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Existence result for a nonlinear elliptic problem by topological degree in Sobolev spaces with variable exponent

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ABSTRACT. The aim of this paper is to establish the existence of solutions for a nonlinear elliptic problem of the form

$$\begin{cases} A(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $A(u) = -\operatorname{div}(x, u, \nabla u)$ is a Leray-Lions operator and $f \in W^{-1,p'(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$.

Our technical approach is based on topological degree method and the theory of variable exponent Sobolev spaces.

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1. Introduction

In this paper we are concerned with the following problem

$$\begin{cases} A(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$, A is a Leray-Lions operator in the form $A(u) = -\operatorname{div}(x, u, \nabla u)$ defined on a subset of $W_0^{1,p(\cdot)}(\Omega)$ and $f \in W^{-1,p'(\cdot)}(\Omega)$ where p is a continuous function on $\overline{\Omega}$ such that

$$1 < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty.$$

The goal of the paper is to prove the existence of solutions for the problem (1.1) in the variational framework by applying the topological degree constructed by Ait Hammou and Azroul in [1].

The use of topological degree theory is an effective tool in the study of nonlinear equations. As the most important stages in the history of this theory, Brouwer [11] had introduced it for continuous maps in finite dimensional Euclidean space. Leray and Schauder [20] developed it for compact operators in infinite dimensional Banach spaces. Browder [12, 13] introduced a degree theory for monotone type operators in reflexive Banach spaces. Berkovits [9] defined a topological degree for operators of generalized (S_+) type as an extension of the Leray-Schauder degree in order to solve abstract Hammerstein equation. This last degree is recently used for the study of some nonlinear elliptic problems in Sobolev spaces with variable exponent without and with weight [2, 3, 4, 5].

Many authors consider the existence of solutions for (1.1) by different methods in some spaces. Existence result of this kinds of problems have been given firstly by Boccardo-Galloüet [10] in the setting of classical Sobolev spaces $W^{1,p}(\Omega)$ where the growth of the function a with respect to u and ∇u was assumed to be polynominal. However, Dong [16] proved the existence of solution to problem (1.1) in the setting of Orlicz-Sobolev space W^1L_M where L_M is an Orlicz space and M , the N-function that defines L_M , is related to the growth of the function a . The same results are also studied by Azroul et al [7] in the framework of Sobolev spaces with variable exponent using the theory of the calculus of variations and pseudo-monotonicity, in the unilateral case. In [6], the authors proved the existence of solutions for the problem (1.1) in the setting of Musielak-Orlicz-Sobolev space $W^1L_\varphi(\Omega)$, under the assumption that the conjugate function ψ of φ has the Δ_2 property.

Our proposal to study this problem in the context of Sobolev spaces with variable exponent is motivated by his modeling of several physical phenomena, especially those related to non-Newtonian fluids [22].

This article is organized as follows: In section 2, we recall some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. The third section we introduce some classes of operators and topological degree. In section 4, we state and prove our main results.

2. Mathematical Preliminaries

In the sequel, let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with a smooth boundary $\partial\Omega$.

We recall in this section some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ (see [8, 15, 17, 18] for more details).

Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}); \inf_{x \in \overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^- := \min\{h(x), x \in \overline{\Omega}\}, \quad h^+ := \max\{h(x), x \in \overline{\Omega}\}.$$

In this paper, we suppose that $p \in C_+(\overline{\Omega})$ such that

$$1 < p^- \leq p(x) \leq p^+ < \infty$$

and $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there exists $\alpha > 0$ such that for all $x, y \in \Omega$, $x \neq y$, one has

$$|p(x) - p(y)| \leq \frac{\alpha}{\log(e + 1/|x - y|)}$$

We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \rho_{p(\cdot)}(u) < \infty\}.$$

where

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

We define a norm, the so-called *Luxemburg norm*, on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 / \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

Endowed with the Luxemburg norm, $L^{p(\cdot)}$ is a Banach spaces, separable, reflexive and uniformly convex.

The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, Hölder inequality holds

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

If $p(\cdot), q(\cdot) \in C_+(\overline{\Omega})$, $q(\cdot) \leq p(\cdot)$ a.e. in Ω then there exists the continuous embedding $L^{p(\cdot)}(\Omega) \rightarrow L^{q(\cdot)}(\Omega)$.

We say that a sequence $(u_n) \subset L^{p(\cdot)}(\Omega)$ converges to $u \in L^{p(\cdot)}(\Omega)$ in the modular sense, denote $u_n \xrightarrow{\text{mod}} u$ in $L^{p(\cdot)}$, if there exists $\lambda > 0$ such that

$$\rho_{p(\cdot)}\left(\frac{u_n - u}{\lambda}\right) \rightarrow 0, \text{ when } n \rightarrow \infty.$$

Next, we define

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) / |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

and $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. By the Poincaré inequality: there is a positive constant C , such that

$$\|u\|_{p(\cdot)} \leq C \|\nabla u\|_{p(\cdot)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega),$$

$W_0^{1,p(\cdot)}(\Omega)$ has a norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)},$$

which is equivalent to the norm $\|\cdot\|_{W^{1,p(\cdot)}}$.

The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a Banach space separable and reflexive .

We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding

$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = +\infty$ if $p(x) \geq N$. Moreover we have the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$.

We say that a sequence $(u_n) \subset W^{1,p(\cdot)}(\Omega)$ converges to $u \in W^{1,p(\cdot)}(\Omega)$ in the modular sense, denote $u_n \xrightarrow{\text{mod}} u$ in $W^{1,p(\cdot)}$, if there exists $\lambda > 0$ such that

$$\rho_{p(\cdot)}\left(\frac{\nabla u_n - \nabla u}{\lambda}\right) \rightarrow 0, \text{ when } n \rightarrow \infty.$$

For any $v \in [W_0^{1,p(\cdot)}(\Omega)]'$, there exists $(v_0, v) \in L^{p'(\cdot)} \times [L^{p'(\cdot)}]^N$ such that, for any $u \in W_0^{1,p(\cdot)}(\Omega)$,

$$\langle v, u \rangle = \int_{\Omega} (v_0 u + v \cdot \nabla u) dx.$$

In other words, the dual space of $W_0^{1,p(\cdot)}(\Omega)$, denoted $W^{-1,p'(\cdot)}(\Omega)$, is equipped with the norm

$$\|v\| = \inf \left\{ \|v_0\|_{p'(\cdot)} + \sum_{i=1}^N \|v_i\|_{p'(\cdot)} \right\},$$

where the infimum is taken on all possible decompositions $v = v_0 - \text{div} F$ with $v_0 \in L^{p'(\cdot)}(\Omega)$ and $F = (v_1, \dots, v_N) \in (L^{p'(\cdot)}(\Omega))^N$.

We say that a sequence $(u_n) \subset W^{-1,p'(\cdot)}(\Omega)$ converges to $u \in W^{-1,p'(\cdot)}(\Omega)$ in the modular sense, denote $u_n \xrightarrow{\text{mod}} u$ in $W^{-1,p'(\cdot)}$, if u_n and u have representations

$$u_n = g_0^{(n)} - \text{div} g^{(n)}, \quad u = g_0 - \text{div} g$$

such that $g_i^{(n)}, g_i \in L^{p'(\cdot)}(\Omega)$ and $g_i^{(n)} \xrightarrow{\text{mod}} g_i$ in $L^{p'(\cdot)}$ for $i = 1, 2, \dots, N$.

If $(u_n) \subset L^{p(\cdot)}(\Omega)$ and $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold true

$$\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+} \quad (2.1)$$

$$\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$$

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{p(\cdot)} = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \rho_{p(\cdot)}(u_n - u) = 0 \quad (2.2)$$

From (2.2), we can deduce that norm convergence and modular convergence are equivalent.

We will need to use the following two lemmas

Lemma 2.1. *If $(u_n) \subset L^{p(\cdot)}(\Omega)$, $(v_n) \subset L^{p'(\cdot)}(\Omega)$, $u_n \rightarrow u \in L^{p(\cdot)}(\Omega)$ in $L^{p(\cdot)}(\Omega)$ and $v_n \rightarrow v \in L^{p'(\cdot)}(\Omega)$ a.e. and $v_n \rightharpoonup v$ in $L^{p'(\cdot)}(\Omega)$, then $u_n v_n \rightarrow uv$ in $L^1(\Omega)$.*

Lemma 2.2. *If $(u_n) \subset L^{p(\cdot)}(\Omega)$ with $u_n \rightarrow u \in L^{p(\cdot)}(\Omega)$ a.e., $u_n, u \geq 0$ a.e. and $|cu_n(x)|^{p(x)} \leq h(x)$ a.e. for some $h \in L^1(\Omega)$ and $c > 0$ then $u_n \rightarrow u$ in $L^{p(\cdot)}(\Omega)$.*

3. Some classes of operators and topological degree

Definition 3.1. *A mapping $F : D_F \subset W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ is*

- *pseudomonotone, denote $F \in (PM)$, if the conditions*
 $(u_n) \subset D_F$, $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$, $F(u_n) \rightharpoonup \chi$ in $W^{-1,p'(\cdot)}(\Omega)$ and $\limsup_{n \rightarrow \infty} \langle F(u_n), u_n \rangle \leq \langle \chi, u \rangle$
imply that $u \in D_F$, $\chi = F(u)$ and $\langle F(u_n), u_n \rangle \rightarrow \langle F(u), u \rangle$.
- *of the class (MOD), denote $F \in (MOD)$, if the conditions*
 $(u_n) \subset D_F$, $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$, $F(u_n) \rightharpoonup \chi$ in $W^{-1,p'(\cdot)}(\Omega)$ and $\limsup_{n \rightarrow \infty} \langle F(u_n), u_n \rangle \leq \langle \chi, u \rangle$
imply that $u \in D_F$, $\chi = F(u)$ and there exists a subsequence $(u_{n'})$ such that $u_{n'} \xrightarrow{\text{mod}} u$ in $W_0^{1,p(\cdot)}(\Omega)$ and $F(u_{n'}) \xrightarrow{\text{mod}} F(u)$ in $W^{-1,p'(\cdot)}(\Omega)$.
- *strongly quasibounded, denote $F \in (QB)$, if the conditions*
 $(u_n) \subset D_F$ bounded and $\limsup_{n \rightarrow \infty} \langle F(u_n), u_n - \bar{u} \rangle$ is bounded from above for some $\bar{u} \in W_0^{1,p(\cdot)}(\Omega)$
imply that $(F(u_n))$ is bounded in $W^{-1,p'(\cdot)}(\Omega)$.

Let $Y = W_0^{1,p(\cdot)}(\Omega)$ and $Z = Y^* = W^{-1,p'(\cdot)}(\Omega)$. We define the class $\mathcal{F}$ of *admissible mappings* and the class $\mathcal{H}$ of *admissible homotopies* as follows:

$F : D_F \subset Y \rightarrow Z$ belongs to $\mathcal{F}$, if

(a): F is a strongly quasibounded mapping of the class (MOD).

$F : D_F \subset Y \rightarrow Z$ belongs to $\mathcal{F}^a$, if $F \in \mathcal{F}$ and the following conditions hold:

- (b): if $(u_n) \subset D_F$ is bounded, $t_n \rightarrow 0^+$ and $\langle t_n F(u_n), u_n - \bar{u} \rangle$ is bounded from above for some $\bar{u} \in Y$, then $(t_n F(u_n)) \subset Z$ is bounded,
- (c): if $(u_n) \subset D_F$, $u_n \rightharpoonup u \in Y$, $t_n \rightarrow 0^+$, $t_n F(u_n) \rightharpoonup \chi \in Z$ and $\limsup \langle t_n F(u_n), u_n \rangle \leq \langle \chi, u \rangle$, then $\langle t_n F(u_n), u_n \rangle \rightarrow \langle \chi, u \rangle$,
- (d): if $(u_n) \subset D_F$, $u_n \rightarrow u$ in Y , $t_n \rightarrow 0^+$, $t_n F(u_n) \rightharpoonup \chi$ in Z and $\limsup \langle t_n F(u_n), u_n \rangle \leq \langle \chi, u \rangle$, then $t_n F(u_n) \rightarrow 0$ in Z .

The homotopy $H : D_H \rightarrow Z$ belongs to $\mathcal{H}$, if H is a strongly quasibounded homotopy of the class (MOD).

Lemma 3.1 ([1, Lemma 3.4]). *If $F, G \in \mathcal{F}^a$, then $H(t, u) = tF(u) + (1 - t)G(u)$ belongs to $\mathcal{H}$ with*

$$D_{H_t} = \begin{cases} D_F \cap D_G & \text{if } 0 < t < 1, \\ D_G & \text{if } t = 0, \\ D_F & \text{if } t = 1. \end{cases}$$

Theorem 3.1 ([1, Remark 3.5]). *For $F \in \mathcal{F}$, $G \subset Y$ open and bounded in Y , $f \in Z$ and $f \notin F(\partial_Y G)$ there exists an integer $d(F, G, f)$ (which is the degree function) satisfying the conditions:*

- (1) (Existence) *if $d(F, G, f) \neq 0$, then $f \in F(G)$,*
- (2) (Additivity) *if $G_1, G_2 \in G$ are open and bounded, $f \notin F(\bar{G} \setminus (G_1 \cup G_2))$, $G_1 \cap G_2 = \emptyset$, then*

$$d(F, G, f) = d(F, G_1, f) + d(F, G_2, f),$$
- (3) (Homotopy invariance) *if $H \in \mathcal{H}$, $f \in Z$ and $f \notin H([0, 1] \times \partial_Y G)$, then*

$$d(H(t, \cdot), G, f) = \text{constant for all } t \in [0, 1],$$
- (4) (Normalization) *There exists a normalising map $K \in \mathcal{F}^a$ such that if $f \in Z$, $f \notin K(\partial_Y G)$ and $f \in K(G)$, then*

$$d(K, G, f) = 1.$$

Any mapping $K \in \mathcal{F}^a$ satisfying

$$\langle K(u), u \rangle > 0, \text{ when } u \neq 0, \text{ and } K(0) = 0$$

can be chosen as a normalising map.

Using the conditions (1) – (4) for the degree function, we can deduce, as in [1], some standard properties of the degree.

Property 1 ([1, Property 1]). *Let $F, T \in \mathcal{F}^a$, $G \subset Y$ open and bounded in Y , $F/\partial_Y G = T/\partial_Y G$ and $f \in Z$. If $f \notin F(\partial_Y G)$, then*

$$d(F, G, f) = d(T, G, f).$$

Property 2 ([1, Property 3]). *Let $F \in \mathcal{F}$, $G \subset Y$ open and bounded in Y and $u_0 \in G$. Define a mapping*

$s : Y \rightarrow Y$, $s(u) = u - u_0$. If $0 \notin F(\partial_Y G)$, then

$$d(F, G, 0) = d(F \circ s^{-1}, s(G), 0).$$

4. Main result

4.1. Basic assumptions and technical Lemmas. Let $A : W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ be the nonlinear operator of Leray-Lions, which is defined by

$$A(u) = -\operatorname{div} a(x, u, \nabla u).$$

i.e.

$$\langle A(u), v \rangle = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v \, dx \text{ for all } v \in W_0^{1,p(\cdot)}(\Omega). \quad (4.1)$$

The function a is assumed to satisfy the conditions:

(A₁): $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Caratheodory function, i.e., $a(\cdot, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $a(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$.

(A₂): There exist a positive function $b(x)$ in $L^{p'(\cdot)}(\Omega)$ and constant $c > 0$ such that

$$|a(x, \eta, \xi)| \leq b(x) + c(|\eta|^{p(x)-1} + |\xi|^{p(x)-1}) \text{ for all } (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N \text{ and a.e. } x \in \Omega.$$

(A₃): $(a(x, \eta, \xi) - a(x, \eta, \xi')) \cdot (\xi - \xi') > 0$ a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N, \xi \neq \xi'$.

(A₄): There exist a constant $c' > 0$ such that $a(x, \eta, \xi) \cdot \xi \geq c' |\xi|^{p(x)}$.

Lemma 4.1. Assume that (A₁) – (A₄) hold, and let (u_n) be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$ and $u_n \rightarrow u$ a.e in Ω and

$$\int_{\Omega} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \cdot \nabla (u_n - u) dx \rightarrow 0. \quad (4.2)$$

Then $u_n \rightarrow u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$.

Proof. See the proof of Lemma 3.2 in [19] taking the weight function $\rho \equiv 1$.

Lemma 4.2. Assume that the conditions (A₁) – (A₃) hold. Then the operator A defined by (4.1) is pseudomonotone.

Proof. Let (u_n) be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p(\cdot)}(\Omega) \\ Au_n \rightharpoonup \chi & \text{in } W^{-1,p'(\cdot)}(\Omega) \\ \limsup_{n \rightarrow \infty} \langle Au_n, u_n \rangle \leq \langle \chi, u \rangle. \end{cases} \quad (4.3)$$

We will prove that

$$\chi = Au \text{ and } \langle Au_n, u_n \rangle \longrightarrow \langle \chi, u \rangle \text{ as } n \rightarrow +\infty.$$

Since (u_n) is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$, then, by the reflexivity of $W_0^{1,p(\cdot)}(\Omega)$ and the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, there is a subsequence still denoted by (u_n) such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega),$$

$$u_n \rightarrow u \text{ in } L^{p(\cdot)}(\Omega) \text{ and a.e in } \Omega.$$

By the growth condition (A₂), the sequence $(a(x, u_n, \nabla u_n))$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, therefore there exists a function $\varphi \in (L^{p'(\cdot)}(\Omega))^N$ such that

$$a(x, u_n, \nabla u_n) \rightharpoonup \varphi \text{ in } (L^{p'(\cdot)}(\Omega))^N \text{ as } n \rightarrow \infty. \quad (4.4)$$

It is clear that, for all $v \in W_0^{1,p(\cdot)}(\Omega)$,

$$\begin{aligned} \langle \chi, v \rangle &= \lim_{n \rightarrow \infty} \langle Au_n, v \rangle \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla v \, dx \\ &= \int_{\Omega} \varphi \cdot \nabla v \, dx. \end{aligned} \quad (4.5)$$

by combining (4.3) and (4.5), we have

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n \rangle = \limsup_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \leq \int_{\Omega} \varphi \cdot \nabla u dx.$$

Therefore

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \leq \int_{\Omega} \varphi \cdot \nabla u dx. \quad (4.6)$$

On the other hand, thanks to assumption (A_3) we have

$$\int_{\Omega} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u) dx > 0. \quad (4.7)$$

then

$$\begin{aligned} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx &\geq - \int_{\Omega} a(x, u_n, \nabla u) \cdot \nabla u dx + \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u dx \\ &\quad + \int_{\Omega} a(x, u_n, \nabla u) \cdot \nabla u_n dx, \end{aligned}$$

by (4.4), we get

$$\liminf_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \geq \int_{\Omega} \varphi \cdot \nabla u dx.$$

This implies, by using (4.6) that

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx = \int_{\Omega} \varphi \cdot \nabla u dx. \quad (4.8)$$

By combining (4.5) and (4.8), we obtain

$$\langle Au_n, u_n \rangle \longrightarrow \langle \chi, u \rangle \text{ as } n \rightarrow +\infty.$$

On the other hand, by (4.8), and the fact that $a(x, u_n, \nabla u) \longrightarrow a(x, u, \nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$ we deduce that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u) dx = 0,$$

by Lemma 4.1, we obtain

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p(\cdot)}(\Omega),$$

then, for a subsequence still denoted (u_n) we have $\nabla u_n \rightarrow \nabla u$ a.e. in Ω .

Since a is Carathéodory function we can write

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{in } (L^{p'(\cdot)}(\Omega))^N,$$

by consequent, $\chi = Au$ which allows to conclude that the operator A is pseudomonotone.

Lemma 4.3. Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$ is bounded, $(t_n) \subset [0, 1]$ and $(\langle t_n A(u_n), u_n - \bar{u} \rangle)$ is bounded from above some $\bar{u} \in W_0^{1,p(\cdot)}(\Omega)$, then the sequence $(t_n a(x, u_n, \nabla u_n))$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$.

Proof. Let $w \in (L^{p(\cdot)}(\Omega))^N$ be arbitrary. By condition (A_3) we have:

$$t_n(a(x, u_n, \nabla u_n) - a(x, u_n, w)) \cdot (\nabla u_n - w) \geq 0.$$

Consequently:

$$\int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot (w - \nabla \bar{u}) dx \leq \langle t_n A(u_n), u_n - \bar{u} \rangle + \int_{\Omega} t_n a(x, u_n, w) \cdot (w - \nabla u_n) dx \quad (4.9)$$

The first term on the right remains bounded from above by the assumption. Moreover, by (A_2) , we have:

$$|a(x, u_n, w)|^{p'(x)} \leq \text{const}(|b(x)|^{p'(x)} + |u_n|^{p(x)} + |w|^{p(x)}). \quad (4.10)$$

If $\|a(x, u_n, w)\|_{p'(\cdot)} \leq 1$, then $\|t_n a(x, u_n, w)\|_{p(\cdot)} \leq 1$.

Suppose that $\|a(x, u_n, w)\|_{p'(\cdot)} > 1$, then we have by (2.1) and (4.10)

$$\begin{aligned} \|a(x, u_n, w)\|_{p'(\cdot)}^{p'^-} &\leq \rho_{p'(\cdot)}(a(x, u_n, w)) \\ &\leq \text{const}(\rho_{p'(\cdot)}(b) + \rho_{p(\cdot)}(u_n) + \rho_{p(\cdot)}(w)). \end{aligned}$$

Hence $\{a(x, u_n, w)\}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, which implies that the second term on the right in (4.9) is also bounded. By the theorem of Banach-Steinhaus, the sequence $(t_n a(x, u_n, \nabla u_n))$ remains bounded in $(L^{p'(\cdot)}(\Omega))^N$.

Lemma 4.4. Assume that the conditions $(A_1) - (A_2)$ hold. If the sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$, $u_n \rightarrow u \in W_0^{1,p(\cdot)}$ in $W_0^{1,p(\cdot)}$, then $t_n a(x, u_n, \nabla u_n) \rightarrow ta(x, u, \nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$.

Proof. Since $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$, then $u_n \rightarrow u$ and $\nabla u_n \rightarrow \nabla u$ in $L^{p(\cdot)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(\cdot)}(\Omega)$ and g in $(L^{p(\cdot)}(\Omega))^N$ such that

$$\begin{aligned} u_k(x) &\rightarrow u(x) \text{ and } \nabla u_k(x) \rightarrow \nabla u(x), \\ |u_k(x)| &\leq h(x) \text{ and } |\nabla u_k(x)| \leq g(x) \end{aligned}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since a satisfies the Carathéodory condition and $t_k \rightarrow t$, we obtain that

$$t_k a(x, u_k(x), \nabla u_k(x)) \rightarrow ta(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega. \quad (4.11)$$

It follows from (A_2) that

$$|t_k a(x, u_k(x), \nabla u_k(x))| \leq b(x) + c(|h(x)|^{p(x)-1} + |g(x)|^{p(x)-1}) \quad (4.12)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. Since

$$b + |h|^{p(x)-1} + |g(x)|^{p(x)-1} \in L^{p'(x)}(\Omega), \quad (4.13)$$

and taking into account the equality

$$\rho_{p'(\cdot)}(t_k a(x, u_k(x), \nabla u_k(x)) - ta(x, u(x), \nabla u(x))) = \int_{\Omega} |t_k a(x, u_k(x), \nabla u_k(x)) - ta(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the dominated convergence theorem and the equivalence (2.2) imply that

$$t_k a(x, u_k, \nabla u_k) \rightarrow ta(x, u, \nabla u) \text{ in } (L^{p'(\cdot)}(\Omega))^N$$

By contradiction argument,

$$t_n a(x, u_n, \nabla u_n) \rightarrow ta(x, u, \nabla u) \text{ in } (L^{p'(\cdot)}(\Omega))^N.$$

Lemma 4.5. *Assume that the conditions $(A_1) - (A_2)$ hold. If the sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$, $u_n \rightarrow u \in L^{p(\cdot)}$ a.e., $\nabla u_n \rightarrow \nabla u \in L^{p(\cdot)}$ a.e., $\nabla u_n \rightarrow \nabla u$ in $L^{p(\cdot)}(\Omega)$, $t_n \rightarrow t \in [0, 1]$ and $t_n a(x, u_n, \nabla u_n) \rightarrow ta(x, u, \nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$ then*

$$t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n \rightarrow ta(x, u, \nabla u) \cdot \nabla u$$

in $L^1(\Omega)$.

Proof. Denote $v_n(x) = t_n a(x, u_n, \nabla u_n)$ and $v = ta(x, u, \nabla u)$.

Since $u_n \rightarrow u$ a.e., $\nabla u_n \rightarrow \nabla u$ a.e. and $t_n \rightarrow t$ and in virtue to the assumption (A_2) , we deduce that $v_n \rightarrow v$ a.e.

We have also $v_n \rightharpoonup v$ in $(L^{p'(\cdot)}(\Omega))^N$ and $\nabla u_n \rightarrow \nabla u$ in $L^{p(\cdot)}(\Omega)$.

Then, by Lemma 2.1, $v_n \cdot \nabla u_n \rightarrow v \cdot \nabla u$ in $L^1(\Omega)$ i.e.

$$t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n \rightarrow ta(x, u, \nabla u) \cdot \nabla u$$

in $L^1(\Omega)$.

Lemma 4.6. *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$, $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$, $\{t_n\} \subset [0, 1]$, $t_n \rightarrow t$, $t_n A(u_n) \rightharpoonup \chi$ in $W^{-1,p'(\cdot)}(\Omega)$ and $\limsup \langle t_n A(u_n), u_n \rangle \leq \langle \chi, u \rangle$, then*

$$\langle t_n A(u_n), u_n \rangle \rightarrow \langle \chi, u \rangle.$$

Proof. By a demonstration similar to that of Lemma 4.2, we can show that there exists a function $\varphi \in (L^{p'(\cdot)}(\Omega))^N$ such that

$$\limsup_{n \rightarrow \infty} \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx \leq \int_{\Omega} \varphi \cdot \nabla u \, dx.$$

So that it is enough to prove that

$$\liminf \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n \geq \int_{\Omega} \varphi \cdot \nabla u.$$

Denote

$$\Omega_k = \{x \in \Omega / |\nabla u(x)| \leq k\}$$

and

$$E_k(x) = \begin{cases} 1 & , \text{ when } x \in \Omega_k \\ 0 & , \text{ otherwise.} \end{cases}$$

By condition (A_3) we have

$$\int_{\Omega} t_n [a(x, u_n, E_k(x) \nabla u) - a(x, u_n, \nabla u_n)] \cdot [E_k(x) \nabla u - \nabla u_n] \geq 0.$$

Consequently,

$$\begin{aligned} \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n &\geq - \int_{\Omega} t_n a(x, u_n, E_k(x) \nabla u) \cdot E_k(x) \nabla u \\ &\quad + \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot E_k(x) \nabla u + \int_{\Omega} t_n a(x, u_n, E_k(x) \nabla u) \cdot \nabla u_n. \end{aligned} \quad (4.14)$$

From (4.11)–(4.13), we can deduce that the first term on the right in (4.14) tend towards

$$- \int_{\Omega} t a(x, u, E_k(x) \nabla u) \cdot E_k(x) \nabla u,$$

and the third term on the right in (4.14) will tend towards

$$\int_{\Omega} t a(x, u, E_k(x) \nabla u) \cdot \nabla u.$$

Consequently,

$$\begin{aligned} \liminf \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n &\geq \liminf \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot E_k(x) \nabla u + \int_{\Omega \setminus \Omega_k} t a(x, u, 0) \cdot \nabla u \\ &= \int_{\Omega_k} \chi \cdot \nabla u + \int_{\Omega \setminus \Omega_k} t a(x, u, 0) \cdot \nabla u, \end{aligned}$$

Letting $k \rightarrow \infty$, then $\Omega_k \rightarrow \Omega$, and hence

$$\liminf \int_{\Omega} t_n a(x, u_n, \nabla u_n) \cdot \nabla u_n \geq \int_{\Omega} \chi \cdot \nabla u.$$

Lemma 4.7. Assume that the conditions $(A_1) - (A_4)$ hold. If the sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$, $u_n \rightharpoonup u \in W_0^{1,p(\cdot)}(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$, $A(u_n) \rightharpoonup \chi$ in $W^{-1,p'(\cdot)}(\Omega)$ and $\limsup \langle A(u_n), u_n \rangle \leq \langle \chi, u \rangle$, then $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$ for some subsequence.

Proof. We deduce as in [21] that $\nabla u_n(x) \rightarrow \nabla u(x)$ a.e. for some subsequence. According to Lemma 4.3 we may assume that

$$a(x, u_n, \nabla u_n) \rightharpoonup w(x) \text{ in } L^{p'(\cdot)}.$$

Since

$$a(x, u_n, \nabla u_n) \rightarrow a(x, u, \nabla u) \text{ a.e.,}$$

then $w(x) = a(x, u, \nabla u)$ a.e. A is pseudomonotone.

Hence

$$u \in W_0^{1,p(\cdot)}(\Omega), \chi = A(u) \text{ and } \langle A(u_n), u_n \rangle \rightarrow \langle A(u), u \rangle.$$

By Lemma 4.5,

$$a(x, u_n, \nabla u_n) \cdot \nabla u_n \rightarrow a(x, u, \nabla u) \cdot \nabla u$$

in $L^1(\Omega)$, and by condition (A_4) ,

$$c' |\nabla u_n(x)|^{p(x)} \leq (a(x, u_n(x), \nabla u_n(x)) \cdot \nabla u_n(x)).$$

The right hand side converges in $L^1(\Omega)$ in accordance with Lemma 2.1. Lemma 2.2 implies $\nabla u_n \rightarrow \nabla u$ in $L^{p(\cdot)}(\Omega)$, and hence $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$.

Theorem 4.1. *If the conditions $(A_1) - (A_4)$ hold, then the mapping A defined by (4.1) belong to the class $\mathcal{F}^a$.*

Proof. Strong quasiboundedness and condition b) of class $\mathcal{F}^a$ follow immediately from Lemma 4.3. Lemma 4.6 implies condition c), and condition d) follows from Lemmas 4.6, 4.5 and 4.4. Suppose that

$$(u_n) \subset W_0^{1,p(\cdot)}(\Omega), u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega), A(u_n) \rightharpoonup \chi \text{ in } W^{-1,p'(\cdot)}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n), u_n \rangle \leq \langle \chi, u \rangle.$$

By pseudomonotonicity, $u \in W_0^{1,p(\cdot)}(\Omega)$, $\chi = A(u)$ and $\langle A(u_n), u_n \rangle \rightarrow \langle A(u), u \rangle$. If (A_4) holds, then, by Lemma 4.7, $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$ for some subsequence. Choosing $t_n = 1$ in Lemma 4.5 and Lemma 4.4 we may deduce that $A(u_n) \xrightarrow{\text{mod}} A(u)$ in $W^{-1,p'(\cdot)}(\Omega)$. Hence $A \in (\text{MOD})$.

4.2. Normalising maps. Let $Y = W_0^{1,p(\cdot)}(\Omega)$, $Z = W^{-1,p'(\cdot)}(\Omega)$. We start with an abstract existence theorem.

Theorem 4.2. *Let $G \subset Y$ be open and bounded in Y , $0 \in G$, $f \in Z$ and $F \in \mathcal{F}^a$. Suppose that there exists a normalising map $K \in \mathcal{F}^a$ such that $K(0) = 0$ and $\langle K(u), u \rangle \geq b > 0$ for all $u \in \partial_Y G$. Choose a constant $a \geq 0$ such that*

$$a \leq \inf_{u \in \partial_Y G} \frac{\langle K(u), u \rangle}{\|K(u)\|}.$$

If $\langle F(u) - f, u \rangle + \|F(u) - f\|a > 0$ for all $u \in \partial_Y G$, then $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. Since $\langle K(u), u \rangle \geq b > 0$ for all $u \in \partial_Y G$, it is clear that $0 \notin \overline{K(\partial_Y G)}$. Define by

$$H(t, u) = tF(u) + (1 - t)K(u).$$

a homotopy which belong to the class $\mathcal{H}$ by Lemma 3.1. We show that $H(t, u) \neq 0$ for all $t \in [0, 1], u \in \partial_Y G$. If $0 \in H([0, 1] \times \partial_Y G)$, then

$$tF(u) + (1 - t)K(u) = 0$$

for some $u \in \partial_Y G$ and $t \in [0, 1]$. It is clear that $t \neq 0$. Thus

$$t\|F(u)\| = (1 - t)\|K(u)\|,$$

implying

$$1 - \frac{1}{t} = -\frac{\|F(u)\|}{\|K(u)\|}.$$

On the other hand,

$$t\langle F(u), u \rangle + (1 - t)\langle K(u), u \rangle = 0,$$

and therefore

$$\langle F(u), u \rangle = (1 - \frac{1}{t})\langle K(u), u \rangle = -\|F(u)\| \frac{\langle K(u), u \rangle}{\|K(u)\|} \leq -\|F(u)\|a,$$

which is a contradiction. Hence $H(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_Y G$. By homotopy invariance,

$$d(F, G, 0) = f(K, G, 0).$$

Since $0 \in K(G)$, we have $d(K, G, 0) = 1$.

Corollary 4.1. *Let $G \subset Y$ be open and bounded in Y , $\bar{u} \in G$, $f \in Z$ and $F \in \mathcal{F}^a$. Suppose that there exists a normalising map $K \in \mathcal{F}^a$ such that $\tilde{K}(u) = K(u + \bar{u}) - K(\bar{u})$ is also a normalising map in the class $\mathcal{F}^a$ and*

$$\langle K(u) - K(\bar{u}), u - \bar{u} \rangle \geq b > 0 \text{ for all } u \in \partial_Y G.$$

Choose a constant $a \geq 0$ such that

$$a \leq \inf_{u \in \partial_Y G} \frac{\langle K(u) - K(\bar{u}), u - \bar{u} \rangle}{\|K(u) - K(\bar{u})\|}.$$

If $\langle F(u) - f, u - \bar{u} \rangle + \|F(u) - f\|a > 0$ for all $u \in \partial_Y G$, then $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. It follows from the above assumptions that the degree $d(F, G, 0)$ is defined. Define $s(u) = u - \bar{u}$. Then, by Property 2,

$$d(F, G, 0) = d(Fos^{-1}, s(G), 0).$$

Let $u \in \partial_{Ys}(G)$ be arbitrary. Denote $u = u' - \bar{u}$, where $u' \in \partial_Y G$. Now $\langle \tilde{K}(u), u \rangle = \langle K(u') - K(\bar{u}), u' - \bar{u} \rangle \geq b > 0$ and

$$\frac{\langle \tilde{K}(u), u \rangle}{\|\tilde{K}(u)\|} = \frac{\langle K(u') - K(\bar{u}), u' - \bar{u} \rangle}{\|K(u') - K(\bar{u})\|} \geq a \geq 0.$$

Moreover,

$$\langle Fos^{-1}(u), u \rangle + \|Fos^{-1}(u)\|a = \langle F(u'), u' - \bar{u} \rangle + \|F(u')\|a.$$

Therefore the assumptions of Theorem 4.2 hold. Hence

$$d(Fos^{-1}(u), s(G), 0) = 1.$$

Remark 4.1. *The mapping, so-called $p(\cdot)$ -Laplacian, $K : W_0^{1,p(\cdot)}(\Omega) \rightarrow W^{-1,p'(\cdot)}(\Omega)$ defined by*

$$K(u) = -\Delta_{p(\cdot)}(u) = -\operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u(x)).$$

i.e.

$$\langle Ku, v \rangle = \int_{\Omega} |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) dx \text{ for all } v \in W_0^{1,p(\cdot)}(\Omega),$$

belongs to the class $\mathcal{F}^a$, $\langle K(u), u \rangle > 0$ for $u \in W_0^{1,p(\cdot)}(\Omega)$, $u \neq 0$ and $K(0) = 0$.

Proof. Denote $a(x, \eta, \xi) = |\xi|^{p(x)-2} \xi$, then

$$\langle Ku, v \rangle = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx \text{ for all } v \in W_0^{1,p(\cdot)}(\Omega).$$

It is obvious that the conditions (A_2) and (A_4) holds. By [14, Theorem 3.1], K is continuous and strictly monotone, hence the conditions (A_1) and (A_3) holds.

Moreover,

$$\langle K(u), u \rangle = \int_{\Omega} |\nabla u(x)|^{p(x)} dx \geq 0,$$

and equality holds only if $\nabla u(x) = 0$ a.e., which implies that $u = 0$ a.e.

4.3. Existence results.

Theorem 4.3. Assume that the conditions $(A_1) - (A_4)$ hold. Define the mapping A as in (4.1). Let $f \in W^{-1,p'(\cdot)}(\Omega)$.

If

$$\liminf_{\|u\|_{1,p(\cdot)} \rightarrow \infty} \langle A(u) - f, u - \bar{u} \rangle \geq 0 \text{ for some } \bar{u} \in W_0^{1,p(\cdot)}(\Omega), \quad (4.15)$$

then the problem (1.1) is almost solvable, i.e., $f \in \overline{A(W_0^{1,p(\cdot)}(\Omega))}$.

If

$$\liminf_{\|u\|_{1,p(\cdot)} \rightarrow \infty} \langle A(u) - f, u - \bar{u} \rangle > 0 \text{ for some } \bar{u} \in W_0^{1,p(\cdot)}(\Omega), \quad (4.16)$$

then the problem (1.1) is solvable, i.e., $f \in A(W_0^{1,p(\cdot)}(\Omega))$.

Proof. Suppose first that (4.15) holds. By theorem 4.1, $A \in \mathcal{F}^a$. If

$$\inf_{\|u - \bar{u}\|_{1,p(\cdot)} = R} (\langle A(u) - f, u - \bar{u} \rangle + \|A(u) - f\| \frac{R}{4N\sqrt{N}}) \leq 0 \text{ for all } R > 0,$$

then, by (4.15), there exists a sequence $(u_n) \subset W_0^{1,p(\cdot)}(\Omega)$ such that $\|u_n\|_{1,p(\cdot)} \rightarrow \infty$ and $\|A(u_n) - f\| \rightarrow 0$. Hence

$$f \in \overline{A(W_0^{1,p(\cdot)}(\Omega))}.$$

Suppose that

$$\inf_{\|u - \bar{u}\|_{1,p(\cdot)} = R} (\langle A(u) - f, u - \bar{u} \rangle + \|A(u) - f\| \frac{R}{4N\sqrt{N}}) > 0 \text{ for some } R > 0.$$

There exists a normalising map K satisfying the assumptions of Corollary 4.1 with $b = \frac{R}{2\sqrt{N}}$ and $a = \frac{R}{4N\sqrt{N}}$.

Denote

$$B_R(\bar{u}) = \{u \in W_0^{1,p(\cdot)}(\Omega) / \|u - \bar{u}\|_{1,p(\cdot)} \leq R\}.$$

By Corollary 4.1, $d(A, B_R(\bar{u}), f) = 1$. By property (1) of the degree function,

$$f \in A(B_R(\bar{u})) \subset A(W_0^{1,p(\cdot)}(\Omega)).$$

If (4.16) holds, then we clearly have

$$\inf_{\|u - \bar{u}\|_{1,p(\cdot)} = R} (\langle A(u) - f, u - \bar{u} \rangle + \|A(u) - f\| \frac{R}{4N\sqrt{N}}) > 0 \text{ for some } R > 0,$$

and proceeding as above we obtain

$$f \in A(W_0^{1,p(\cdot)}(\Omega)).$$

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