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Controlled K -Fusion Frame for Hilbert Spaces

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ABSTRACT. K -fusion frames are a generalization of fusion frames in frame theory. In this paper, we extend the concept of controlled fusion frames to controlled K -fusion frames, and we develop some results on the controlled K -fusion frames for Hilbert spaces, which generalize some well known results of controlled fusion frame case. Also we discuss some characterizations of controlled Bessel K -fusion sequences and of controlled K -fusion frames. Further, we analyze stability conditions of controlled K -fusion frames under perturbation.

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1. Introduction

Frames are more flexible than bases to solve some problems in Hilbert spaces. They were firstly introduced by Duffin and Schaeffer [11] to study nonharmonic Fourier series in 1952, and widely studied by Daubechies, Grossman and Meyer [10] in 1986. More results of frames are in [7].

Fusion frames as a generalisation of frames were introduced by Casazza and Kutyniok in [6] and further there were developed in their joint paper [9] with Li. The theory for fusion frames

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is available in arbitrary separable Hilbert spaces (finite-dimensional or not). The motivation behind fusion frames comes from signal processing, more precisely, the desire to process and analyze large data sets efficiently. A natural idea is to split such data sets into suitable smaller "blocks" which can be treated independently. From a pure mathematical point of view, fusion frames are special cases of the g -frames [22]. However, the connection to concrete applications is less apparent from the more abstract definition of g -frames. In 2012, L. Gavrutu [15] introduced the notions of K -frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K . Controlled frames in Hilbert spaces have been introduced by P. Balaz [1] to improve the numerical efficiency of iterative algorithms for inventing the frame operator. Further A. Khosravi [17] generalized this concept to the case of fusion frames. He has showed that controlled fusion framee as a generalization of fusion frames give a generalized way to obtain numerical advantage in the sense of preconditioning to check the fusion frame condition. In 2015 Rahimi [20] defined the concept of controlled K -frames in Hilbert spaces and showed that controlled K - frames are equivalent to K -frames.

Motivated by the above literature, we introduce and investigate some properties of controlled K -fusion frames, we also generalize some known results for controlled fusion frames to controlled K -fusion frames. Finaly, we present perturbation result for controlled K -fusion frames. This paper is organized as follows. In Section 2, we recall several definitions about fusion frames, K -fusion frames and controlled fusion frames. Then, we give a basic properties about a bounded linear operator. In Section 3, we introduce the concept of controlled K -fusion frames and discuss their properties. In section 4, we analyze stability conditions of controlled K -fusion frames under perturbation.

2. Preliminaries and Notations

Throughout this paper, we will adopt the following notations. $\mathcal{H}$ is a separable Hilbert space, $\{W_i\}_{i \in I}$ is a sequence of closed subspaces of $\mathcal{H}$, where I is a countable index set. the family of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. We denote $\mathcal{R}_T$, $\mathcal{N}_T$, range and null space of a bounded linear operator T , respectively. $GL(\mathcal{H})$ is the set of all bounded invertible operators on $\mathcal{H}$ with bounded inverse, and $GL(\mathcal{H})^+$ denotes the set of all positive operators in $GL(\mathcal{H})$. π_{W_i} is the orthogonal projection from $\mathcal{H}$ into W_i , and $\{w_i\}_{i \in I}$ is a family of weights, i.e. $w_i > 0$, for any $i \in I$.

The space $(\oplus_{i \in I} \mathcal{H})_{l^2}$ which is defined by

$$(\oplus_{i \in I} \mathcal{H})_{l^2} = \{\{f_i\}_{i \in I} : f_i \in \mathcal{H}, i \in I, \sum_{i \in I} \|f_i\|^2 < \infty\},$$

with the inner product as

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_{\mathcal{H}},$$

is a separable Hilbert space [17].

2.1. Fusion frames.

Definition 2.1. [9] Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$. Let $\{w_i\}_{i \in I}$ be a family of weights, the family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is called a K -fusion frame for $\mathcal{H}$, if there exist positive constants $A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, f \in \mathcal{H}. \quad (2.1)$$

A and B are called the lower and upper bounds of fusion frame, respectively. If only the right inequality of (2.1) holds, we call the family $\{W_i, w_i\}_{i \in I}$ a fusion bessel sequence.

2.2. K -fusion frames.

Definition 2.2. [3] Let $K \in B(\mathcal{H})$, let $\{W_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$, and let $\{w_i\}_{i \in I}$ be a family of weights. Then the family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is called a K -fusion frame for $\mathcal{H}$, if there exist positive constants $A \leq B < \infty$ such that

$$A\|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, f \in \mathcal{H}. \quad (2.2)$$

Where K^* is the adjoint operator of K .

A and B are called the lower and upper bounds of K -fusion frame, respectively.

suppose that $\{W_i, w_i\}_{i \in I}$ is a fusion Bessel sequence for $\mathcal{H}$, then the synthesis operator of $\{W_i, w_i\}_{i \in I}$ is defined by $T_W : (\sum_{i \in I} \oplus W_i)_{l^2} \longrightarrow \mathcal{H}$,

$$T_W(\{f_i\}_{i \in I}) = \sum_{i \in I} w_i f_i, \quad \{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus W_i)_{l^2}.$$

Where

$$(\sum_{i \in I} \oplus W_i)_{l^2} = \{\{f_i\}_{i \in I} : f_i \in W_i, i \in I, \sum_{i \in I} \|f_i\|^2 < \infty\}.$$

Its adjoint operator, which is called the analysis operator $T_W^* : \mathcal{H} \longrightarrow (\sum_{i \in I} \oplus W_i)_{l^2}$, is defined by

$$T_W^*(f) = \{w_i \pi_{W_i} f\}_{i \in I}, \quad f \in \mathcal{H}.$$

And the K -fusion frame operator associated is $S_W : \mathcal{H} \longrightarrow \mathcal{H}$.

$$S_W(f) = \sum_{i \in I} w_i^2 \pi_{W_i} f, \quad f \in \mathcal{H}. \quad (2.3)$$

2.3. Controlled fusion frame.

Definition 2.3. [17] Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$, let $\{w_i\}_{i \in I}$ be a family of weights, and let $T, U \in GL(\mathcal{H})$. Then the family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is called a (T, U) -controlled fusion frame for $\mathcal{H}$, if there exist positive constants $A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} T f, \pi_{W_i} U f \rangle \leq B\|f\|^2, f \in \mathcal{H}. \quad (2.4)$$

A and B are called the lower and upper bounds of (T, U) -controlled fusion frame, respectively. For further information in K -fusion frame and controlled fusion frame theory we refer the reader to [3], [17] [18] and [9].

In theory of frames, often use the following theorem, which describes some properties of the adjoint operator.

Theorem 1. [7] *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and suppose that $U \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then,*

- i) $U^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ and $\|U^*\| = \|U\|$.
- ii) U is surjective if and only if $\exists A > 0$ such that $\|U^*h\|_{\mathcal{H}_2} \geq A\|h\|_{\mathcal{H}_1}$.

It is well-Known that not all bounded operator U on a Hilbert space $\mathcal{H}$ is invertible: an operator U needs to be injective and surjective in order to be invertible. For doing this, one can use right-inverse operator. The following lemma shows that if an operator U has closed range, there exists a "right-inverse operator" $U^\dagger$ in the following sense:

Lemma 2. [7] *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ be a closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}_U, \quad (2.5)$$

and

$$(U^*)^\dagger = (U^\dagger)^*.$$

The operator $U^\dagger$ is called the Pseudo-inverse of U .

In the literature, one will often see the pseudo-inverse of an operator U with closed range defined as the unique operator $U^\dagger$ satisfying that

$$\mathcal{N}_{U^\dagger} = \mathcal{R}_U^\perp, \quad UU^\dagger x = x, \quad x \in \mathcal{R}_U.$$

The following lemma is necessary for our results.

Lemma 3. [14] *Let $V \subseteq \mathcal{H}$ be a closed subspace, and T be a linear bounded operator on $\mathcal{H}$. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}. \quad (2.6)$$

If T is a unitary (i.e. $T^*T = TT^* = Id_{\mathcal{H}}$), then

$$\pi_{\overline{TV}} T = T \pi_V. \quad (2.7)$$

Proposition 2.1. [21] *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. Then the following condition are equivalent:*

- (1) *There exist $m > 0$ and $M < \infty$, such that $mI \leq T \leq MI$;*
- (2) *T is positive and there exist $m > 0$ and $M < \infty$, such that $m\|f\|^2 \leq \|T^{\frac{1}{2}}f\|^2 \leq M\|f\|^2$ for all $f \in \mathcal{H}$;*
- (3) *T is positive and $T^{\frac{1}{2}} \in GL(\mathcal{H})$;*
- (4) *There exists a self-adjoint operator $A \in GL(\mathcal{H})$, such that $A^2 = T$;*
- (5) *$T \in GL^+(\mathcal{H})$.*

The following lemma will be used in the sequel.

Lemma 4. [12] Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be Hilbert spaces. Let $T \in B(\mathcal{F}, \mathcal{G})$ and $T' \in B(\mathcal{H}, \mathcal{G})$ with $\overline{\mathcal{R}_{T^*}}$ be orthogonally complemented.

Then the following statements are equivalent:

- i) $T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$.
- ii) There exists $\mu > 0$ such that $\|T'^*z\| \leq \mu\|T^*z\|$, for all $z \in \mathcal{G}$.

Theorem 5. [13] Let T be a positive linear bounded operator on $\mathcal{H}$. T possesses a unique positive bounded square root which commutes with every bounded operator that commutes with T .

3. Controlled K -fusion frame

In this section, we introduce the notion of controlled K -fusion frames in Hilbert spaces and we discuss some their properties.

Definition 3.1. Let $K \in B(\mathcal{H})$, and $\{W_i\}_{i \in I}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$. Also, let $\{w_i\}_{i \in I}$ be a family of weights, and let $C, C' \in GL(\mathcal{H})$. Then $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is called a K -fusion frame controlled by C and C' or (C, C') -controlled K -fusion frame if there exist two constantants

$$0 < A_{CC'} \leq B_{CC'} < \infty$$

such that

$$A_{CC'}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} Cf, \pi_{W_i} C'f \rangle \leq B_{CC'}\|f\|^2, f \in \mathcal{H}. \quad (3.1)$$

Where K^* is the adjoint operator of K .

$A_{CC'}$ and $B_{CC'}$ are called lower and upper bounds of a (C, C') -controlled K -fusion frame respectively.

- (1) We call $\mathcal{W}$ a (C, C') -controlled Parsval K -fusion frame if $A_{CC'} = B_{CC'} = 1$.
- (2) If only the second inequality (3.1) is required, we call $\mathcal{W}$ a (C, C') -controlled Bessel K -fusion sequence with Bessel bound B .

Remark 6. i) If $K = I$ (where is the identity operator), then every (C, C') -controlled K -fusion frame is a (C, C') -controlled fusion frame.

ii) If $C = C' = I$, then every (C, C') -controlled K -fusion frame is a K -fusion frame.

iii) Every (C, C') -controlled fusion frame is a (C, C') -controlled K -fusion frame. Indeed, by definition (3.1) there exist constants $0 < A_{CC'} \leq B_{CC'}$, such that for all $f \in \mathcal{H}$, we have

$$A_{CC'}\|f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} Cf, \pi_{W_i} C'f \rangle \leq B_{CC'}\|f\|^2.$$

Therefore, for $\|K\| > 0$, one has

$$A_{CC'}\|K^*f\|^2 \leq A_{CC'}\|K^*\|^2\|f\|^2 \leq A_{CC'}\|K\|^2\|f\|^2,$$

that is,

$$\frac{A_{CC'}}{\|K\|^2}\|K^*f\|^2 \leq A_{CC'}\|f\|^2,$$

it follows that,

$$\frac{A_{CC'}}{\|K\|^2} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.$$

Hence, the family $\mathcal{W}$ is a (C, C') -controlled K -fusion frame for $\mathcal{H}$.

The next example shows that in general, frames may be controlled K -fusion frame without being a controlled fusion frame.

Example 3.1. Let $\mathcal{H} = l_2(\mathbb{C}) = \{\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \mid \sum_{n=0}^{+\infty} |a_n|^2 < \infty\}$ be a Hilbert space, with respect to the inner product

$$\langle \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} a_n \bar{b}_n,$$

equipped with the norm

$$\|\{a_n\}_{n \in \mathbb{N}}\|_{l_2(\mathbb{C})} = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Consider two operators C and C' defined by

$$\begin{aligned} C : \mathcal{H} &\longrightarrow \mathcal{H} \\ \{a_n\}_{n \in \mathbb{N}} &\longmapsto \{\alpha a_n\}_{n \in \mathbb{N}} \\ \text{resp. } C' : \mathcal{H} &\longrightarrow \mathcal{H} \\ \{a_n\}_{n \in \mathbb{N}} &\longmapsto \{\beta a_n\}_{n \in \mathbb{N}} \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}_+^*$.

It is easy to see that:

- C and C' are positives.
- C and C' are invertibles.

Then, invertible operators are given respectively by:

$$\begin{aligned} C^{-1} : \mathcal{H} &\longrightarrow \mathcal{H} \\ \{a_n\}_{n \in \mathbb{N}} &\longmapsto \{\alpha^{-1} a_n\}_{n \in \mathbb{N}}, \end{aligned}$$

and

$$\begin{aligned} C'^{-1} : \mathcal{H} &\longrightarrow \mathcal{H} \\ \{a_n\}_{n \in \mathbb{N}} &\longmapsto \{\beta^{-1} a_n\}_{n \in \mathbb{N}}. \end{aligned}$$

Let $E_i = \{a_j\}_{j \in \mathbb{N}}$, where $a_j = \{\delta_i^j\}_{j \in \mathbb{N}}$ (where δ_i^j is the Kronecker symbol). Let $\{W_i\}_{i \in \mathbb{N}}$ be a closed subspaces of $\mathcal{H}$ such that $W_i = \mathbb{C}E_i$, and let $w_i = \frac{1}{\sqrt{i+1}}$, for all $i \in \mathbb{N}$.

The family $\mathcal{W} = \{W_i, w_i\}_{i \in \mathbb{N}}$ is a (C, C') -controlled Bessel fusion sequence. Indeed for each $\{a_n\}_{n \in \mathbb{N}} \in \mathcal{H}$, we have

$$\sum_{i \in \mathbb{N}} w_i^2 \langle \pi_{W_i} C(\{a_n\}_{n \in \mathbb{N}}), \pi_{W_i} C'(\{a_n\}_{n \in \mathbb{N}}) \rangle = \alpha \beta \sum_{i \in \mathbb{N}} \frac{1}{i+1} |a_i|^2$$

$$\begin{aligned}
&\leq \alpha\beta \sum_{i \in \mathbb{N}} |a_i|^2 \\
&= \alpha\beta \|\{a_n\}_{n \in \mathbb{N}}\|_{\mathcal{H}}^2.
\end{aligned}$$

But is not (C, C') -controlled fusion frame, For this, assume the contrary that exists $A_{CC'} > 0$ such that:

$$A_{CC'} \sum_{i \in \mathbb{N}} |a_i|^2 \leq \sum_{i \in \mathbb{N}} \frac{\alpha\beta}{i+1} |a_i|^2. \quad (3.2)$$

Hence

$$\sum_{i \in \mathbb{N}} |a_i|^2 < \infty \implies \lim_{i \rightarrow +\infty} a_i = 0$$

So, we have

$$\begin{aligned}
&|a_j| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty; \\
&\frac{\alpha\beta}{i+1} \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \\
&\implies \begin{cases} \forall \varepsilon \geq 0 \quad \exists N \in \mathbb{N} : j \geq N \implies |a_j| < \varepsilon, \\ \forall \gamma \geq 0 \quad \exists N \in \mathbb{N} : j \geq M \implies \frac{\alpha\beta}{j+1} < \gamma. \end{cases} \\
&\implies \sum_{i \in \mathbb{N}} \frac{\alpha\beta - (j+1)A_{CC'}}{j+1} |a_j|^2 \geq 0.
\end{aligned}$$

By fixing $\varepsilon = \gamma$, there exist $N, M \in \mathbb{N}^*$, such that

$$\begin{cases} j \geq N \implies |a_j| < \varepsilon, \\ j \geq M \implies \frac{\alpha\beta}{j+1} < \varepsilon. \end{cases}$$

Now, let $N_1 = \max(N, M)$, then $\forall j \geq N_1, |a_j| < \varepsilon$ and $\frac{\alpha\beta}{i+1} < \varepsilon$. Hence

$$\sum_{i=0}^{N_1-1} \frac{\alpha\beta - (i+1)A_{CC'}}{i+1} |a_i|^2 + \varepsilon^2 \sum_{i=N_1}^{\infty} (\varepsilon - A_{CC'}) \geq 0.$$

Now, for $\varepsilon = \frac{A_{CC'}}{2}$, we obtain

$$\sum_{i=0}^{N_1-1} \frac{\alpha\beta - (i+1)A_{CC'}}{i+1} |a_i|^2 + \left(\frac{A_{CC'}}{2}\right)^2 \sum_{i=N_1}^{\infty} \left(-\frac{A_{CC'}}{2}\right) \geq 0.$$

absurde.

Now if we considere the operator

$$\begin{aligned}
K: \mathcal{H} &\longrightarrow \mathcal{H} \\
\{a_n\}_{n \in \mathbb{N}} &\longmapsto \left\{ \frac{a_n}{\sqrt{n+1}} \right\}_{n \in \mathbb{N}}.
\end{aligned}$$

Then, K is a bounded linear operator on $\mathcal{H}$. Furthermore, for each $\{a_n\}_{n \in \mathbb{N}} \in \mathcal{H}$, we have

$$\langle K^*(\{a_n\}_{n \in \mathbb{N}}), K^*(\{a_n\}_{n \in \mathbb{N}}) \rangle = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

So,

$$\sum_{n=0}^{\infty} \frac{\alpha\beta}{2} \frac{|a_n|^2}{n+1} \leq \sum_{n=0}^{\infty} \frac{\alpha\beta}{n+1} |a_n|^2 \leq \alpha\beta \sum_{n=0}^{\infty} |a_n|^2,$$

The following proposition provides a relation between controlled K -fusion frames and controlled fusion frames.

Proposition 7. *Let $K \in \mathcal{B}(\mathcal{H})$ be a closed range operator. Then, every (C, C') -controlled K -fusion frame is a (C, C') -controlled fusion frame for $\mathcal{R}_K$.*

Proof. Let $\mathcal{W} = \{(W_i, w_i)\}_{i \in I}$ be a (C, C') -controlled K -fusion frame with frame bounds $A_{CC'}$ and $B_{CC'}$. Then for all $f \in \mathcal{R}_K$, we have

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.$$

Therefore, via lemma 2, we have

$$A_{CC'} \|f\|^2 \leq A_{CC'} \|(K^*)^\dagger f\|^2 \|K^* f\|^2.$$

Hence,

$$\frac{A_{CC'}}{\|(K^*)^\dagger\|^2} \|f\|^2 \leq A_{CC'} \|K^* f\|^2.$$

Thus,

$$\frac{A_{CC'}}{\|(K^*)^\dagger\|^2} \|f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.$$

So, we have the result.

If $\mathcal{W}$ is a (C, C') -controlled K -fusion frame and $C'^* \pi_{W_i} C$ is a positive operator for each $i \in I$, then $C'^* \pi_{W_i} C = C^* \pi_{W_i} C'$ and we have

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \|(C'^* \pi_{W_i} C)^{\frac{1}{2}} f\|^2 \leq B_{CC'} \|f\|^2, f \in \mathcal{H}.$$

Indeed,

$$\begin{aligned} \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle &= \sum_{i \in I} w_i^2 \langle C'^* \pi_{W_i} C f, f \rangle \\ &= \sum_{i \in I} w_i^2 \langle (C'^* \pi_{W_i} C)^{\frac{1}{2}} f, (C'^* \pi_{W_i} C)^{\frac{1}{2}} f \rangle \\ &= \sum_{i \in I} w_i^2 \|(C'^* \pi_{W_i} C)^{\frac{1}{2}} f\|^2. \end{aligned}$$

We define the controlled analysis operator by

$$T_{CC'} : \mathcal{H} \longrightarrow \mathcal{K}$$

$$f \longmapsto T_{CC'}(f) := (w_i(C'^* \pi_{W_i} C)^{\frac{1}{2}} f)_{i \in I},$$

Where

$$\mathcal{K} = \{(w_i(C'^* \pi_{W_i} C)^{\frac{1}{2}} f)_{i \in I} | f \in \mathcal{H}\} \subseteq (\oplus_{i \in I} \mathcal{H})_{l^2}.$$

$\mathcal{K}$ is closed [17] and $T_{CC'}$ is well defined. Moreover $T_{CC'}$ is a bounded linear operator. Its adjoint operator is given by

$$\begin{aligned} T_{CC'}^* : \mathcal{K} &\longrightarrow \mathcal{H} \\ (w_i(C'^* \pi_{W_i} C)^{\frac{1}{2}} f)_{i \in I} &\longmapsto T_{CC'}^*((w_i(C'^* \pi_{W_i} C)^{\frac{1}{2}} f)_{i \in I}) := \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, \end{aligned}$$

and is called the controlled synthesis operator.

Therefore, we define the controlled K -fusion frame operator $S_{CC'}$ on $\mathcal{H}$ by

$$S_{CC'} = T_{CC'}^* T_{CC'}(f) = \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, \quad f \in \mathcal{H}. \quad (3.3)$$

In fact, many of the properties of the ordinary K -fusion frames are valid in this case.

Lemma 8. *Let $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ be a (C, C') -controlled K -fusion frame with bounds $A_{CC'}$ and $B_{CC'}$. Then the operator $S_{CC'}$ (3.3) is a well defined, linear, positive, bounded and self-adjoint operator. Furthermore, we have*

$$A_{CC'} K K^* \leq S_{CC'} \leq B_{CC'} Id_{\mathcal{H}}. \quad (3.4)$$

Proof. • By definition, $S_{CC'}$ is a linear bounded and well defined operator, and it is clear to see that $S_{CC'}$ is a positive and self-adjoint operator.

- The family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is a (C, C') -controlled K -fusion frame for $\mathcal{H}$ with bounds $A_{CC'}$ and $B_{CC'}$ if and only if

$$A_{CC'} \|K^* f\|^2 \leq \langle S_{CC'} f, f \rangle = \langle \sum_{i \in I} w_i^2 C'^* \pi_{W_i} C f, f \rangle \leq B_{CC'} \|f\|^2, \quad f \in \mathcal{H},$$

that is,

$$A_{CC'} \langle K K^* f, f \rangle \leq \langle S_{CC'} f, f \rangle \leq B_{CC'} \langle f, f \rangle, \quad f \in \mathcal{H}.$$

Hence,

$$A_{CC'} K K^* \leq S_{CC'} \leq B_{CC'} Id_{\mathcal{H}},$$

so the conclusion holds.

The next theorem generalizes the situation of controlled Bessel K -fusion sequence. Since it has similar procedure, the proof is omitted.

Theorem 9. *$\mathcal{W}$ is a (C, C') -controlled Bessel K -fusion sequence with bound $B_{CC'}$ if and only if $T_{CC'}^*$ is well-defined bounded operator and $\|T_{CC'}^*\| \leq \sqrt{B}$.*

Controlled K -fusion frame operator of (C, C') -controlled K -fusion frame is not invertible in general, but we can show that it is invertible on the subspace $\mathcal{R}_K \subset \mathcal{H}$. In fact, since $\mathcal{R}_K$ is closed

$$KK^\dagger|_{\mathcal{R}_K} = id_{\mathcal{R}_K},$$

so we have

$$id_{\mathcal{R}_K}^* = (K^\dagger|_{\mathcal{R}_K})^* K^*.$$

Hence for any $f \in \mathcal{R}_K$

$$\|f\| = \|(K^\dagger|_{\mathcal{R}_K})^* K^* f\| \leq \|K^\dagger\| \|K^* f\|,$$

that is,

$$\|f\|^2 \leq \|K^\dagger\|^2 \|K^* f\|^2. \quad (3.5)$$

Combined with (3.1) we have

$$\langle S_{CC'} f, f \rangle \geq A_{CC'} \|K^* f\|^2 \geq A_{CC'} \|K^\dagger\|^2 \|f\|^2, \quad \forall f \in \mathcal{R}_K.$$

So from the definition of (C, C') -controlled K -fusion frame, one implies that $S : \mathcal{R}_K \rightarrow S(\mathcal{R}_K)$ is an isomorphism, furthermore we have

$$B_{CC'}^{-1} \|f\| \leq \|S^{-1} f\| \leq A_{CC'}^{-1} \|K^\dagger\|^2 \|f\|, \quad \forall f \in (S(\mathcal{R}_K)).$$

Theorem 10. Let $K \in B(\mathcal{H})$ be a closed range operator, then $\mathcal{W}$ is a (C, C') -controlled K -fusion frame with bounds $A_{CC'}$ and $B_{CC'}$ if and only if $T_{CC'}^*$ is well-defined and surjective.

Proof. Let the sequence $\mathcal{W}$ be a (C, C') -controlled K -fusion frame for $\mathcal{H}$, and let $S_{CC'}$ be its controlled K -fusion frame operator. Then, it is a (C, C') -controlled Bessel K -fusion sequence and therefore, by Theorem 9, the bounded operator $T_{CC'}^*$ is well-defined. It remains to show that By definition, for each $f \in \mathcal{H}$, we have

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.$$

In particular, we have

$$A_{CC'} \|K^* f\|^2 \leq \langle S_{CC'} f, f \rangle \leq \|S_{CC'} f\| \|f\|.$$

Since, $S_{CC'} = T^* T$, then

$$\|S_{CC'} f\| \|f\| \leq \|T\| \|T f\| \|f\|.$$

Hence,

$$A_{CC'} \|T\|^{-1} \|K^* f\|^2 \leq \|T f\| \|f\|.$$

Since $\mathcal{R}_K$ is closed, and via lemma we have

$$KK^\dagger|_{\mathcal{R}_K} = id_{\mathcal{R}_K},$$

so we have

$$id_{\mathcal{R}_K}^* = (K^\dagger|_{\mathcal{R}_K})^* K^*.$$

Hence for any $f \in \mathcal{R}_K$

$$\|f\| = \|(K^\dagger|_{\mathcal{R}_K})^* K^* f\|,$$

that is,

$$\|f\|^2 \leq \|(K^*)^\dagger\|^2 \|K^* f\|^2. \quad (3.6)$$

Therefore, we have

$$A_{CC'} \|f\|^2 \leq A_{CC'} \|(K^*)^\dagger\|^2 \|K^* f\|^2.$$

Hence,

$$\begin{aligned} \frac{A_{CC'}}{\|(K^*)^\dagger\|^2} \|f\|^2 &\leq A_{CC'} \|K^* f\|^2. \\ \|Tf\| &\geq \frac{A_{CC'}}{\|T\| \|(K^*)^\dagger\|^2} \|f\|. \end{aligned}$$

Thus, $T_{CC'}^*$ is a surjective operator.

Conversely, let $T_{CC'}^*$ be a well-defined, bounded and surjective, then theorem 9 shows that $\mathcal{W}$ is a (C, C') -controlled Bessel K -fusion sequence for $\mathcal{H}$. Therefore, for each $f \in \mathcal{H}$, since $T_{CC'}^*$ is surjective, then, by Lemma 2, there exists an operator $(T_{CC'}^*)^\dagger : \mathcal{H} \rightarrow \mathcal{K}$, such that

$$T_{CC'}^* (T_{CC'}^*)^\dagger = id.$$

Hence,

$$T_{CC'}^\dagger T_{CC'} = id.$$

So, for each $f \in \mathcal{H}$, we have

$$\begin{aligned} \|K^* f\|^2 &\leq \|K\|^2 \|T_{CC'}^\dagger\|^2 \|T_{CC'} f\|^2 \\ &= \|T_{CC'}^\dagger\|^2 \|K\|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle. \end{aligned}$$

Therefore, $\mathcal{W}$ is a (C, C') -controlled K -fusion frame for $\mathcal{H}$.

Proposition 11. *Let $K \in B(\mathcal{H})$, $C, C' \in GL^+(\mathcal{H})$ and let $\mathcal{W}$ be a (C, C') -controlled K -fusion frame for $\mathcal{H}$ with bounds $A_{CC'}$ and $B_{CC'}$ with $\overline{\mathcal{R}_{T^*}}$ is orthogonally complemented. If $T \in B(\mathcal{H})$ with $\mathcal{R}_T \subset \mathcal{R}_K$. Then $\mathcal{W}$ is a (C, C') -controlled T -fusion frame for $\mathcal{H}$.*

Proof. Assume that $\mathcal{W}$ be a (C, C') -controlled K -fusion frame for $\mathcal{H}$ with bounds $A_{CC'}$ and $B_{CC'}$. Then for each $f \in \mathcal{H}$, we have

$$A_{CC'} \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \langle f, f \rangle.$$

Since $\mathcal{R}_T \subset \mathcal{R}_K$, so by using lemma 4, there exists some $\lambda > 0$ such that

$$TT^* \leq \lambda KK^*.$$

This implies that for all $f \in \mathcal{H}$, we have

$$A_{CC'} \langle T^* f, T^* f \rangle \leq A_{CC'} \lambda \langle K^* f, K^* f \rangle.$$

Therefore,

$$\frac{A_{CC'}}{\lambda} \langle T^* f, T^* f \rangle \leq A_{CC'} \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \langle f, f \rangle.$$

Then, $\mathcal{W}$ is a (C, C') -controlled T -fusion frame for $\mathcal{H}$ with bounds $\frac{A_{CC'}}{\lambda}$ and $B_{CC'}$.

Theorem 12. Let $K_1, K_2 \in B(\mathcal{H})$ such that $\mathcal{R}_{K_1^*} \perp \mathcal{R}_{K_2^*}$. If $\mathcal{W}$ is a (C, C') -controlled K_i -fusion frame for $\mathcal{H}$ ($i = 1, 2$). Then $\mathcal{W}$ is a (C, C') -controlled $(\alpha K_1 + \beta K_2)$ -fusion frame for $\mathcal{H}$, where $\alpha, \beta \in \mathbb{C}$.

Proof. Since $\mathcal{W}$ is a (C, C') -controlled K_i -fusion frame for $\mathcal{H}$ ($i = 1, 2$), there exist $A_{CC'}^j, B_{CC'}^j > 0$, such that for all $f \in \mathcal{H}$, $j = 1, 2$, we have

$$A_{CC'}^j \langle K_j^* f, K_j^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'}^j \langle f, f \rangle.$$

Then for any $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f \rangle &= \langle \bar{\alpha} K_1^* f + \bar{\beta} K_2^* f, \bar{\alpha} K_1^* f + \bar{\beta} K_2^* f \rangle \\ &= |\alpha|^2 \langle K_1^* f, K_1^* f \rangle + \bar{\alpha} \beta \langle K_1^* f, K_2^* f \rangle + \alpha \bar{\beta} \langle K_2^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle. \end{aligned}$$

Since $\mathcal{R}_{K_1} \perp \mathcal{R}_{K_2}$, then, for any $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle K_1^* f, K_2^* f \rangle &= 0, \\ \langle K_2^* f, K_1^* f \rangle &= 0. \end{aligned}$$

Thus,

$$\langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f \rangle = |\alpha|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle.$$

Therefore, for any $f \in \mathcal{H}$, we have

$$\begin{aligned} |\alpha|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle &\leq \left(\frac{|\alpha|^2}{A_{CC'}^1} + \frac{|\beta|^2}{A_{CC'}^2} \right) \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \\ &\leq \frac{(A_{CC'}^2 |\alpha|^2 + A_{CC'}^1 |\beta|^2 + 1)}{A_{CC'}^1 A_{CC'}^2} \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle. \end{aligned}$$

Then,

$$\begin{aligned} 0 &< \frac{A_{CC'}^1 A_{CC'}^2}{(A_{CC'}^2 |\alpha|^2 + A_{CC'}^1 |\beta|^2 + 1)} (|\alpha|^2 \langle K_1^* f, K_1^* f \rangle + |\beta|^2 \langle K_2^* f, K_2^* f \rangle) \\ &\leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle. \end{aligned}$$

Then, we have

$$\frac{A_{CC'}^1 A_{CC'}^2 \langle (\alpha K_1 + \beta K_2)^* f, (\alpha K_1 + \beta K_2)^* f \rangle}{(A_{CC'}^2 |\alpha|^2 + A_{CC'}^1 |\beta|^2 + 1)} \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle,$$

then,

$$\begin{aligned} \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle &= \frac{1}{2} \left(\sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle + \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \right) \\ &\leq \frac{B_{CC'}^1 + B_{CC'}^2}{2} \langle f, f \rangle. \end{aligned}$$

Thus, $\mathcal{W}$ is a (C, C') -controlled $\alpha K_1 + \beta K_2$ -fusion frame with bounds $\frac{A_{CC'}^1 A_{CC'}^2}{(|\alpha|^2 A_{CC'}^2 + |\beta|^2 A_{CC'}^1 + 1)}$ and $\frac{B_{CC'}^1 + B_{CC'}^2}{2}$.

Lemma 13. Let $K \in B(\mathcal{H})$, and $C, C' \in GL^+(\mathcal{H})$ such that $CC' = C'C$. Assume that $CK = KC$, $C'K = KC'$; $SC = CS$, $SC' = C'S$ and $\pi_{W_i} C = C \pi_{W_i}$, $\pi_{W_i} C' = C' \pi_{W_i}$. Then, $\mathcal{W}$ is a (C, C') -controlled K -fusion frame for $\mathcal{H}$ if and only if $\mathcal{W}$ is a K -fusion frame for $\mathcal{H}$. Where S is the K -fusion frame operator (2.3), defined by

$$Sf = \sum_{i \in I} w_i^2 \pi_{W_i} f, \quad f \in \mathcal{H}.$$

Proof. Assume that $\mathcal{W}$ is a K -fusion frame with bounds A and B . Then for each $f \in \mathcal{H}$, we have

$$A \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 \leq B \|f\|^2.$$

Since, C and C' are linear bounded operators, applying 2.1, there exist constants m, m', M and $M' > 0$ such that

$$\begin{cases} mI \leq C \leq MI, \\ m'I \leq C' \leq M'I. \end{cases}$$

$$\langle SCf, f \rangle = \langle f, CSf \rangle.$$

Then,

$$mKK^* \leq CS \leq MS \leq MBI.$$

We deduce that

$$mm'KK^* \leq C'SC \leq MM'BI.$$

Therefore, for each $f \in \mathcal{H}$, we have

$$mm' A \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq MM' B \|f\|^2.$$

Thus, $\mathcal{W}$ is a (C, C') -controlled K -fusion frame.

Conversely, Assume that $\mathcal{W}$ is a (C, C') -controlled K -fusion frame with bounds A and B . Then for each $f \in \mathcal{H}$, we have

$$A_{CC'} \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} C f, \pi_{W_i} C' f \rangle \leq B_{CC'} \|f\|^2.$$

By assumption, $C, C' \in GL^+(\mathcal{H})$, $CC' = C'C$ and via theorem 5, we have

$$\begin{aligned} ((C)^{\frac{1}{2}}(C')^{-\frac{1}{2}})^* &= (C'^*)^{-\frac{1}{2}}(C^*)^{\frac{1}{2}} \\ &= (C')^{-\frac{1}{2}}(C)^{\frac{1}{2}} \\ &= (C)^{\frac{1}{2}}(C')^{-\frac{1}{2}}. \end{aligned}$$

Then, for each $f \in \mathcal{H}$, we have

$$\begin{aligned} A_{CC'} \langle K^* f, K^* f \rangle &= A_{CC'} \langle (CC')^{\frac{1}{2}}(CC')^{-\frac{1}{2}} K^* f, (CC')^{\frac{1}{2}}(CC')^{-\frac{1}{2}} K^* f \rangle \\ &\leq \| (CC')^{\frac{1}{2}} \|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i} C (CC')^{-\frac{1}{2}} f, \pi_{W_i} C' (CC')^{-\frac{1}{2}} f \rangle \\ &= \| (CC')^{\frac{1}{2}} \|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i} (C)^{\frac{1}{2}} (C')^{-\frac{1}{2}} f, \pi_{W_i} (C')^{\frac{1}{2}} (C)^{-\frac{1}{2}} f \rangle \\ &= \| (CC')^{\frac{1}{2}} \|^2 \sum_{i \in I} w_i^2 \langle ((C')^{\frac{1}{2}}(C)^{-\frac{1}{2}})^* \pi_{W_i} (C)^{\frac{1}{2}} (C')^{-\frac{1}{2}} f, f \rangle \\ &= \| (CC')^{\frac{1}{2}} \|^2 \sum_{i \in I} w_i^2 \langle ((C')^{\frac{1}{2}}(C)^{-\frac{1}{2}})^* \pi_{W_i} (C)^{\frac{1}{2}} (C')^{-\frac{1}{2}} f, f \rangle \\ &= \| (CC')^{\frac{1}{2}} \|^2 \sum_{i \in I} w_i^2 \langle \pi_{W_i} (C')^{\frac{1}{2}} (C)^{-\frac{1}{2}} (C)^{\frac{1}{2}} (C')^{-\frac{1}{2}} f, f \rangle \\ &= \| (CC')^{\frac{1}{2}} \|^2 \langle \sum_{i \in I} w_i^2 \pi_{W_i} f, f \rangle. \end{aligned}$$

$$\implies A_{CC'} \| (CC')^{\frac{1}{2}} \|^2 \langle K^* f, K^* f \rangle \leq \sum_{i \in I} w_i^2 \langle \pi_{W_i} f, \pi_{W_i} f \rangle.$$

In the other hand

$$\sum_{i \in I} w_i^2 \langle \pi_{W_i} f, \pi_{W_i} f \rangle = \langle Sf, f \rangle,$$

where $Sf = \sum_{i \in I} w_i^2 \pi_{W_i} f$.

$$\begin{aligned} \langle Sf, f \rangle &= \langle (CC')^{-\frac{1}{2}}(CC')^{\frac{1}{2}} Sf, f \rangle \\ &= \langle (CC')^{\frac{1}{2}} Sf, ((CC')^{-\frac{1}{2}})^* f \rangle \\ &= \langle (CC')^{\frac{1}{2}} Sf, (CC')^{-\frac{1}{2}} f \rangle \\ &= \langle CC' (CC')^{-\frac{1}{2}} Sf, (CC')^{-\frac{1}{2}} f \rangle \\ &= \langle C' SC (CC')^{-\frac{1}{2}} f, (CC')^{-\frac{1}{2}} f \rangle \\ &\leq B_{CC'} \| (CC')^{-\frac{1}{2}} \|^2 \| f \|^2. \end{aligned}$$

Thus, $\mathcal{W}$ is a K -fusion frame with bounds $A_{CC'} \| (CC')^{\frac{1}{2}} \|^2$ and $B_{CC'} \| (CC')^{-\frac{1}{2}} \|^2$.

Theorem 14. Let $K \in B(\mathcal{H})$, let $\mathcal{W}$ be a (C, C) -controlled K -fusion frame with bounds A_{CC} and B_{CC} . If $U \in \mathcal{B}(\mathcal{H})$ is an invertible operator such that $U^*C = CU^*$ and $K^*(U^*)^{-1} = (U^*)^{-1}K^*$, then $(UW_i, w_i)_{i \in I}$ is a (C, C) -controlled K -fusion frame for $\mathcal{H}$.

Proof. Assume that $\mathcal{W}$ is a (C, C) -controlled K -fusion frame with bounds A_{CC} and B_{CC} . By definition, for each $f \in \mathcal{H}$, we have

$$A_{CC}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i} Cf\|^2 \leq B_{CC}\|f\|^2.$$

Now, let $f \in \mathcal{H}$. Via lemma 3 and since UW_i is closed, we have

$$\begin{aligned} \|\pi_{W_i} CU^*f\| &= \|\pi_{W_i} U^* Cf\| = \|\pi_{W_i} U^* \pi_{\overline{UW_i}} Cf\| = \|\pi_{W_i} U^* \pi_{UW_i} Cf\| \\ &\leq \|U\| \|\pi_{UW_i} Cf\|. \end{aligned}$$

Therefore,

$$\begin{aligned} A_{CC}\|K^*U^*f\|^2 &\leq \sum_{i \in I} w_i^2 \|\pi_{W_i} CU^*f\|^2 \\ &\leq \|U\|^2 \sum_{i \in I} w_i^2 \|\pi_{UW_i} Cf\|^2. \end{aligned}$$

$$\begin{aligned} \|K^*f\|^2 &= \|K^*(U^*)^{-1}U^*f\|^2 = \|(U^*)^{-1}K^*U^*f\|^2 \\ &\leq \|U^{-1}\|^2 \|K^*U^*f\|^2. \end{aligned}$$

Then, we have

$$\frac{A_{CC}}{\|U^{-1}\|^{-2}\|U\|^{-2}} \|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{UW_i} Cf\|^2. \quad (3.7)$$

On the other hand, via lemma 3, we obtain with U^{-1} instead of T :

$$\pi_{UW_i} = \pi_{UW_i}(U^*)^{-1} \pi_{W_i} U^*.$$

Thus,

$$\begin{aligned} \|\pi_{UW_i} Cf\|^2 &= \|\pi_{UW_i}(U^*)^{-1} \pi_{W_i} U^* Cf\|^2 \\ &\leq \|U^{-1}\|^2 \|\pi_{W_i} U^* Cf\|^2, \end{aligned}$$

and it follows

$$\sum_{i \in I} w_i^2 \|\pi_{UW_i} Cf\|^2 \leq \|U^{-1}\|^2 \sum_{i \in I} w_i^2 \|\pi_{W_i} U^* Cf\|^2.$$

hence,

$$\sum_{i \in I} w_i^2 \|\pi_{UW_i} Cf\|^2 \leq B_{CC} \|U^{-1}\|^2 \|U\|^2 \|f\|^2.$$

Thus, $\mathcal{W}$ is a (C, C) -controlled K -fusion frame with bounds $A_{CC} \|U^{-1}\|^{-2} \|U\|^{-2}$ and $B_{CC} \|U^{-1}\|^2 \|U\|^2$.

Corollary 15. Let $K \in B(\mathcal{H})$, let $\mathcal{W}$ be a (C, C) -controlled K -fusion frame with bounds A_{CC} and B_{CC} . If $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator such that $U^{-1}C = CU^{-1}$ and $K^*U = UK^*$, then $(UW_i, w_i)_{i \in I}$ is a (C, C) -controlled K -fusion frame for $\mathcal{H}$.

Proof. The result follows from Theorem 14.

4. Perturbation on Controlled K -fusion frame

The following result provides a sufficient condition on a family of closed subspaces of $\mathcal{H}$ to be a controlled K -fusion frame, in the presence of another controlled K -fusion frame. In fact it is a generalisation of Proposition 2.4 in [2], Proposition 4.6 in [8] and Proposition 2.6 in [17].

Proposition 16. Let $K \in B(\mathcal{H})$ be a closed range operator $\mathcal{R}_K$, let $T, U \in GL(\mathcal{H})$ and let $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ be a (C, C') -controlled K -fusion frame for $\mathcal{H}$ with lower and upper bounds $A_{CC'}$ and $B_{CC'}$, respectively. Let $\{V_i\}_{i \in I}$ be a family of closed subspaces of $\mathcal{H}$. If there exists a number $0 < R < A_{CC'}$ such that

$$0 < \sum_{i \in I} w_i^2 \langle C'^*(\pi_{V_i} - \pi_{W_i})Cf, f \rangle \leq R\|f\|^2, \forall f \in \mathcal{H}, \quad (4.1)$$

then $\mathcal{V} = \{V_i, w_i\}_{i \in I}$ is a (C, C') -controlled Bessel K -fusion sequence for $\mathcal{H}$ and a (C, C') -controlled K -fusion frame for $\mathcal{R}_K$.

Proof. Let $f \in \mathcal{H}$. Considering that the family $\mathcal{W} = \{W_i, w_i\}_{i \in I}$ is a (C, C') -controlled K -fusion frame for $\mathcal{H}$, we have

$$A_{CC'}\|K^*f\|^2 \leq \sum_{i \in I} w_i^2 \langle C'^*\pi_{W_i}Cf, f \rangle \leq B_{CC'}\|f\|^2.$$

Firstly, let us prove that $\{V_i, w_i\}_{i \in I}$ is a (C, C') -controlled Bessel K -fusion sequence for $\mathcal{H}$. We have

$$\begin{aligned} \sum_{i \in I} w_i^2 \langle C'^*\pi_{V_i}Cf, f \rangle &= \sum_{i \in I} w_i^2 \langle C'^*(\pi_{V_i} - \pi_{W_i})Cf, f \rangle + \sum_{i \in I} w_i^2 \langle C'^*\pi_{W_i}Cf, f \rangle \\ &\leq R\|f\|^2 + B_{CC'}\|f\|^2, \end{aligned}$$

consequently,

$$\sum_{i \in I} w_i^2 \langle C'^*\pi_{V_i}Cf, f \rangle \leq (R + B_{CC'})\|f\|^2.$$

Now, let us establish for $\{V_i, w_i\}_{i \in I}$ the left-hand side. We obtain

$$\begin{aligned} \sum_{i \in I} w_i^2 \langle C'^*\pi_{V_i}Cf, f \rangle &= \sum_{i \in I} w_i^2 \langle C'^*\pi_{W_i}Cf, f \rangle + \sum_{i \in I} w_i^2 \langle C'^*(\pi_{V_i} - \pi_{W_i})Cf, f \rangle \\ &\geq \sum_{i \in I} w_i^2 \langle C'^*\pi_{W_i}Cf, f \rangle - \sum_{i \in I} w_i^2 \langle C'^*(\pi_{V_i} - \pi_{W_i})Cf, f \rangle \\ &\geq A_{CC'}\|K^*f\|^2 - R\|f\|^2. \end{aligned} \quad (4.2)$$

Therefore, for any $f \in \mathcal{R}_K$, we have

$$\|f\| = \|(K^\dagger|_{\mathcal{R}_K})^* K^* f\| \leq \|K^\dagger\| \|K^* f\|,$$

that is, $\|K^* f\|^2 \geq \|K^\dagger\|^{-2} \|f\|^2$.

$$-R\|f\|^2 \geq -R\|K^\dagger\|^{-2} \|K^* f\|^2. \quad (4.3)$$

Then, according to (4.2) and (4.3), we have

$$\sum_{i \in I} w_i^2 \langle C'^* \pi_{V_i} C f, f \rangle \geq (A_{CC'} - R\|K^\dagger\|^{-2}) \|K^* f\|^2.$$

Which completes the proof.

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