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On the quasi-Fredholm and Saphar spectrum of strongly continuous Cosine operator function

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ABSTRACT. Let $(C(t))_{t \in \mathbb{R}}$ be a strongly continuous cosine family and A be its infinitesimal generator. In this work, we prove that, if $C(t) - \cosh \lambda t$ is Saphar (resp. quasi-Fredholm) operator and $\lambda t \notin i\pi\mathbb{Z}$, then $A - \lambda^2$ is also Saphar (resp. quasi-Fredholm) operator. We show by counter-example that the converse is false in general.

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1. Introduction

Let X be a complex Banach space, $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X and $\mathcal{C}(X)$ the set of all linear closed operators from X to X . We write $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{R}^\infty(A) = \bigcap_{n \geq 0} \mathcal{R}(A^n)$, $\mathcal{N}(T)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_r(T)$ respectively for the domain, the range, the hyper-range, the kernel, the resolvent, the spectrum, the point spectrum and residual spectrum of an operator $T \in \mathcal{C}(X)$. The function resolvent of $T \in \mathcal{C}(X)$ is defined for all $\lambda \in \rho(T)$ by $R(\lambda, T) = (\lambda - T)^{-1}$.

Recall that a closed operator T is said to be a semi-regular operator or semi-regular if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) \subseteq \mathcal{R}^\infty(T)$, see [6, Theorem 2.10].

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A closed operator S is called a generalized inverse of T if $\mathcal{R}(T) \subseteq \mathcal{D}(S)$, $\mathcal{R}(S) \subseteq \mathcal{D}(T)$, $TST = T$ on $\mathcal{D}(T)$ and $STS = S$ on $\mathcal{D}(S)$, see [3, Definition 1.1].

A closed operator T is called a Saphar operator if T has a generalized inverse and $\mathcal{N}(T) \subseteq \mathcal{R}^\infty(T)$.

Let $T \in \mathcal{C}(X)$ and let $\Delta(T) = \{n \in \mathbb{N} : \forall m \geq n, \mathcal{R}(T^n) \cap \mathcal{N}(T) = \mathcal{R}(T^m) \cap \mathcal{N}(T)\}$. The degree of stable iteration $dis(T)$ of T is defined as $dis(T) = \inf \Delta(T)$ with $dis(T) = \infty$ if $\Delta(T) = \emptyset$, see [4]. T is called a quasi-Fredholm operator of degree d if there exists an integer $d \in \mathbb{N}$ such that

- (1) $dis(T) = d$;
- (2) $\mathcal{R}(T^n)$ is closed in X for all $n \geq d$;
- (3) $\mathcal{R}(T) + \mathcal{N}(T^n)$ is closed in X for all $n \geq d$.

The semi-regular, Saphar and quasi-Fredholm spectra are defined by:

$$\sigma_K(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-regular}\},$$

$$\sigma_{Sap}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Saphar}\},$$

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not quasi-Fredholm}\}.$$

Consider in X the well-posed Cauchy problem

$$(P) \begin{cases} u''(t) = Au(t), & t \in \mathbb{R} \\ u(0) = u_0, \\ u'(0) = u_1. \end{cases}$$

Where $A : X \rightarrow X$ is a densely defined closed operator with nonempty resolvent set $\rho(A)$. The problem (P) is (see [2], [5] and [10]) well-posed if and only if A generates a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$, i.e., a family of operators satisfying the following conditions:

- (1) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.
- (2) $C(0) = I$ (the identity operator).
- (3) $t \rightarrow C(t)$ is continuous on $\mathbb{R}$ with respect to the operator norm topology on $\mathcal{B}(X)$.

There exist some $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

If $(C(t))_{t \in \mathbb{R}}$ is a strongly continuous cosine operator function, then the infinitesimal generating operator A is defined by

$$D(A) = \left\{ x \in X : \lim_{s \rightarrow 0} \frac{2(C(s)x - x)}{s^2} \text{ exists} \right\}$$

and

$$Ax = \lim_{s \rightarrow 0} \frac{2(C(s)x - x)}{s^2} = C''(0).$$

A solution of problem (P) is given with the help of a strongly continuous cosine operator function by the formula $u(t) = C(t)u_0 + S(t)u_1$ for $t \in \mathbb{R}$, where $S(t)$ is the sine operator function associated with the $(C(t))_{t \in \mathbb{R}}$ and is defined as $S(t)x := \int_0^t C(s)x ds$, $t \in \mathbb{R}$, $x \in X$. In this work we will use the theory of integration in the sense of Bochner.

If $(C(t))_{t \in \mathbb{R}}$ is a uniformly continuous operator cosine function then there is an $A \in \mathcal{B}(X)$ with $C(t) = \cosh t\sqrt{A}$, $t \in \mathbb{R}$. For $t \in \mathbb{R}$, the function $f : z \in \mathbb{C} \mapsto \cosh t\sqrt{z}$ defines an entire function. Thus, according to the spectral mapping theorem, we have $\cosh t\sqrt{\sigma_*(A)} = \sigma_*(C(t))$, for all $t \in \mathbb{R}$, with σ_* the spectrum corresponding to regularity in the sense of V. Müller [7, Definition 6.1].

The following spectral inclusion $\cosh t\sqrt{\sigma(A)} \subseteq \sigma(C(t))$, $t \in \mathbb{R}$ was obtained by B. Nagy; he also gave an example where the reverse inclusion fails [8] and he showed that $\sigma_*(C(t)) = \cosh t\sqrt{\sigma_*(A)}$, $t \in \mathbb{R}$, with $\star \in \{p, r\}$.

However there are several large classes of generators A for which the spectrum of $C(t)$ can be expressed in terms of $\sigma(A)$, namely $\sigma(C(t)) = \overline{\cosh t\sqrt{\sigma(A)}}$, $t \in \mathbb{R}$, if A is the generator of a uniformly bounded cosine function on a Hilbert space [11] or of a cosine function of normal operators [12].

In this paper, we continue to study the spectral theory of strongly continuous cosine operator function. we prove that, if $C(t) - \cosh \lambda t$ is Saphar (resp. quasi-Fredholm) operator and $\lambda t \notin i\pi\mathbb{Z}$, then $A - \lambda^2$ is also Saphar (resp. quasi-Fredholm) operator. We show by counter-example that the converse is false in general.

2. Main results

Now, the following theorem state the main result of this article.

Theorem 1. *Let $(C(t))_{t \in \mathbb{R}}$ be a strongly continuous cosine operator function on X with generator A . For all $t \neq 0$, we have,*

- (1) *if $\lambda^2 \in \sigma_{\text{Sap}}(A)$ and $\lambda t \notin i\pi\mathbb{Z}$, then $\cosh \lambda t \in \sigma_{\text{Sap}}(C(t))$,*
- (2) *if $\lambda^2 \in \sigma_{qF}(A)$ and $\lambda t \notin i\pi\mathbb{Z}$, then $\cosh \lambda t \in \sigma_{qF}(C(t))$.*

Theorem 1 then directly yields the following important result:

Corollary 2. *Let A be the generator of a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $t \neq 0$,*

$$\cosh t\sqrt{\sigma_*(A)} \cup \{-1, 1\} \subseteq \sigma_*(C(t)) \cup \{-1, 1\},$$

with $\star \in \{\text{Sap}, qF\}$.

Remark 3. *Let X be the complex ℓ^2 space. For $(z_n)_{n \in \mathbb{N}} \in \ell^2$ and $s \in \mathbb{R}$, we put $C(s)(z_n)_n = (\cos(ns)z_n)_n$. Then $A(z_n)_n = (-n^2 z_n)_n$ with $\mathcal{D}(A) = \{(z_n)_n \in \ell^2 : \sum_{n=1}^{\infty} n^4 |z_n|^2 < \infty\}$ and*

$\sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}^\}$. Then $\cosh \sqrt{\sigma_{\text{Sap}}(A)}$ is countable. From [8], $\sigma(C(1))$ contains the set $[-1, 1] \setminus \{\cos n : n \in \mathbb{N}^*\}$ which is uncountable set. Then $\sigma(C(1))$ is uncountable. By [9, Corollary 2.10], we have $\sigma_{\text{Sap}}(C(1))$ and $\sigma_{qF}(C(1))$ are uncountable. This shows that all of the above inclusions are strict.*

The following lemmas are the most widely used results of this paper.

Lemma 4. [8, Lemma. 4] Let A be the generator of a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$. Let A be the generator of the cosine operator function C . For $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, let

$$S_\lambda(t)x := \int_0^t \sinh \lambda(t-s)C(s)x ds, \quad x \in X.$$

Then $S_\lambda(t) \in \mathcal{B}(X)$ is an operator that commutes with A , and

$$(A - \lambda^2)S_\lambda(t)x = \lambda(C(t) - \cosh \lambda t)x,$$

for all $x \in X$.

Lemma 5. Let A be the generator of a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, there exist two operators $L_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)$ such that

$$(A - \lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I.$$

Moreover, the operators $L_\lambda(t), S_\lambda(t), G_\lambda(t)$ and $A - \lambda^2$ are mutually commuting.

Proof. For all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, let $K_\lambda(t)x := \int_0^t \sinh \lambda(t-s)S_\lambda(s)x ds$. It is clear that $K_\lambda(t)$ is a bounded linear operator on X . We consider the function $f : s \in [0, t] \rightarrow \sinh \lambda(t-s)S_\lambda(s)x$, then f is Bochner integrable and $f(s) \in D(A)$ for all $s \in [0, t]$. Moreover, $(A - \lambda^2)f(s) = \lambda \sinh \lambda(t-s)(C(t) - (\cosh \lambda t)I)$ is Bochner integrable. From [1, Proposition 1.1.7], $K_\lambda(t)x \in D(A)$ and

$$\begin{aligned} (A - \lambda^2)K_\lambda(t)x &= \int_0^t \lambda \sinh \lambda(t-s)(C(s) - \cosh \lambda s)x ds \\ &= \lambda \int_0^t \sinh \lambda(t-s)C(s)x ds - \lambda \int_0^t \sinh \lambda(t-s) \cosh \lambda s x ds \\ &= \lambda S_\lambda(t)x - t\lambda \sinh \lambda t x. \end{aligned}$$

We put $L_\lambda(t) := (-t\lambda \sinh \lambda t)^{-1}K_\lambda(t)$ and $G_\lambda(t) := (t \sinh \lambda t)^{-1}I$. Then we have $(A - \lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I$. Furthermore, the operators $L_\lambda(t), S_\lambda(t), G_\lambda(t)$ and $A - \lambda^2$ are mutually commuting. Indeed, since $C(s)C(t) = C(t)C(s)$ for all $s, t \geq 0$, then $S_\lambda(s)C(t) = C(t)S_\lambda(s)$. Hence $S_\lambda(s)S_\lambda(t) = S_\lambda(t)S_\lambda(s)$. Thus, we deduce that $L_\lambda(s)S_\lambda(t) = S_\lambda(t)L_\lambda(s)$. On the other hand, for all $x \in D(A)$, we have

$$\begin{aligned} K_\lambda(t)(A - \lambda^2)x &= \int_0^t \sinh \lambda(t-s)S_\lambda(s)(A - \lambda^2)x ds \\ &= \lambda \int_0^t \sinh \lambda(t-s)(C(s) - (\cosh \lambda s)I)x ds \\ &= \lambda S_\lambda(t)x - t\lambda \sinh \lambda t x \\ &= (A - \lambda^2)K_\lambda(t)x. \end{aligned}$$

Then, $L_\lambda(t)$ commute with each one of $S_\lambda(t), G_\lambda(t)$ and $(A - \lambda^2)$.

Lemma 6. *Let A be the generator of a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $n \in \mathbb{N}^*$, $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, there exist two operators $F_{\lambda,n}(t), H_{\lambda,n}(t) \in \mathcal{B}(X)$ such that,*

$$(A - \lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t) S_\lambda^n(t) = I.$$

Moreover, the operators $F_{\lambda,n}(t)$, $H_{\lambda,n}(t)$, $S_\lambda^n(t)$ and $(A - \lambda^2)^n$ are mutually commuting.

Proof. By lemma 5, there exist two operators $L_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)$ such that

$$(A - \lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I.$$

For all $n \geq 1$ and $x \in X$, we have $L_\lambda^n(t)x \in D(A^n)$. In fact, the proof is by induction. For $n = 1$, from lemma 5 $L_\lambda(t)x \in D(A)$. Suppose that $L_\lambda^{n-1}(t)x \in D(A^{n-1})$, so $L_\lambda^n(t)x \in D(A^{n-1})$ and

$$\begin{aligned} (A - \lambda^2)^{n-1} L_\lambda^n(t)x &= [(A - \lambda^2)L_\lambda(t)]^{n-1} L_\lambda(t)x \\ &= L_\lambda(t)[(A - \lambda^2)L_\lambda(t)]^{n-1}x \in D(A), \end{aligned}$$

hence, $L_\lambda^n(t)x \in D(A^n)$. Furthermore,

$$\begin{aligned} (A - \lambda^2)^n L_\lambda^n(t) &= [(A - \lambda^2)L_\lambda(t)]^n \\ &= [I - G_\lambda(t)S_\lambda(t)]^n \\ &= I - T_{\lambda,n}(t)S_\lambda(t), \end{aligned}$$

with $T_{\lambda,n}(t) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} G_\lambda^k(t) S_\lambda^{k-1}(t)$. So $(A - \lambda^2)^n L_\lambda^n(t) + T_{\lambda,n}(t)S_\lambda(t) = I$. Similarly, we have

$$\begin{aligned} T_{\lambda,n}^n(t) S_\lambda^n(t) &= [I - (A - \lambda^2)^n L_\lambda^n(t)]^n \\ &= I - (A - \lambda^2)^n \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (A - \lambda^2)^{n(k-1)} L_\lambda^{nk}(t). \end{aligned}$$

We define $F_{\lambda,n}(t) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (A - \lambda^2)^{n(k-1)} L_\lambda^{nk}(t)$ and $H_{\lambda,n}(t) = T_{\lambda,n}^n(t)$. Then $(A - \lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t) S_\lambda^n(t) = I$. Moreover the operators $(A - \lambda^2)^n$, $F_{\lambda,n}(t)$, $H_{\lambda,n}(t)$ and $S_\lambda^n(t)$ are pairwise commuting.

Lemma 7. *Let A be the generator of a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $q \in \mathbb{N}$, $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, if $\mathcal{R}(C(t) - \cosh \lambda t)^q$ is closed, then $\mathcal{R}(A - \lambda^2)^q$ is also closed.*

Proof. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{R}(A - \lambda^2)^q$ converging to $y \in X$. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(A^q)$ satisfying $(A - \lambda^2)^q x_n = y_n$. By Lemma 6, we obtain

$$(A - \lambda^2)^q F_{\lambda,q}(t) y_n + H_{\lambda,q}(t) S_\lambda^q(t) y_n = y_n.$$

Hence, we conclude that

$$\lambda^q (C(t) - \cosh \lambda t)^q H_{\lambda,q}^q(t) x_n = S_\lambda^q(t) (A - \lambda^2)^q H_{\lambda,q}^q(t) x_n$$

$$\begin{aligned}
&= H_{\lambda,q}^q(t) S_{\lambda}^q(t) (A - \lambda^2)^q x_n \\
&= H_{\lambda,q}^q(t) S_{\lambda}^q(t) y_n \\
&= y_n - (A - \lambda^2)^q F_{\lambda,q}(t) y_n.
\end{aligned}$$

Thus, $y_n - (A - \lambda^2)^q F_{\lambda,q}(t) y_n \in \mathcal{R}(C(t) - \cosh \lambda t)^q$. Since $\mathcal{R}(C(t) - \cosh \lambda t)^q$ is closed and $(A - \lambda^2)^q F_{\lambda,q}(t)$ a bounded linear operator, it follows that the sequence $y_n - (A - \lambda^2)^q F_{\lambda,q}(t) y_n$ converges to $y - (A - \lambda^2)^q F_{\lambda,q}(t) y$ as n tends to ∞ and

$$y - (A - \lambda^2)^q F_{\lambda,q}(t) y \in \mathcal{R}(C(t) - \cosh \lambda t)^q \subseteq \mathcal{R}(A - \lambda^2)^q.$$

We obtain $y \in \mathcal{R}(A - \lambda^2)^q$, which completes the proof.

Corollary 8. *Let $(C(t))_{t \in \mathbb{R}}$ be a strongly continuous cosine operator function on X with generator A . Then, if $C(t) - \cosh \lambda t$ is semi-regular operator and $\lambda t \notin i\pi\mathbb{Z}$, then $A - \lambda^2$ is also semi-regular operator.*

Proof. If $C(t) - \cosh \lambda t$ is semi-regular operator, then $\mathcal{R}(C(t) - \cosh \lambda t)$ is closed and $\mathcal{N}(C(t) - \cosh \lambda t) \subseteq \mathcal{R}^\infty(C(t) - \cosh \lambda t)$. By lemma 7, $\mathcal{R}(A - \lambda^2)$ is closed and $N(A - \lambda^2) \subseteq N(C(t) - \cosh \lambda t) \subseteq \mathcal{R}^\infty(C(t) - \cosh \lambda t) \subseteq \mathcal{R}^\infty(A - \lambda^2)$. Hence $A - \lambda^2$ is semi-regular operator.

- (1) Assume that $C(t) - \cosh \lambda t$ is a Saphar operator and $\lambda t \notin i\pi\mathbb{Z}$, then $C(t) - \cosh \lambda t$ has a generalized inverse and $C(t) - \cosh \lambda t$ is semi-regular. By corollary 8 $C(t) - \cosh \lambda t$ is semi-regular. It remains to prove that $A - \lambda^2$ has a generalized inverse. Since $C(t) - \cosh \lambda t$ has a generalized inverse, then by [7, Proposition 1] there exists $R \in \mathcal{B}(X)$ such that,

$$(C(t) - \cosh \lambda t) R (C(t) - \cosh \lambda t) = C(t) - \cosh \lambda t.$$

According to lemma 5, we have $(A - \lambda^2)L_{\lambda}(t) + G_{\lambda}(t)S_{\lambda}(t) = I$, then

$$\begin{aligned}
(A - \lambda^2) &= (A - \lambda^2)L_{\lambda}(t)(A - \lambda^2) + S_{\lambda}(t)G_{\lambda}(t)(A - \lambda^2) \\
&= (A - \lambda^2)L_{\lambda}(t)(A - \lambda^2) + (C(t) - \cosh \lambda t) G_{\lambda}(t) \\
&= (A - \lambda^2)L_{\lambda}(t)(A - \lambda^2) + (C(t) - \cosh \lambda t) R (C(t) - \cosh \lambda t) G_{\lambda}(t) \\
&= (A - \lambda^2)L_{\lambda}(t)(A - \lambda^2) + (A - \lambda^2)S_{\lambda}(t)R(A - \lambda^2)S_{\lambda}(t)G_{\lambda}(t) \\
&= (A - \lambda^2)L_{\lambda}(t)(A - \lambda^2) + (A - \lambda^2)S_{\lambda}(t)RS_{\lambda}(t)G_{\lambda}(t)(A - \lambda^2) \\
&= (A - \lambda^2)[L_{\lambda}(t) + S_{\lambda}(t)RS_{\lambda}(t)G_{\lambda}(t)](A - \lambda^2).
\end{aligned}$$

Let $H_{\lambda} = L_{\lambda}(t) + S_{\lambda}(t)RS_{\lambda}(t)G_{\lambda}(t)$. Then H_{λ} is a bounded linear operator, $R(H_{\lambda}) \subseteq \mathcal{D}(A)$ and $(A - \lambda^2)H_{\lambda}(A - \lambda^2) = A - \lambda^2$. Let $K_{\lambda} = H_{\lambda}(A - \lambda^2)H_{\lambda}$, it follows that K_{λ} and $R(K_{\lambda}) \subseteq \mathcal{D}(A)$. Moreover, we have

$$\begin{aligned}
(A - \lambda^2)K_{\lambda}(A - \lambda^2) &= (A - \lambda^2)H_{\lambda}(A - \lambda^2)H_{\lambda}(A - \lambda^2), \\
&= (A - \lambda^2)H_{\lambda}(A - \lambda^2),
\end{aligned}$$

$$= A - \lambda^2,$$

and

$$\begin{aligned} K_\lambda(A - \lambda^2)K_\lambda &= H_\lambda(A - \lambda^2)H_\lambda(A - \lambda^2)H_\lambda, \\ &= H_\lambda(A - \lambda^2)H_\lambda, \\ &= K_\lambda. \end{aligned}$$

Hence $A - \lambda^2$ has a generalized inverse. Consequently $A - \lambda^2$ is a Saphar operator.

- (2) Assume that $C(t) - \cosh \lambda t$ is quasi-Fredholm. First, we prove the $\text{dis}(A - \lambda^2)$ is finite. Since $\text{dis}(C(t) - \cosh \lambda t)$ is finite, then there exists $d \in \mathbb{N}$ for all $n \geq d$,

$$\mathcal{R}(C(t) - \cosh \lambda t)^n \cap \mathcal{N}(C(t) - \cosh \lambda t) = \mathcal{R}(C(t) - \cosh \lambda t)^d \cap \mathcal{N}(C(t) - \cosh \lambda t).$$

We show that for all $n \geq d$,

$$\mathcal{R}(A - \lambda^2)^n \cap \mathcal{N}(A - \lambda^2) = \mathcal{R}(A - \lambda^2)^d \cap \mathcal{N}(A - \lambda^2).$$

Let $y \in \mathcal{R}(A - \lambda^2)^d \cap \mathcal{N}(A - \lambda^2)$, then there exists $x \in \mathcal{D}(A^d)$ such that $y = (A - \lambda^2)^d x$. Then, according to lemma 5, there exist two operators $F_{\lambda,d}(t)$ and $H_{\lambda,d}(t)$ such that

$$(A - \lambda^2)^d F_{\lambda,d}(t) + S_\lambda^d(t) H_{\lambda,d}(t) = I.$$

Therefore,

$$\begin{aligned} y &= (A - \lambda^2)^d F_{\lambda,d}(t) y + S_\lambda^d(t) H_{\lambda,d}(t) y, \\ &= F_{\lambda,d}(t) (A - \lambda^2)^d y + (C(t) - \cosh \lambda t)^d H_{\lambda,d}(t) x, \\ &= (C(t) - \cosh \lambda t)^d H_{\lambda,d}(t) x. \end{aligned}$$

Therefore,

$$y \in \mathcal{R}(C(t) - \cosh \lambda t)^d \cap \mathcal{N}(C(t) - \cosh \lambda t) \subseteq \mathcal{R}(C(t) - \cosh \lambda t)^n \subseteq \mathcal{R}(\lambda - A)^n.$$

Consequently $y \in \mathcal{R}(\lambda - A)^n \cap \mathcal{N}(A - \lambda^2)$. Hence $\text{dis}(A - \lambda^2)$ is finite.

Next we show that $\mathcal{R}(A - \lambda^2) + \mathcal{N}(A - \lambda^2)^n$ is closed in X , for all $n \geq d$. If $n = 0$, then from lemma 7, $\mathcal{R}(A - \lambda^2)$ is closed. If $n \geq 1$ let $y_p = (A - \lambda^2)x_p + z_p$ be a sequence which converges to y ($p \rightarrow \infty$), with $x_p \in X$ and $z_p \in \mathcal{N}(A - \lambda^2)^n$. As $S_\lambda^n(t)y_p = S_\lambda^n(t)(A - \lambda^2)x_p + S_\lambda^n(t)z_p \in \mathcal{R}(C(t) - \cosh \lambda t) + \mathcal{N}(C(t) - \cosh \lambda t)^n$, then $S_\lambda^n(t)y \in \mathcal{R}(C(t) - \cosh \lambda t) + \mathcal{N}(C(t) - \cosh \lambda t)^n$. There exist $x \in X$ and $z \in \mathcal{N}(C(t) - \cosh \lambda t)^n$ such that $S_\lambda^n(t)y = (C(t) - \cosh \lambda t)x + z$. So $S_\lambda^{2n}(t)y = S_\lambda^n(t)(C(t) - \cosh \lambda t)x + S_\lambda^n(t)z$ where $S_\lambda^n(t)z \in \mathcal{N}(A - \lambda^2)^n$, which implies that,

$$\begin{aligned} y &= (A - \lambda^2)^n F_{\lambda,n}(t) y + S_\lambda^{2n}(t) H_{\lambda,n}(t) y, \\ &= (A - \lambda^2)^n F_{\lambda,n}(t) y + H_{\lambda,n}(t) S_\lambda^n(t) (C(t) - \cosh \lambda t) x + H_{\lambda,n}(t) S_\lambda^n(t) z, \\ &= \left((A - \lambda^2)^n F_{\lambda,n}(t) y + H_{\lambda,n}(t) S_\lambda^n(t) \right) + H_{\lambda,n}(t) S_\lambda^n(t) z. \end{aligned}$$

Therefore, $y \in \mathcal{R}(A - \lambda^2) + \mathcal{N}(A - \lambda^2)^n$. Hence $\mathcal{R}(A - \lambda^2) + \mathcal{N}(A - \lambda^2)^n$ is closed.

Proposition 9. Let $(C(t))_{t \in \mathbb{R}}$ be a strongly continuous cosine operator function on X with generator A satisfying $\|C(t)\| = o(t^2)$ ($t \rightarrow \infty$). The following assertions are equivalent:

- (1) A is bounded below,
- (2) A is semi-regular.

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Let $y \in \mathcal{N}(A)$. Since A is semi-regular, then $\mathcal{N}(A) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A)$ is closed. So there exists $x \in \mathcal{D}(A)$ such that $y = Ax$. From [8, Theorem.7], $C(t)x = x + \frac{t^2}{2!}Ax + \int_0^t \frac{(t-s)^3}{3!}C(s)A^2x ds$, for all $t \neq 0$. As $\|C(t)\| = o(t^2)$, then $y = 0$. Which implies that A is bounded below.

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