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Domination and Kwapien's factorization theorems for positive Cohen p -nuclear m -linear operators

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ABSTRACT. In this paper, we introduce and study the concept of positive Cohen p -nuclear multilinear operators between Banach lattice spaces. We prove a natural analog to the Pietsch domination theorem for this class. Moreover, we give like the Kwapien's factorization theorem. Finally, we investigate some relations with another known classes.

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1. Introduction and notation

For a Banach space X , X^* will denote its topological dual and B_X will denote its closed unit ball. For a Banach lattice E , E^+ will denote its positive cone, and for each $x \in E$, x^+ and x^- will denote the positive part and negative part respectively, and we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Throughout this paper, $X, X_1, \dots, X_m, Y$ will be Banach spaces and $E, E_1, \dots, E_m, F, F_1, \dots, F_m$ and G will be Banach lattices. We will denote by $\mathcal{L}(X_1, \dots, X_m; Y)$ the

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space of all continuous m -linear operators from $X_1 \times \cdots \times X_m$ into Y . If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, \dots, X_m)$. In the case $X_1 = \cdots = X_m = X$, we will simply write $\mathcal{L}(^m X; Y)$.

Recall that an m -linear operator $T : E_1 \times \cdots \times E_m \longrightarrow F$ is called positive if $T(x^1, \dots, x^m) \in F^+$, whenever $(x^1, \dots, x^m) \in E_1^+ \times \cdots \times E_m^+$.

For $1 \leq p \leq \infty$, let p^* be its conjugate, that is, $1/p + 1/p^* = 1$. For a Banach space X and a finite sequence $(x_i)_{i=1}^n$

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}},$$

and

$$w_p((x_i)_{i=1}^n) = \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n(X)} = \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, \varphi \rangle|^p \right)^{\frac{1}{p}}.$$

Consider the case where X is replaced by a Banach lattice E , and define

$$\ell_{p,|weak|}^n(E) := \left\{ (x_i)_{i=1}^n : (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E) \right\}$$

(see [8, 17] for details). Moreover, if $B_{E^*}^+ = \{\xi \in B_{E^*} : \xi \geq 0\}$ denotes the unit ball in $B_{E^*} \cap E^{*+}$, it follows that

$$\begin{cases} \|(x_i)_{i=1}^n\|_{\ell_{1,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \sum_{i=1}^n \langle |x_i|, \xi \rangle \\ \|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \left(\sum_{i=1}^n \langle |x_i|, \xi \rangle^p \right)^{\frac{1}{p}}, 1 < p < \infty \\ \|(x_i)_{i=1}^n\|_{\ell_{\infty,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \sup_{1 \leq i \leq n} \langle |x_i|, \xi \rangle. \end{cases}$$

If $x_1, \dots, x_n \in E^+$, we have that

$$\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x_i, \xi \rangle^p \right)^{\frac{1}{p}}. \quad (1.1)$$

Q. Bu and G. Buskes in [9], and Fremlin in [15, 16] introduced a lattice tensor product, called the positive projective tensor product. The projective cone on the tensor product $E_1 \otimes \cdots \otimes E_m$ is defined as:

$$E_1^+ \otimes \cdots \otimes E_m^+ = \left\{ \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m : x_i^j \in E_j^+, j = 1, \dots, m, n \in \mathbb{N} \right\}.$$

The positive projective tensor norm on $E_1 \otimes \cdots \otimes E_m$ is defined as:

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\| : x_i^j \in E_j^+, n \in \mathbb{N}, |u| \leq \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m \right\},$$

for every $u \in E_1 \otimes \cdots \otimes E_m$. By $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ will denote the complete positive m -fold projective tensor product of $E_1, \dots, E_m$ (Fremlin projective tensor product). Then $E_1 \widehat{\otimes}_{|\pi|} \cdots$

$\widehat{\otimes}_{|\pi|} E_m$ with the above norm, is a Banach lattice; we will use the next notation $\widehat{E} = E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$.

Every $T \in \mathcal{L}(E_1, \dots, E_m; F)$ has an associated linear operator $T_L \in \mathcal{L}(\widehat{E}; F)$ defined by for all $x^j \in E_j^+$ ($j = 1, \dots, m$), $T_L(x^1 \otimes \cdots \otimes x^m) = T(x^1, \dots, x^m)$ (see [9] for details). Let $\delta_m : E_1 \times \cdots \times E_m \longrightarrow \widehat{E}$ denote the canonical m -linear operator, that is, $\delta_m(x^1, \dots, x^m) = x^1 \otimes \cdots \otimes x^m$. Then,

$$T = T_L \circ \delta_m. \quad (1.2)$$

In the work of Achour et al. [1], one can represent the class $\mathcal{N}_p^m$ as follows; $\mathcal{N}_p^m = \mathcal{D}_p^m \circ (\Pi_p, \dots, \Pi_p)$, where $\mathcal{D}_p^m, \Pi_p$ are the Banach spaces of Cohen strongly p -summing multilinear operators and p -summing linear operators, respectively and in [1, Theorem 2.5], they characterized this class by Pietsch's domination theorem. In our work, we generalize this notion for positive operators and prove, among other results, the domination theorem for this new class of operators.

We recall some definitions which we need in the sequel.

An operator $u : E \longrightarrow Y$ is said to be positive p -summing ($1 \leq p < \infty$) if there exists a constant $C > 0$ such that the inequality

$$\|u(x_i)_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)}, \quad (1.3)$$

holds for every positive integer n and for all $x_1, \dots, x_n \in E$.

For $p = \infty$,

$$\sup \|u(x_i)\| \leq C \sup_{\xi \in B_{E^*}^+} \sup_{1 \leq i \leq n} \langle |x_i|, \xi \rangle.$$

We denote by $\Pi_p^+(E, Y)$, the space of positive p -summing operators from E into Y . $\Pi_p^+(E, Y)$ becomes a Banach space with the norm $\pi_p^+(\cdot)$, $\pi_p^+(u)$ given by the infimum of the constant $C > 0$ that verifies the inequality (1.3).

Let $1 \leq p \leq \infty$. An operator $u : X \longrightarrow F$ is positive strongly p -summing if there exists a constant $C > 0$, such that for all finite sets $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset F^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq C \|(x_i)_{i=1}^n\|_p \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}. \quad (1.4)$$

The class of all positive strongly p -summing operators between X and F is denoted by $\mathcal{D}_p^+(X, F)$ (see [2]). The infimum of all the constant C in the inequality (1.4) defines a norm $d_p^+(\cdot)$ on $\mathcal{D}_p^+(X, F)$. We have $\mathcal{D}_1^+(X, F) = \mathcal{L}(X, F)$.

Now, we continue in specifying definition of the Cohen positive strongly p -summing m -linear operators and the positive multiple p -summing were introduced in [7], [10] respectively.

Definition 1.1 [7] Let $1 \leq p \leq +\infty$. An m -linear operator $T : X_1 \times \cdots \times X_m \longrightarrow F$ is Cohen positive strongly p -summing multilinear operator, if there is a constant $C > 0$, such that for any $x_1^j, \dots, x_n^j \in X_j$; $1 \leq j \leq m$, and any $y_1^*, \dots, y_n^* \in F^*$

$$\left\| \left\langle T(x_i^1, \dots, x_i^m), y_i^* \right\rangle \right\|_{\ell_1^n} \leq C \left(\sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|_{X_j}^p \right)^{\frac{1}{p}} \| (y_i^*)_{i=1}^n \|_{\ell_{p^*, |weak|}^n(F^*)}. \quad (1.5)$$

The class of all Cohen positive strongly p -summing m -linear operators from $X_1 \times \dots \times X_m$ into F , denoted by $\mathcal{D}_p^{m+}(X_1, \dots, X_m; F)$, is a Banach space with the norm $d_p^{m+}(T)$, which is the smallest constant C such that the (1.5) holds.

A multilinear operator $T \in \mathcal{L}(X_1, \dots, X_m; F)$ is Cohen positive strongly p -summing ($1 < p \leq \infty$) if, and only if, there is a Radon probability measure μ on $B_{F^{**}}^+$ such that for all $(x^1, \dots, x^m) \in X_1 \times \dots \times X_m$ and $y^* \in F^{*+}$ we have

$$\left| \left\langle T(x^1, \dots, x^m), y^* \right\rangle \right| \leq d_p^{m+}(T) \prod_{j=1}^m \|x^j\| \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \mu)}. \quad (1.6)$$

Definition 1.2 [10] Let $1 \leq p < \infty$. An m -linear operator $T : E_1 \times \dots \times E_m \rightarrow Y$ is called positive multiple p -summing if there exists a constant $K > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{n_j} \subseteq E_j^+$ ($1 \leq j \leq m$),

$$\left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} \|T(x_{i_1}^1, \dots, x_{i_m}^m)\|^p \right)^{\frac{1}{p}} \leq K \prod_{j=1}^m \left\| (x_i^j)_{i=1}^{n_j} \right\|_{\ell_{p, weak}^n(E_j)}. \quad (1.7)$$

In this case, we define the positive multiple p -summing norm of T by

$$\Lambda_p(T) = \inf \{K : K \text{ verifies the inequality (1.7)}\}.$$

It is easily verified that the class $\Lambda_p^{mult}(E_1, \dots, E_m; Y)$ of positive multiple p -summing m -linear operators, with its associated norm $\Lambda_p(\cdot)$, is a Banach space.

For $1 \leq p < \infty$, an m -linear operator $T : X_1 \times \dots \times X_m \rightarrow F$ is called multiple p -convex if there exists a constant $C > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{n_j} \subset X_j$; $1 \leq j \leq m$.

$$\left\| \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} |T(x_{i_1}^1, \dots, x_{i_m}^m)|^p \right)^{\frac{1}{p}} \right\| \leq C \prod_{j=1}^m \left(\sum_{i=1}^{n_j} \|x_i^j\|_{X_j}^p \right)^{\frac{1}{p}}.$$

Or

$$\left\| \bigvee_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} |T(x_{i_1}^1, \dots, x_{i_m}^m)| \right\| \leq C \prod_{j=1}^m \max_{1 \leq i \leq n_j} \|x_i^j\|_{X_j}, \text{ whenever } p = \infty.$$

In this case, we define the multiple p -convex norm of T by

$$C_p^{vex}(T) = \inf \{C : C \text{ vérifies the inequality above}\}.$$

It is easily verified that the class $\mathcal{C}_p^{vex, mult}(X_1, \dots, X_m; F)$ of multiple p -convex m -linear operators, with its associated norm $C_p^{vex}(\cdot)$, is a Banach space.

The next definition is due to D. Achour and A. Alouani [1], an m -linear operator $T : X_1 \times \cdots \times X_m \longrightarrow Y$ is Cohen p -nuclear ($1 < p < \infty$) if there is a constant $C > 0$ such that for any $x_1^j, \dots, x_n^j \in X_j$ ($1 \leq j \leq m$) and any $y_1^*, \dots, y_n^* \in Y^*$, we have

$$\left| \sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \right| \leq C \left(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m |\langle x_i^j, x^{j*} \rangle|^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \|y_i^*(y)\|_{\ell_{p^*}^n}. \quad (1.8)$$

The class of all Cohen p -nuclear m -linear operators from $X_1 \times \cdots \times X_m$ into Y , is denoted by $\mathcal{N}_p^m(X_1, \dots, X_m; Y)$. The infimum of all the constant C in the inequality (1.8) defines a norm $n_p^m(\cdot)$ on $\mathcal{N}_p^m(X_1, \dots, X_m; Y)$.

For $p = \infty$, (1.8) becomes

$$\left| \sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \right| \leq C \left(\sup_{1 \leq i \leq n} \prod_{j=1}^m \|x_i^j\|_{X_j} \right) \sup_{y \in B_Y} \|y_i^*(y)\|_{\ell_1^n}.$$

2. Positive Cohen p -nuclear multilinear operator

In this section, we introduce the positive Cohen p -nuclear multilinear operator. We prove a natural analog to the Pietsch domination theorem for this class and characterize their conjugate. The original motivation for our research is to give a multilinear version of Kwapien's factorization theorem : $\mathcal{N}_p^{m+} = \mathcal{D}_p^{m+} \circ (\Pi_p^+, \dots, \Pi_p^+)$. As an application, we study a relation between some classes of positive m -linear operators (Multiple p -summing and Cohen strongly p -summing).

Definition 2.1 An m -linear operator $T : E_1 \times \cdots \times E_m \longrightarrow F$ is positive Cohen p -nuclear ($1 < p < \infty$) if there is a constant $C > 0$ such that for any $x_1^j, \dots, x_n^j \in E_j$ ($1 \leq j \leq m$) and any $y_1^*, \dots, y_n^* \in F^*$, we have

$$\left| \sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \right| \leq C \left(\sup_{\substack{x^{j*} \in B_{E_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m |\langle x_i^j, x^{j*} \rangle|^p \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*, |weak|}^n(F^*)}. \quad (2.1)$$

Moreover, the class of all positive Cohen p -nuclear m -linear operators from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$ (or $\mathcal{N}_p^+(E; F)$ if $m = 1$, the space of positive Cohen p -nuclear operators). Our space is a Banach space with the norm $n_p^{m+}(\cdot)$, which is the smallest constant C such that (2.1) holds. For $p = \infty$, we have $\mathcal{N}_\infty^{m+}(E_1, \dots, E_m; F) = \mathcal{D}_\infty^{m+}(E_1, \dots, E_m; F)$.

Proposition 2.3 Let $T \in \mathcal{L}(E_1, \dots, E_m; F)$, then T is positive Cohen p -nuclear multilinear operator if, and only if, there is a constant $C > 0$ such that the inequality

$$\sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \leq C \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle x_i^j, x^{j*} \rangle^p \right)^{\frac{1}{p}} w_{p^*}((y_i^*)_{i=1}^n), \quad (2.2)$$

holds for every $x_1^j, \dots, x_n^j \in E_j^+$ ($1 \leq j \leq m$) and any $y_1^*, \dots, y_n^* \in F^{*+}$.

Proof. For the first part use the formula (1.1).

The second part, for convenience, we prove the inequality for $m = 2$ only. Let $(x_i^1)_{i=1}^n \subset E_1$, $(x_i^2)_{i=1}^n \subset E_2$ and $(y_i^*)_{i=1}^n \subset F^*$, we have

$$\begin{aligned} \left| \sum_{i=1}^n \langle T(x_i^1, x_i^2), y_i^* \rangle \right| &= \left| \sum_{i=1}^n \langle T(x_i^{1+} - x_i^{1-}, x_i^2), y_i^* \rangle \right| \\ &\leq \left| \sum_{i=1}^n \langle T(x_i^{1+}, x_i^2), y_i^* \rangle \right| + \left| \sum_{i=1}^n \langle T(x_i^{1-}, x_i^2), y_i^* \rangle \right| \\ &\leq \left| \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2+}), y_i^* \rangle \right| + \left| \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2-}), y_i^* \rangle \right| \\ &\quad + \left| \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2+}), y_i^* \rangle \right| + \left| \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2-}), y_i^* \rangle \right| \\ &\leq \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2+}), y_i^{*+} \rangle + \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2+}), y_i^{*-} \rangle \\ &\quad + \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2-}), y_i^{*+} \rangle + \sum_{i=1}^n \langle T(x_i^{1+}, x_i^{2-}), y_i^{*-} \rangle \\ &\quad + \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2+}), y_i^{*+} \rangle + \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2+}), y_i^{*-} \rangle \\ &\quad + \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2-}), y_i^{*+} \rangle + \sum_{i=1}^n \langle T(x_i^{1-}, x_i^{2-}), y_i^{*-} \rangle \end{aligned}$$

Using the inequality (2.2) and simplifying the calculation, we find

$$\left| \sum_{i=1}^n \langle T(x_i^1, x_i^2), y_i^* \rangle \right| \leq K \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq 2}} \sum_{i=1}^n \prod_{j=1}^2 \langle |x_i^j|, x^{j*} \rangle^p \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*, |\text{weak}|}^n(F^*)}.$$

Proposition 2.4 Let $T \in \mathcal{L}(E_1, \dots, E_m; F)$, R a positive operator in $\mathcal{L}(F, G)$ and S_j a positive operator in $\mathcal{L}(F_j, E_j)$ ($1 \leq j \leq m$).

(1) If T is positive Cohen p -nuclear, then RT is positive Cohen p -nuclear from $E_1 \times \dots \times E_m$ to G and $n_p^{m+}(RT) \leq n_p^{m+}(T) \|R\|$.

(2) If T is positive Cohen p -nuclear, then $T \circ (S_1, \dots, S_m)$ is positive Cohen p -nuclear from $F_1 \times \dots \times F_m$ to F and

$$n_p^{m+}(T \circ (S_1, \dots, S_m)) \leq n_p^{m+}(T) \prod_{j=1}^m \|S_j\|.$$

Proof. (1) Let $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$. We have by (2.2),

$$\begin{aligned} \sum_{i=1}^n \langle RT(x_i^1, \dots, x_i^m), y_i^* \rangle &= \sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), R^*(y_i^*) \rangle \\ &\leq n_p^{m+}(T) \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle x_i^j, x^{j*} \rangle^p \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n \langle R^*(y_i^*), y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq n_p^{m+}(T) \|R\| \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle x_i^j, x^{j*} \rangle^p \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n \left\langle y_i^*, \frac{R^{**}(y^{**})}{\|R^{**}\|} \right\rangle^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq n_p^{m+}(T) \|R\| \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle x_i^j, x^{j*} \rangle^p \right)^{\frac{1}{p}} \sup_{\psi \in B_{G^{**}}^+} \left(\sum_{i=1}^n \langle y_i^*, \psi \rangle^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned}$$

So $RT \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; G)$ and $n_p^{m+}(RT) \leq n_p^{m+}(T) \|R\|$.

(2) Let $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$. Then

$$\begin{aligned} \sum_{i=1}^n \langle T \circ (S_1, \dots, S_m)(x_i^1, \dots, x_i^m), y_i^* \rangle &\leq n_p^{m+}(T) \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle S_j x_i^j, x^{j*} \rangle^p \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n} \\ &\leq n_p^{m+}(T) \prod_{j=1}^m \|S_j\| \left(\sup_{\substack{x^{j*} \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \left\langle x_i^j, \frac{S_j^*(x^{j*})}{\|S_j\|} \right\rangle^p \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n} \end{aligned}$$

$$\leq n_p^{m+}(T) \prod_{j=1}^m \|S_j\| \left(\sup_{\substack{\varphi_j \in B_{F_j^*}^+ \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \langle x_i^j, \varphi_j \rangle^p \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n}.$$

Thus, $T \circ (S_1, \dots, S_m)$ is positive Cohen p -nuclear multilinear operator and

$$n_p^{m+}(T \circ (S_1, \dots, S_m)) \leq n_p^{m+}(T) \prod_{j=1}^m \|S_j\|.$$

The main result of this section is the next extension of Pietsch's domination theorem for the class of positive Cohen p -nuclear multilinear operator.

Theorem 2.5 *An m -linear operators $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive Cohen p -nuclear ($1 < p \leq \infty$) if, and only if, there exists a positive constant $C > 0$ and Radon probability measures μ_j on $B_{E_j^*}^+$ ($1 \leq j \leq m$) and λ on $B_{F^{**}}^+$ such that for all $(x^1, \dots, x^m) \in E_1^+ \times \dots \times E_m^+$ and $y^* \in F^{*+}$*

$$\langle T(x^1, \dots, x^m), y^* \rangle \leq C \prod_{j=1}^m \|x^j\|_{L_p(B_{E_j^*}^+, \mu_j)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)}. \quad (2.3)$$

To prove this theorem, we use the same technique as that used in [1] and follow the idea of [7].

We also have the following result:

Theorem 2.6 *Let $1 < p \leq +\infty$. A mapping $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive Cohen p -nuclear ($1 < p \leq \infty$) if, and only if, there exists a positive constant C and Radon probability measures μ_1 on $B_{E_1^*}^+ \times \dots \times B_{E_m^*}^+$ and μ_2 on $B_{F^{**}}^+$ such that*

$$\langle T(x^1, \dots, x^m), y^* \rangle \leq C \left(\int_{B_{E_1^*}^+ \times \dots \times B_{E_m^*}^+} \left(\prod_{j=1}^m \langle x^j, \varphi^j \rangle \right)^p d\mu_1 \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle y^*, \varphi \rangle^{p^*} d\mu_2 \right)^{\frac{1}{p^*}} \quad (2.4)$$

for all $(x^1, \dots, x^m) \in E_1^+ \times \dots \times E_m^+$ and $y^* \in F^{*+}$.

For the proof of the Domination Theorem we use the full general Pietsch Domination Theorem recently presented by Pellegrino et al. in [22].

Let $X_1, \dots, X_m, Y$ and $Z_1, \dots, Z_k$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X_1 \times \dots \times X_m$ to Y . Let also $K_1, \dots, K_t$ be compact Hausdorff topological spaces, $G_1, \dots, G_t$ be Banach spaces and suppose that the maps

$$\begin{cases} R_j : K_j \times Z_1 \times \dots \times Z_k \times G_j \longrightarrow [0, +\infty), j = 1, \dots, t \\ S : \mathcal{H} \times Z_1 \times \dots \times Z_k \times G_1 \times \dots \times G_t \longrightarrow [0, +\infty). \end{cases}$$

satisfy:

(1) For each $x^l \in Z_l$ and $b \in G_j$, with $(j, l) \in \{1, \dots, t\} \times \{1, \dots, k\}$ the mapping

$$(R_j)_{x^1, \dots, x^k, b} : K_j \longrightarrow [0, +\infty),$$

defined by

$$(R_j)_{x^1, \dots, x^k, b}(\varphi_j) = R_j(\varphi_j, x^1, \dots, x^k, b)$$

is continuous.

(2) The following inequalities hold:

$$\begin{cases} R_j(\varphi_j, x^1, \dots, x^k, \eta_j b^j) \leq \eta_j R_j(\varphi_j, x^1, \dots, x^k, b^j) \\ S(f, x^1, \dots, x^k, \alpha_1 b^1, \dots, \alpha_t b^t) \geq \alpha_1 \dots \alpha_t S(f, x^1, \dots, x^k, b^1, \dots, b^t), \end{cases}$$

for every $\varphi_j \in K_j, x^l \in Z_l, 0 \leq \eta_j, \alpha_j \geq 1, b^j \in G_j$, with $(j, l) \in \{1, \dots, t\} \times \{1, \dots, k\}$ and $f \in \mathcal{H}$.

Definition 2.7 [22]. If $0 \leq p_1, \dots, p_t, q < \infty$, with $\frac{1}{q} = \frac{1}{p_1} + \dots + \frac{1}{p_t}$, a mapping $f : X_1 \times \dots \times X_m \rightarrow Y$ in $\mathcal{H}$ is said to be $R_1, \dots, R_t$ -S-abstract $(p_1, \dots, p_t)$ -summing if there is a constant $C > 0$ so that

$$\left(\sum_{i=1}^n S(f, x_i^1, \dots, x_i^k, b_i^1, \dots, b_i^t)^q \right)^{\frac{1}{q}} \leq C \prod_{j=1}^t \sup_{\varphi_j \in K_j} \left(\sum_{i=1}^n R_j(\varphi_j, x_i^1, \dots, x_i^k, b_i^j)^{p_j} \right)^{\frac{1}{p_j}}, \quad (2.5)$$

for all $(x_i^s)_{i=1}^n \subset Z_s, (b_i^j)_{i=1}^n \subset G_j, n \in \mathbb{N}$ and $(s, j) \in \{1, \dots, k\} \times \{1, \dots, t\}$.

Theorem 2.8 [22]. A map $f \in \mathcal{H}$ is $R_1, \dots, R_t$ -S-abstract $(p_1, \dots, p_t)$ -summing if, and only if there is a constant $C > 0$ and Borel probability measures μ_j on K_j such that

$$S(f, x^1, \dots, x^k, b^1, \dots, b^t) \leq C \prod_{j=1}^t \left(\int_{K_j} R_j(\varphi_j, x^1, \dots, x^k, b^j)^{p_j} d\mu_j \right)^{\frac{1}{p_j}}, \quad (2.6)$$

for all $x^l \in Z_l, l \in \{1, \dots, k\}$ and $b^j \in G_j$ with $j = 1, \dots, t$.

Proof of Theorem 2.6. By choosing the parameters

$$\left\{ \begin{array}{l} t = 2, k = m \\ Z_l = E_l^+, l = 1, \dots, m \\ K_1 = B_{E_1^*}^+ \times \dots \times B_{E_m^*}^+ \text{ and } K_2 = B_{F^{**}}^+ \\ G_1 = \mathbb{K}, G_2 = F^* \\ \mathcal{H} = \mathcal{L}(E_1, \dots, E_m; F) \\ q = 1, p_1 = p, p_2 = p^* \\ S(T, x^1, \dots, x^m, b, y^*) = |b| \langle T(x^1, \dots, x^m), y^* \rangle \\ R_1((\varphi^1, \dots, \varphi^m), x^1, \dots, x^m, b) = |b| \prod_{j=1}^m \langle x^j, \varphi^j \rangle \\ R_2(\varphi, x^1, \dots, x^k, y^*) = \langle |y^*|, \varphi \rangle, \end{array} \right.$$

we can easily conclude that $T : E_1 \times \dots \times E_m \rightarrow F$ is positive Cohen p -nuclear if, and only if T is R_1, R_2 -S-abstract (p, p^*) -summing. Theorem 2.8 tells us that T is R_1, R_2 -S-abstract (p, p^*) -summing if, and only if there is a $C > 0$ and there are probability measures μ_1 on K_1 and μ_2 on K_2 such that

$$S(T, x^1, \dots, x^m, b, y^*) \leq C \left(\int_{K_1} \left(\prod_{j=1}^m \langle x^j, \varphi^j \rangle \right)^p d\mu_1 \right)^{\frac{1}{p}} \left(\int_{K_2} \langle |y^*|, \varphi \rangle^{p^*} d\mu_2 \right)^{\frac{1}{p^*}} \quad (2.7)$$

and we recover (2.4).

Therefore Theorem 2.5 and Theorem 2.6 are equivalent.

Proposition 2.9. *An m -linear operators $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive Cohen p -nuclear ($1 < p \leq \infty$) if, and only if, there exists a positive constant $C > 0$ and Radon probability measures μ_j on $B_{E_j}^+$ ($1 \leq j \leq m$) and λ on $B_{F^{**}}^+$ such that the inequality*

$$|\langle T(x^1, \dots, x^m), y^* \rangle| \leq n_p^{m+}(T) \prod_{j=1}^m \left(\int_{B_{E_j}^+} \langle |x^j|, x^{j*} \rangle^p d\mu_j(x^{j*}) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}};$$

holds for every $(x^1, \dots, x^m) \in E_1 \times \dots \times E_m$ and $y^* \in F^*$.

3. Kwapien's Factorization Theorem

We present now the main result, Kwapien's Factorization Theorem, of this section. For the proof, we will use the following lemma due to D. Achour and A. Belacel, see [2, Lemma 3.5]. For more informations about $J_{p,0}$ and $L_0^p(\mu)$, see [2, p792-793].

Lemma 3.1. The operator $J_{p,0} \circ i_E : E \longrightarrow i_E(E) \longrightarrow L_0^p(\mu)$ is positive p -summing and $\pi_p^+(J_{p,0} \circ i_E) \leq 1$, such that for all $x \in E$

$$\|\langle x, \cdot \rangle\|_{L_0^p(\mu)} = \|J_{p,0} \circ i_E\|_{L_0^p(\mu)}.$$

Where μ is a probability measure on the set $B_{E^*}^+$.

Theorem 3.2. Let $1 < p \leq \infty$ and $j = 1, \dots, m$. Then $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive Cohen p -nuclear if, and only if, there exist Banach spaces $X_1, \dots, X_m$, positive p -summing linear operators $u_j \in \mathcal{L}(E_j, X_j)$ and a Cohen positive strongly p -summing multilinear operator $S \in \mathcal{L}(X_1, \dots, X_m; F)$ such that $T = S(u_1, \dots, u_m)$, i.e. $\mathcal{N}_p^{m+} = \mathcal{D}_p^{m+} \circ (\Pi_p^+, \dots, \Pi_p^+)$, in addition

$$n_p^{m+}(T) = \inf \left\{ d_p^{m+}(S) \prod_{j=1}^m \pi_p^+(u_j) : T = S(u_1, \dots, u_m) \right\}.$$

Proof. To prove the first implication, we take $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$. Then, by (2.3), there exist Radon probability measures μ_j on $B_{E_j}^+$ ($1 \leq j \leq m$) and λ on $B_{F^{**}}^+$ such that for all $(x^1, \dots, x^m) \in E_1^+ \times \dots \times E_m^+$ and $y^* \in F^{*+}$

$$\langle T(x^1, \dots, x^m), y^* \rangle \leq C \prod_{j=1}^m \|x^j\|_{L_p(B_{E_j}^+, \mu_j)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)}.$$

Now consider the diagram

$$\begin{array}{ccc}
E_1 & \times \cdots \times & E_m \\
\downarrow i_{E_1} & & \downarrow i_{E_m} \\
i_{E_1}(E_1) & \times \cdots \times & i_{E_m}(E_m) \\
\cap & & \cap \\
C(B_{E_1}^+) & & C(B_{E_m}^+)
\end{array}
\begin{array}{c}
\begin{array}{ccc}
& \xrightarrow{T} & \\
& & F \\
& \uparrow S & \\
& L_0^p(\mu_1) \times \cdots \times L_0^p(\mu_m) &
\end{array} \\
\longrightarrow \\
(J_{p,0}, \dots, J_{p,0})
\end{array}$$

where $i_{E_j} : E_j \rightarrow C(B_{E_j}^+)$ is the canonical injection. If we denote the range of $J_{p,0} \circ i_{E_j}$ by X_j , and the closure of X_j by $L_0^p(\mu_j)$, the map $X_1 \times \cdots \times X_m \rightarrow F : J_{p,0} \circ i_{E_j}(x^1, \dots, x^m) \mapsto T(x^1, \dots, x^m)$ is well-defined operator. We now apply the Lemma 3.1, we find that the operators $u_j = J_{p,0} \circ i_{E_j} : E_j \rightarrow i_{E_j}(E_j) \rightarrow L_0^p(\mu_j)$ are positive p -summing, and $\pi_p^+(u_j) \leq 1$.

The operator S is defined on $u_1(E_1) \times \cdots \times u_m(E_m)$, where $u_j(x^j) = (J_{p,0} \circ i_{E_j})(x^j)$, by

$$S(u_1(x^1), \dots, u_m(x^m)) := T(x^1, \dots, x^m),$$

this definition makes sense because

$$\begin{aligned}
|\langle S(u_1(x^1), \dots, u_m(x^m)), y^* \rangle| &= |\langle S(u_1(x^1), \dots, u_m(x^m)), y^* \rangle| = |\langle T(x^1, \dots, x^m), y^* \rangle| \\
&\stackrel{\text{By (2.3)}}{\leq} n_p^{m+}(T) \prod_{j=1}^m \|x^j\| \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \\
&= n_p^{m+}(T) \prod_{j=1}^m \|u_j(x^j)\|_{L_0^p(\mu_j)} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.
\end{aligned}$$

From [7, Theorem 2.5] S is a Cohen positive strongly p -summing m -linear operator and $d_p^{m+}(S) \leq n_p^{m+}(T)$.

Conversely, of course, we use Pietsch's domination theorems for positive p -summing linear and Cohen positive strongly p -summing m -linear operators.

In the next proposition, we investigate the relationships between certain classes of multilinear operators.

Proposition 3.3. *Let $1 < p < \infty$. We have*

- 1) *If $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$, then $T \in \mathcal{D}_p^{m+}(E_1, \dots, E_m; F)$ and $d_p^{m+}(T) \leq n_p^{m+}(T)$.*
- 2) *If $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$, then $T \in \mathcal{C}_p^{vex, mult}(E_1, \dots, E_m; F)$.*
- 3) *If $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$, then $T \in \Lambda_p^{mult}(E_1, \dots, E_m; F)$ and $\Lambda_p(T) \leq n_p^{m+}(T)$.*

Proof. 1) If T is positive Cohen p -nuclear. Then, by (2.3), there exist Radon probability measures μ_j on $B_{E_j}^+$ ($1 \leq j \leq m$) and λ on $B_{F^{**}}^+$ such that for all $(x^1, \dots, x^m) \in E_1^+ \times \cdots \times E_m^+$ and $y^* \in F^{*+}$

$$|\langle T(x^1, \dots, x^m), y^* \rangle| = \langle T(x^1, \dots, x^m), y^* \rangle$$

$$\begin{aligned}
&\leq n_p^{m+}(T) \prod_{j=1}^m \left(\int_{B_{E_j^*}^+} \langle x^j, \xi^j \rangle^p d\mu_j(\xi^j) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \\
&\leq n_p^{m+}(T) \prod_{j=1}^m \left(\sup_{\xi^j \in B_{E_j^*}^+} \langle x^j, \xi^j \rangle \right) \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \\
&\leq n_p^{m+}(T) \prod_{j=1}^m \|x^j\| \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.
\end{aligned}$$

By Pietsch's domination theorem for Cohen positive strongly p -summing m -linear operator [7, Theorem 2.5], $T \in \mathcal{D}_p^{m+}(E_1, \dots, E_m; F)$ and $d_p^{m+}(T) \leq n_p^{m+}(T)$.

2) If T is positive Cohen p -nuclear. Then, $T \in \mathcal{D}_p^{m+}(E_1, \dots, E_m; F)$ and we have by [7, Proposition 3.3], $T \in \mathcal{C}_p^{vex, mult}(E_1, \dots, E_m; F)$.

3) Let T be an operator in $\mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$, we have for all $(x_i^j)_{i=1}^{n_j} \subseteq E_j^+$ ($1 \leq j \leq m$) and $y^* \in F^{*+}$

$$\begin{aligned}
\|T(x_{i_1}^1, \dots, x_{i_m}^m)\| &= \sup_{y^* \in B_{F^*}^+} \langle T(x_{i_1}^1, \dots, x_{i_m}^m), y^* \rangle \\
&\leq \sup_{y^* \in B_{F^*}^+} n_p^{m+}(T) \prod_{j=1}^m \left(\int_{B_{E_j^*}^+} \langle x_{i_j}^j, \xi^j \rangle^p d\mu_j(\xi^j) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \\
&\leq n_p^{m+}(T) \prod_{j=1}^m \left(\int_{B_{E_j^*}^+} \langle x_{i_j}^j, \xi^j \rangle^p d\mu_j(\xi^j) \right)^{\frac{1}{p}} \sup_{y^* \in B_{F^*}^+} \|y^*\|
\end{aligned}$$

So

$$\begin{aligned}
\|T(x_{i_1}^1, \dots, x_{i_m}^m)\| &\leq n_p^{m+}(T) \prod_{j=1}^m \left(\int_{B_{E_j^*}^+} \langle x_{i_j}^j, \xi^j \rangle^p d\mu_j(\xi^j) \right)^{\frac{1}{p}}. \text{ Then we have} \\
\left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} \|T(x_{i_1}^1, \dots, x_{i_m}^m)\|^p \right)^{\frac{1}{p}} &\leq n_p^{m+}(T) \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} \prod_{j=1}^m \int_{B_{E_j^*}^+} \langle x_{i_j}^j, \xi^j \rangle^p d\mu_j(\xi^j) \right)^{\frac{1}{p}} \\
&\leq n_p^{m+}(T) \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} \prod_{j=1}^m \sup_{\xi^j \in B_{E_j^*}^+} \langle x_{i_j}^j, \xi^j \rangle^p \right)^{\frac{1}{p}} \\
&= n_p^{m+}(T) \left(\sum_{i_1=1}^{n_1} \sup_{\xi^1 \in B_{E_1^*}^+} \langle x_{i_1}^1, \xi^1 \rangle^p \times \sum_{i_2=1}^{n_2} \sup_{\xi^2 \in B_{E_2^*}^+} \langle x_{i_2}^2, \xi^2 \rangle^p \times \dots \times \sum_{i_m=1}^{n_m} \sup_{\xi^m \in B_{E_m^*}^+} \langle x_{i_m}^m, \xi^m \rangle^p \right)^{\frac{1}{p}}
\end{aligned}$$

$$= n_p^{m+}(T) \prod_{j=1}^m \left\| \left(x_i^j \right)_{i=1}^{n_j} \right\|_{\ell_{p,weak}^m(E_j)}.$$

Again by Definition 1.2, we have $T \in \Lambda_p^{mult}(E_1, \dots, E_m; F)$ and $\Lambda_p(T) \leq n_p^{m+}(T)$.

Proposition 3.4. For $1 < p \leq \infty$.

$$\mathcal{N}_p^m(E_1, \dots, E_m; F) \subseteq \mathcal{N}_p^{m+}(E_1, \dots, E_m; F).$$

Proof. Suppose that $T \in \mathcal{N}_p^m(E_1, \dots, E_m; F)$ by Theorem 3.1 in [1] there exist Banach spaces $X_1, \dots, X_m$ such that $T = S(u_1, \dots, u_m)$ where $S \in \mathcal{D}_p^m(X_1, \dots, X_m; F)$ and $u_j \in \Pi_p(E_j, X_j)$ ($1 \leq j \leq m$). By Theorem 2.10 in [7], $S \in \mathcal{D}_p^{m+}(X_1, \dots, X_m; F)$ and by Proposition 3 in [4], $u_j \in \Pi_p^+(E_j, X_j)$. Again by Theorem 3.2 we have $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$.

4. Some further resultes

For every multilinear positive operator T there is an unique positive linear operator T_L which verify the relation (1.2). The following result justify the introduction of positive Cohen p -nuclear m -linear operators as it establishes a direct connection with their linearization.

Theorem 4.1. Let $1 < p \leq \infty$ and $T \in \mathcal{L}(E_1, \dots, E_m; F)$. Then the following statements are equivalent.

- 1) $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$.
- 2) $T_L \in \mathcal{N}_p^+(\hat{E}; F)$. Moreover, $n_p^+(T_L) = n_p^{m+}(T)$.

Proof. 1) $\implies$ 2) Let $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$. By Theorem 2.6, we have

$$\langle T(x^1, \dots, x^m), y^* \rangle \leq n_p^{m+}(T) \prod_{j=1}^m \|x^j\|_{L_p(B_{E_j^+}^+, \mu_j)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)},$$

for all $(x^1, \dots, x^m) \in E_1^+ \times \dots \times E_m^+$ and $y^* \in F^{*+}$.

We take $z \in \hat{E}$. For all $\varepsilon > 0$, z admits a representation $z = \sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m$ such that

$$\sum_{i=1}^n \left(\|x_i^1\|_{L_p(B_{E_1^+}^+, \mu_1)} \times \dots \times \|x_i^m\|_{L_p(B_{E_m^+}^+, \mu_m)} \right) \leq \|z\|_{L_p(B_{\hat{E}}^+, \mu)} + \varepsilon, \quad (4.1)$$

where $\mu = \mu_1 \otimes \dots \otimes \mu_m$.

Then

$$\begin{aligned}
\langle T_L(z), y^* \rangle &= \left\langle T_L \left(\sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m \right), y^* \right\rangle \\
&= \sum_{i=1}^n \langle T(x_i^1, \dots, x_i^m), y^* \rangle \\
&\leq n_p^{m+}(T) \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|_{L_p(B_{E_j^+}^+, \mu_j)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)} \\
&\leq n_p^{m+}(T) \left(\|z\|_{L_p(B_{(\hat{E})^*}^+, \mu)} + \varepsilon \right) \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)}.
\end{aligned}$$

Then, by (2.3), $T_L \in \mathcal{N}_p^+(\hat{E}; F)$ and $n_p^+(T_L) \leq n_p^{m+}(T)$.

2) $\implies$ 1) Suppose that T_L is positive Cohen p -nuclear. Let $(x^1, \dots, x^m) \in E_1^+ \times \cdots \times E_m^+$ and $y^* \in F^{*+}$, we have

$$\begin{aligned}
\langle T(x^1, \dots, x^m), y^* \rangle &= \langle T_L(x^1 \otimes \cdots \otimes x^m), y^* \rangle \\
&\leq n_p^+(T_L) \|x^1 \otimes \cdots \otimes x^m\|_{L_p(B_{(\hat{E})^*}^+, \mu)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)} \\
&\leq n_p^+(T_L) \left(\int_{B_{(\hat{E})^*}^+} \langle x^1 \otimes \cdots \otimes x^m, \xi^j \rangle^p d\mu(\xi) \right)^{\frac{1}{p}} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)} \\
&\leq n_p^+(T_L) \left(\int_{B_{(\hat{E})^*}^+} \langle x^1 \otimes \cdots \otimes x^m, \xi \rangle^p d(\mu_1 \otimes \cdots \otimes \mu_m)(\xi) \right)^{\frac{1}{p}} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)} \\
&\leq n_p^+(T_L) \prod_{j=1}^m \|x^j\|_{L_p(B_{E_j^+}^+, \mu_j)} \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)}.
\end{aligned}$$

Then, by (2.3), $T \in \mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$. and $n_p^{m+}(T) \leq n_p^+(T_L)$.

The next theorem show the duality of positive Cohen p -nuclear m -linear operator. We need the following definition.

The adjoint of an m -linear operator is defined as follows, if $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, we define the adjoint of T by

$$T^* : Y^* \longrightarrow \mathcal{L}(X_1, \dots, X_m), y^* \longrightarrow T^*(y^*) : X_1 \times \cdots \times X_m \longrightarrow \mathbb{K}$$

with $T^*(y^*)(x^1, \dots, x^m) = y^*(T(x^1, \dots, x^m))$.

Theorem 4.2. Let $1 < p \leq \infty$. Let $T \in \mathcal{L}(E_1, \dots, E_m; F)$ and T^* its adjoint. Then T belongs to $\mathcal{N}_p^{m+}(E_1, \dots, E_m; F)$ if, and only if, T^* belongs to $\mathcal{N}_{p^*}^+(F^*, \mathcal{L}(E_1, \dots, E_m))$, and we have $n_p^{m+}(T) = n_{p^*}^+(T^*)$.

Proof. Assume first that $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is positive Cohen p -nuclear. By above Theorem, its linearization $T_L : \widehat{E} \rightarrow F$ is positive Cohen p -nuclear operator. Then its adjoint $T_L^* : F^* \rightarrow (\widehat{E})^*$ is positive Cohen p^* -nuclear operator. Indeed, there are some Banach space X , a positive p -summing operator $u : \widehat{E} \rightarrow X$ and a Cohen positive strongly p -summing operator $S : X \rightarrow F$ such that $T_L = S \circ u$. By [2, Theorem 4.6], we have $T_L^* = u^* \circ S^*$ such that u^* is a Cohen positive strongly p^* -summing operator and S^* is a positive p^* -summing operator, we have T_L^* is positive Cohen p^* -nuclear. Consider the isometric isomorphism $\Delta_m : \mathcal{L}(E_1, \dots, E_m) \rightarrow (\widehat{E})^*$ given by $\Delta_m(T) = T_L$. Since $T = T_L \circ \delta_m$, by duality we get $T^* = \delta_m^* \circ T_L^* = \Delta_m^{-1} \circ T_L^*$. The ideal property ensures that T^* is positive Cohen p^* -nuclear.

Conversely, assume that T^* is positive Cohen p^* -nuclear. Then the equality $T_L^* = \Delta_m \circ T^*$ and the ideal property gives that T_L^* is positive Cohen p^* -nuclear, then T_L is positive Cohen p -nuclear and by Theorem 4.1 we conclude that T is positive Cohen p -nuclear.

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