j}$ and $B(., .)$ the restriction to $\mathfrak{j}$ of the Killing form of $\mathfrak{g}$ (see [4, 6]).

Let's consider the Hilbert space $\mathfrak{j}^*$ with the scalar product $(., .)_{\mathfrak{j}^*}$ which is the Killing form on $\mathfrak{j}^*$, $\mathcal{F}(E)$ the set of complex functions defined on E , h a function on E with values in $\mathfrak{j}^*$ defined by $h(p) = h_p$, L the map from $\mathfrak{j}^*$ to $\mathcal{F}(E)$ such that for all $p \in E$, $f \in \mathfrak{j}^*$ with $\tilde{f} = Lf$,

$$\tilde{f}(p) = (Lf)(p) = (f, h_p)_{\mathfrak{j}^*} = f(H_p) = B(F, H_p) \text{ where } H_p, F \in \mathfrak{j} \text{ (see [15] p.3).}$$

Let's consider the kernel K defined by:

$$K(p, q) = (h_q, h_p)_{\mathfrak{j}^*} = L(h_q)(p) = B(H_q, H_p) \text{ for all } p, q \in E.$$

Since B is non-degenerate (See [6, 9, 12]), the kernel K is positive definite. Then, by the theorem of Aronszajn, there exists a unique Hilbert space where K is the kernel, denoted by h_K such that for all $p \in E$,

$$g(p) = (g, K(., p))_{h_K}, \text{ for all } g \in h_K.$$

Let $R(L)$ be the range of L . We introduce an inner product in $R(L)$ (see [3]) induced by the norm:

$$\|\tilde{f}\|_{R(L)} = \inf\{\|f\|_{\mathfrak{j}^*}; \tilde{f} = Lf\}.$$

$R(L)$ is a Hilbert space verifying the forgoing assumptions (see [10, Theorem 1 p.21]). By uniqueness of the reproducing kernel Hilbert space, we have $R(L) = h_K$ and

$$(Lf)(p) = \tilde{f}(p) = (\tilde{f}, K(., p))_{h_K}, \text{ for all } f \in \mathfrak{j}^*.$$

Considering the bidual $\mathfrak{j}^\vee$ of the Cartan subalgebra $\mathfrak{j}$, $\mathfrak{j}^\vee$ is the reproducing kernel Cartan subalgebra (RKCS) when E is identified to $\mathfrak{j}^*$. The kernel of $\mathfrak{j}^\vee$ is defined by $K(\alpha, \beta) = (h_\beta, h_\alpha)_{\mathfrak{j}^*}$ for all $\alpha, \beta \in \mathfrak{j}^*$.

In fact, for all $a \in \mathfrak{j}^\vee$, there exists a unique $a^* \in \mathfrak{j}^*$ such that $a(b^*) = B(a^*, b^*)$ for all $b^* \in \mathfrak{j}^*$ (see [12, 15]). Then, we have

$$a(b^*) = (a, K(., b^*))_{\mathfrak{j}^\vee} \text{ for all } b^* \in \mathfrak{j}^*, \text{ where } K(a^*, b^*) = (b, a)_{\mathfrak{j}^\vee}.$$

Note that B is strictly positive definite on $\mathfrak{j}^* \times \mathfrak{j}^*$. Let's consider the real semisimple Lie algebra $\mathfrak{g}^\sharp$ equipped with the involution θ . $\mathfrak{g}^\sharp$ is a Hilbert space with the scalar product $(., .)_\theta$. This Killing form leads to a map $p_\theta(x, y) = (x, y)_\theta = -B(x, \theta(y))$, for all $x, y \in \mathfrak{g}^\sharp$ (see [15]). Therefore, we define the following map: $K_{p_\theta}(x, y) = (x, y)_\theta$. K_{p_θ} is positive definite, thus by the theorem of Aronszajn, there exists a unique Hilbert space H_θ such that for all $x \in \mathfrak{g}^\sharp$, $K_{p_\theta}(., x) \in H_\theta$ and

$$f(x) = (f, K_{p_\theta}(., x))_{H_\theta}, \text{ for all } f \in H_\theta.$$

For all $x, y \in \mathfrak{g}^\sharp$, $K_{p_\theta}(x, y) = K_{p_\theta}(y)(x) = (K_{p_\theta}(., y), K_{p_\theta}(., x))_{H_\theta}$.

This Hilbert space will be called the Hilbert space of type θ and its kernel is of type θ .

In this work, we will prove some results about the Hilbert space H_θ and construct a chain of reproducing kernel Cartan subalgebras ordered by inclusion (as in [10] p.52).

2. Chain by inclusion of reproducing kernel Cartan subalgebras

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{j}^*$ the dual of $\mathfrak{j}$ and $\mathfrak{j}^\vee$, the bidual of $\mathfrak{j}$.

Let $\mathfrak{g}^\sharp$ be a real semisimple Lie algebra with an involution θ . We consider the Hilbert space of type θ and its kernel of type θ . Let U be a representation of $\mathfrak{g}^\sharp$ on H_θ . We consider for $g, x, y \in \mathfrak{g}^\sharp$, $(U_g^1)^* K_{p_\theta}(x, y)$ which means that the representation is applied to $K_{p_\theta}(x, y)$ as function of the first variable x . The function obtained will be considered as function of x but it will depend on y which will act as a parameter, where $(U_g^1)^*$ is the adjoint of U_g^1 . We have

Theorem 2.1. The function defined on $\mathfrak{g}^\sharp \times \mathfrak{g}^\sharp$ by $\Lambda_g^{U^1}(y, z) = (U_g^1)^* K_{p_\theta}(y, z)$ for all $y, z \in \mathfrak{g}^\sharp$ is positive definite if and only if U_g^1 is positive.

For all $g \in \mathfrak{g}^\sharp$, if U_g^1 is positive then $\Lambda_g^{U^1}$ is a positive definite kernel corresponding to U_g^1 and the reproducing kernel space is denoted by $H_{\Lambda_g^{U^1}}$. The kernel corresponding to $(U_g^1)^*$, the adjoint of U_g^1 is $(\Lambda_g^{U^1})^*(y, z) = \overline{(\Lambda_g^{U^1})(z, y)}$ for all $y, z \in \mathfrak{g}^\sharp$ and the kernel corresponding to $U_g^1 \circ U_{g'}^1$ is

$$(\Lambda_{gg'}^{U^1})(y, z) = (\Lambda_g^{U^1}(\cdot, z), \overline{\Lambda_{g'}^{U^1}(y, \cdot)})_{H_\theta} \text{ for all } y, z, g, g' \in \mathfrak{g}^\sharp.$$

Proof. For all $g \in \mathfrak{g}^\sharp$, let's assume that U_g^1 is positive.

Let us find the kernel $(\Lambda_g^{U^1})^*$ corresponding to the adjoint operator $(U_g^1)^*$. For $y, z, g \in \mathfrak{g}^\sharp$, $((U_g^1)^*)^* = U_g^1$, and

$$\begin{aligned} (U_g^1 K_{p_\theta}(\cdot, z), K_{p_\theta}(\cdot, y))_{H_\theta} &= (K_{p_\theta}(\cdot, z), ((U_g^1)^* K_{p_\theta}(\cdot, y))_{H_\theta}, \\ ((\Lambda_g^{U^1})^*(\cdot, z), K_{p_\theta}(\cdot, y))_{H_\theta} &= ((\Lambda_g^{U^1})(\cdot, y), K_{p_\theta}(\cdot, z))_{H_\theta}, \\ \text{then } (\Lambda_g^{U^1})^*(y, z) &= \overline{\Lambda_g^{U^1}(z, y)}. \end{aligned}$$

Consider $y_1, y_2, \dots, y_n \in \mathfrak{g}^\sharp$, $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $n \in \mathbb{N}$.
For $f = \sum_{k=1}^n c_k K_{p_\theta}(\cdot, y_k)$,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j (U_g^1 K_{p_\theta}(\cdot, y_i), K_{p_\theta}(\cdot, y_j))_{H_\theta} &= \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j ((\Lambda_g^{U^1})^*(\cdot, y_i), K_{p_\theta}(\cdot, y_j))_{H_\theta} \\ &= \sum_{i=1}^n \sum_{j=1}^n (\Lambda_g^{U^1})^*(y_j, y_i) c_i \bar{c}_j \\ &= \sum_{i=1}^n \sum_{j=1}^n \overline{\Lambda_g^{U^1}(y_i, y_j)} c_i \bar{c}_j \\ &\geq 0. \end{aligned}$$

We proved that $\overline{\Lambda_g^{U^1}}$ is positive definite thus $\Lambda_g^{U^1}$ is positive definite.

For all $g \in \mathfrak{g}^\sharp$, let's assume that $\Lambda_g^{U^1}$ is positive definite, we notice that $(U_g^1 f, f) \geq 0$ for all f of the form $f = \sum_{k=1}^n c_k K_{p_\theta}(\cdot, y_k)$. As these functions form a dense set in H_θ , every f may be approximated by them and we get $(U_g^1 f, f) \geq 0$ by approaching the limit.

We consider the adjoint operator $(U_g^1)^*$ with $g \in \mathfrak{g}^\sharp$. For $f, k \in H_\theta$, $((U_g^1)^* f, k)_{H_\theta} = (f, U_g^1 k)_{H_\theta}$ and for $x, y \in \mathfrak{g}^\sharp$, $\Lambda_g^{U^1}(x, y) = (U_g^1)^* K_{p_\theta}(x, y)$; $\Lambda_g^{U^1}$ is a function of two points x and y .

If we take any $f \in H_\theta$ and $y \in \mathfrak{g}^\sharp$, then we get

$$\begin{aligned} (f, \Lambda_g^{U^1}(\cdot, y))_{H_\theta} &= (f, (U_g^1)^* K_{p_\theta}(\cdot, y))_{H_\theta} \\ &= (U_g^1 f, K_{p_\theta}(\cdot, y))_{H_\theta} \\ &= U_g^1 f(y). \end{aligned}$$

Thus, $U_g^1 f(y) = (f, \Lambda_g^{U^1}(\cdot, y))_{H_\theta}$. We deduce that each operator U_g^1 corresponds to a kernel $\Lambda_g^{U^1}(x, y)$ for which every y , as function of x belongs to H_θ .

We shall now give the kernel corresponding to the composition $U_g^1 \circ U_{g'}^1$, with $g, g' \in \mathfrak{g}^\sharp$. Since $(U_g^1 \circ U_{g'}^1)^* = (U_{g'}^1)^* \circ (U_g^1)^*$, for $y, z \in \mathfrak{g}^\sharp$,

$$\begin{aligned} (\Lambda_{gg'}^{U^1})(y, z) &= (U_g^1 \circ U_{g'}^1)^* K_{p_\theta}(y, z) \\ &= (U_{g'}^1)^* \circ (U_g^1)^* K_{p_\theta}(y, z) \\ &= (U_{g'}^1)^* \Lambda_g^{U^1}(y, z) \\ &= (\Lambda_g^{U^1}(\cdot, z), (\Lambda_{g'}^{U^1})^*(\cdot, y))_{H_\theta} \\ &= (\Lambda_g^{U^1}(\cdot, z), \overline{(\Lambda_{g'}^{U^1})(y, \cdot)})_{H_\theta}. \end{aligned}$$

Then, $(\Lambda_{gg'}^{U^1})^*(y, z) = (\Lambda_g^{U^1}(\cdot, z), \overline{\Lambda_{g'}^{U^1}(y, \cdot)})_{H_\theta}$. □

Theorem 2.2. If we consider the representation U of $\mathfrak{g}^\sharp$ on H_θ and the kernel $\Lambda_g^{U^1}(x, y)$ with $g, x, y \in \mathfrak{g}^\sharp$, then the necessary and sufficient condition that this kernel corresponds to a bounded operator U_g^1 with lower bound m and upper bound M is: $mK_{p_\theta} \leq \Lambda_g^{U^1} \leq MK_{p_\theta}$.

Proof. We consider $g \in \mathfrak{g}^\sharp$ and we suppose that U_g^1 admits bounds not less than m and not greater than M , that means that we have

$$m(f, f)_{H_\theta} \leq (U_g^1 f, f)_{H_\theta} \leq M(f, f)_{H_\theta}, \text{ for all } f \in H_\theta.$$

It follows that $((U_g^1 - mI)f, f)_{H_\theta} \geq 0$ and $((MI - U_g^1)f, f)_{H_\theta} \geq 0$; that is, the operators $U_g^1 - mI$ and $MI - U_g^1$ are positive. By the previous results, we deduce that $\Lambda_g^{U^1} - mK_{p_\theta}$ and $MK_{p_\theta} - \Lambda_g^{U^1}$ are positive definite.

Suppose that there exists some constants m and M such that

$$mK_{p_\theta}(x, y) \leq \Lambda_g^{U^1}(x, y) \leq MK_{p_\theta}(x, y), \text{ for all } x, y \in \mathfrak{g}^\sharp.$$

Then, $0 \leq \frac{1}{M-m}(\Lambda_g^{U^1}(x, y) - mK_{p_\theta}(x, y)) \leq K_{p_\theta}(x, y)$, for all $x, y \in \mathfrak{g}^\sharp$.

We deduce that the kernel $K_1 = \frac{1}{M-m}(\Lambda_g^{U^1}(x, y) - mK_{p_\theta}(x, y))$ is positive definite and $K_1(x, y) \leq K_{p_\theta}(x, y)$ for all $x, y \in \mathfrak{g}^\sharp$, then there exists a reproducing kernel Hilbert space H_1 where K_1 is the kernel such that $H_1 \subset H_\theta$ and $\|f_1\|_{H_1} \geq \|f_1\|_{H_\theta}$ for $f_1 \in H_1$.

We introduce for all $x, y \in \mathfrak{g}^\sharp$, the operator $((U'_g)^1)f(y) = (f, K_1(\cdot, y))_{H_\theta}$ obtained from another representation U' on $\mathfrak{g}^\sharp$; the operator is positive in H_θ with a bound not greater than 1 (see [2, Theorem III p.356]), that means $0 \leq ((U'_g)^1 f, f)_{H_\theta} \leq (f, f)_{H_\theta}$. Since $(U'_g)^1$ corresponds to K_1 , then $\Lambda_g^{U^1} = (M-m)K_1 + mK_{p_\theta}$ corresponds to $U_g^1 = (M-m)(U'_g)^1 + mI$ and for all $f \in H_\theta$, the last inequalities give:

$$\begin{aligned} m(f, f)_{H_\theta} &\leq (mIf, f)_{H_\theta} + ((M-m)(U'_g)^1 f, f)_{H_\theta} \leq M(f, f)_{H_\theta} \\ &\text{and} \\ m(f, f)_{H_\theta} &\leq (U_g^1 f, f)_{H_\theta} \leq M(f, f)_{H_\theta}. \end{aligned}$$

□

Let T be a bounded linear operator from the reproducing kernel space $\mathfrak{j}^\vee$ into $H_{\Lambda^{U^1}}$. Let's introduce (see [11]) the following inner products:

$$\begin{aligned} (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^+} &= (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathfrak{j}^\vee} + (T\tilde{\beta}_1, T\tilde{\beta}_2)_{H_{\Lambda^{U^1}}} \\ &\text{and} \\ (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^-} &= (\tilde{\beta}_1, \tilde{\beta}_2)_{\mathfrak{j}^\vee} - (T\tilde{\beta}_1, T\tilde{\beta}_2)_{H_{\Lambda^{U^1}}}. \end{aligned}$$

We shall construct the reproducing kernels for $\mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^+$ and $\mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^-$ such that

$$\mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^+ \subset \mathfrak{j}^\vee \subset \mathfrak{j}^\vee[H_{\Lambda^{U^1}}]^-.$$

We consider the linear map defined by $I : H_{\Lambda^{U^1}} \longrightarrow \mathcal{F}(\mathfrak{j}^*)$
 $a \longmapsto I(a) = \check{a}$

such that for all $\beta \in \mathfrak{j}^*$, $\check{a}(\beta) = (a, T(K(\cdot, \beta)))_{H_{\Lambda^{U^1}}}$.

Theorem 2.3. For $\beta \in \mathfrak{j}^*$, $\check{a} = T^*a$ and $\|\check{a}\|_{\mathfrak{j}^\vee}^2 = (a, T(a, TK(\cdot, \beta))_{H_{\Lambda^{U^1}}})_{H_{\Lambda^{U^1}}}$. Furthermore, the following conditions are equivalent:

- 1- $K(\alpha, \beta) \geq (TK(\cdot, \beta), TK(\cdot, \alpha))_{H_{\Lambda^{U^1}}}$.
- 2- $\|T\| \leq 1$.
- 3- $\|\check{a}\|_{\mathfrak{j}^\vee} \leq \|a\|_{H_{\Lambda^{U^1}}}$.

Proof. Let be $\beta \in \mathfrak{j}^*$ and $a \in H_{\Lambda^{U^1}}$. $\check{a}(\beta) = (a, T(K(\cdot, \beta)))_{H_{\Lambda^{U^1}}} = (T^*a, K(\cdot, \beta))_{\mathfrak{j}^\vee}$. Since $\check{a} \in \mathfrak{j}^\vee$, $\check{a}(\beta) = (\check{a}, K(\cdot, \beta))_{\mathfrak{j}^\vee}$, then $\check{a} = T^*a$ and

$$\begin{aligned} \|\check{a}\|_{\mathfrak{j}^\vee}^2 &= (\check{a}, \check{a})_{\mathfrak{j}^\vee} \\ &= (T^*a, (a, T(K(\cdot, \cdot)))_{H_{\Lambda^{U^1}}})_{H_{\Lambda^{U^1}}} \\ &= (a, T(a, TK(\cdot, \cdot)))_{H_{\Lambda^{U^1}}} \end{aligned}$$

Now, we show the equivalences.

1 $\implies$ 3.

We consider the following positive definite kernel K_T defined by:

$K_T(\alpha, \beta) = (TK(\cdot, \beta), TK(\cdot, \alpha))_{H_{\Lambda U^1}}$ on $j^* \times j^*$, the reproducing kernel space where K_T is the kernel will be denoted by h_{K_T} . Since the range of I verifies $R(I) = h_{K_T}$, we obtain $\|\check{a}\|_{h_{K_T}} \leq \|a\|_{H_{\Lambda U^1}}$. If 1 is true, we have $\|\check{a}\|_{j^\vee} \leq \|\check{a}\|_{h_{K_T}}$ because we know that for two positive definite kernels $K^{(1)}(p, q)$ and $K^{(2)}(p, q)$ defined on any set $E \times E$, if $K^{(1)}(p, q) \leq K^{(2)}(p, q)$ on $E \times E$, then for the reproducing kernels Hilbert spaces $H_{K^{(1)}}$ and $H_{K^{(2)}}$, we have $\|f\|_{H_{K^{(1)}}} \geq \|f\|_{H_{K^{(2)}}}$ for any $f \in H_{K^{(1)}}$ (see [2, Theorem I p.354]). So, we get $\|\check{a}\|_{j^\vee} \leq \|\check{a}\|_{h_{K_T}} \leq \|a\|_{H_{\Lambda U^1}}$.

3 $\implies$ 2

Assume 3. For $a \in H_{\Lambda U^1}$ and $\check{a} \in j^\vee$ such that $\|\check{a}\|_{j^\vee} \leq \|a\|_{H_{\Lambda U^1}}$, we get $\|T\| \leq 1$ (See [3]).

2 $\implies$ 1

Assume 2. We know that for two positive definite kernels $K^{(1)}(p, q)$ and $K^{(2)}(p, q)$ on $E \times E$, $H_{K^{(1)}} \subset H_{K^{(2)}}$ if and only if there exists a positive constant γ , such that $K^{(1)}(p, q) \leq \gamma^2 K^{(2)}(p, q)$ and the minimum of such γ coincides with the norm of the inclusion map from $H_{K^{(1)}}$ into $H_{K^{(2)}}$ (see [14, Theorem 5.1 p.66]). By this result, we have that the map $\check{a} = T^*a$ is into $j^\vee$ and onto $j^\vee$ if and only if $K_T(\alpha, \beta) = (TK(\cdot, \beta), TK(\cdot, \alpha))_{H_{\Lambda U^1}} \leq K(\alpha, \beta)$ (see [10, Lemna 3 p.51]), hence 1 is proved. $\square$

For $\check{a} \in j^\vee[H_{\Lambda U^1}]^+$, $\|\check{a}\|_{j^\vee[H_{\Lambda U^1}]^+} \geq \|\check{a}\|_{j^\vee}$, then there exists a reproducing kernel $K_{H_{\Lambda U^1}}^+$ for the space $j^\vee[H_{\Lambda U^1}]^+$ such that $K_{H_{\Lambda U^1}}^+(\alpha, \beta) \leq K(\alpha, \beta)$ on $j^* \times j^*$ (see [2, Theorem II p.355]). Let's consider the kernel K defined on $j^* \times j^*$ by $K(\alpha, \beta) = (h_\beta, h_\alpha)_{j^*} = (K(\cdot, \beta), K(\cdot, \alpha))_{j^\vee}$.

By the following theorem, we construct the reproducing kernel $K_{H_{\Lambda U^1}}^+(\alpha, \beta)$ using $K(\alpha, \beta)$ for $\alpha, \beta \in j^*$.

Theorem 2.4. Given that $\{\tilde{K}(\cdot, \beta), \beta \in j^*\} \subset j^\vee$, the set of functions satisfying the following equation

$$K(\alpha, \beta) = \tilde{K}(\alpha, \beta) + (T\tilde{K}(\cdot, \beta), TK(\cdot, \alpha))_{H_{\Lambda U^1}} \quad (1).$$

Then the kernel $\tilde{K}(\alpha, \beta)$ is a solution of the functional equation (1) if and only if $\tilde{K}(\alpha, \beta)$ is the reproducing kernel of $j^\vee[H_{\Lambda U^1}]^+$.

Proof. For any $\gamma, \alpha \in j^*$, we have

$$(K(\cdot, \gamma), K(\cdot, \alpha))_{j^\vee} = (\tilde{K}(\cdot, \gamma), K(\cdot, \alpha))_{j^\vee} + (T\tilde{K}(\cdot, \gamma), T\tilde{K}(\cdot, \alpha))_{j^*}$$

$$\Leftrightarrow K(\cdot, \gamma) = \tilde{K}(\cdot, \gamma) + T^*T\tilde{K}(\cdot, \gamma).$$

For all $\tilde{\beta} \in j^\vee$,

$$(\tilde{\beta}, K(\cdot, \gamma))_{j^\vee} = (\tilde{\beta}, \tilde{K}(\cdot, \gamma))_{j^\vee} + (\tilde{\beta}, T^*T\tilde{K}(\cdot, \gamma))_{j^\vee}$$

$$\Leftrightarrow \tilde{\beta}(\gamma) = (\tilde{\beta}, \tilde{K}(\cdot, \gamma))_{j^\vee[H_{\Lambda U^1}]^+}.$$

If for any $\tilde{\beta} \in j^\vee$, $\|\tilde{\beta}\|_{j^\vee} \geq \|T\tilde{\beta}\|_{H_{\Lambda U^1}}$ with the equality if and only if $\tilde{\beta} = 0$, then we can introduce the pre-Hilbert space H' equipped with the inner product

$$(\tilde{\beta}_1, \tilde{\beta}_2)_{H'} = (\tilde{\beta}_1, \tilde{\beta}_2)_{j^\vee} - (T\tilde{\beta}_1, T\tilde{\beta}_2)_{H_{\Lambda U^1}}.$$

For any fixed $\alpha \in j^*$, $\tilde{\beta}(\alpha)$ the evaluation of $\tilde{\beta}$ at point α is bounded on H' and for any Cauchy sequence $\{\tilde{\beta}_n\}$ in H' , the condition $\tilde{\beta}_n \rightarrow 0$ on j^* implies $\|\tilde{\beta}_n\|_{H'} \rightarrow 0$. By the completion, $\overline{H'}$ is unique and admits the reproducing kernel.

We shall denote the completion of H' by $j^\vee[H_{\Lambda U^1}]^-$, if the conditions above are satisfied.

Then, for any $\tilde{\beta} \in j^\vee$, $\|\tilde{\beta}\|_{j^\vee[H_{\Lambda U^1}]^-} \leq \|\tilde{\beta}\|_{j^\vee}$ so $j^\vee$ is a subspace of $j^\vee[H_{\Lambda U^1}]^-$. We shall denote the reproducing kernel of $j^\vee[H_{\Lambda U^1}]^-$ by $K_{[H_{\Lambda U^1}]}^-$. For this kernel, we obtain a similar result by exchanging + and - in (1). □

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