Moroccan J. of Pure and Appl. Anal. (MJPAA)

Volume 7(1), 2021, Pages 134–172

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2021-0012](https://doi.org/10.2478/mjpaa-2021-0012)

Complete Approximations by Multivariate Generalized Gauss-Weierstrass Singular Integrals

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ABSTRACT. This research and survey article deals exclusively with the study of the approximation of generalized multivariate Gauss-Weierstrass singular integrals to the identity-unit operator. Here we study quantitatively most of their approximation properties. The multivariate generalized Gauss-Weierstrass operators are not in general positive linear operators. In particular we study the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. These are given via Jackson type inequalities and by the use of multivariate high order modulus of smoothness of the high order partial derivatives of the involved function. Also we study the global smoothness preservation properties of these operators. These multivariate inequalities are nearly sharp and in one case the inequality is attained, that is sharp. Furthermore we give asymptotic expansions of Voronovskaya type for the error of multivariate approximation. The above properties are studied with respect to L_p norm, $1 \leq p \leq \infty$.

Mathematics Subject Classification (2020). 26A15, 26D15, 41A17, 41A35.

Key words and phrases. Approximations by multivariate Gauss-Weierstrass singular integral, Multivariate moduli of smoothness.

1. Introduction

We start with our motivation for this work which comes from [6].

In the next we introduce and deal with the smooth Gauss-Weierstrass singular integral operators $W_{r,\tilde{\epsilon}}(f, x)$ defined as follows.

Received : October 10, 2020 - Accepted: November 25, 2020.

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For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \quad (1)$$

that is $\sum_{j=0}^r \alpha_j = 1$.

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$W_{r,\xi}(f; x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x + jt) \right) e^{-\frac{t^2}{\xi}} dt. \quad (2)$$

Note 1.1. The operators $W_{r,\xi}$ are not, in general, positive.

Let $r = 2$, $n = 3$. Then $\alpha_0 = \frac{23}{8}$, $\alpha_1 = -2$, $\alpha_2 = \frac{1}{8}$. Consider $f(t) = t^2 \geq 0$ and $x = 0$. Then

$$W_{r,\xi}(t^2; 0) = -\frac{3\xi}{4} < 0.$$

We notice by $\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1$, that $W_{r,\xi}(c, x) = c$, c constant, and

$$W_{r,\xi}(f; x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \quad (3)$$

We need the r th L_p -modulus of smoothness

$$\omega_r(f^{(n)}, h)_p := \sup_{|t| \leq h} \left\| \Delta_t^r f^{(n)}(x) \right\|_{p,x}, \quad h > 0, \quad (4)$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jt), \quad (5)$$

see [8], p. 44. Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

We need to introduce

$$\delta_k := \sum_{j=1}^r \alpha_j j^k, \quad k = 1, \dots, n \in \mathbb{N}, \quad (6)$$

and denote by $\lfloor \cdot \rfloor$ the integral part. Call

$$r(w, x) := \sum_{j=0}^r \alpha_j j^n f^{(n)}(x + jw) - \delta_n f^{(n)}(x). \quad (7)$$

According to [2], p. 306, [1], we get

$$\tau(w, x) = \Delta_w^r f^{(n)}(x). \quad (8)$$

Thus

$$\|\tau(w, x)\|_{p, x} \leq w_r \left(f^{(n)}, |w| \right)_p, \quad w \in \mathbb{R}. \quad (9)$$

Define

$$R_n(0, t, x) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!} \tau(w, x) dw, \quad n \in \mathbb{N}. \quad (10)$$

Using the above terminology ([6]) we obtain

$$\Delta(x) := W_{r, \xi}(f; x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x) \delta_{2m} \frac{(2m-1)(2m-3) \cdots 3 \cdot 1}{(2m)!} \left(\frac{\xi}{2} \right)^m = R_n^*(x), \quad (11)$$

where

$$R_n^*(x) := \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} R_n(0, t, x) e^{-\frac{t^2}{\xi}} dt, \quad n \in \mathbb{N}. \quad (12)$$

In $\Delta(x)$, see (11), the sum collapses when $n = 1$.

We mention

Theorem 1.2. ([6]) *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ and the rest as above. Then*

$$\|\Delta(x)\|_p \leq \frac{2^{\frac{p+1}{2p}} \tau^{\frac{1}{p}} \xi^{\frac{n}{2}}}{q^{\frac{1}{2q}} \pi^{\frac{1}{2p}} (rp+1)^{\frac{1}{p}} ((n-1)!) (q(n-1)+1)^{\frac{1}{q}}} \omega_r \left(f^{(n)}, \sqrt{\xi} \right)_p, \quad (13)$$

where

$$\tau := \left[\int_0^{\infty} (1+u)^{rp+1} u^{np-1} e^{-\frac{pu^2}{2}} du - \int_0^{\infty} u^{np-1} e^{-\frac{pu^2}{2}} du \right] < \infty. \quad (14)$$

Hence as $\xi \rightarrow 0$ we obtain $\|\Delta(x)\|_p \rightarrow 0$.

If additionally $f^{(2m)} \in L_p(\mathbb{R})$, $m = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ then $\|W_{r, \xi}(f) - f\|_p \rightarrow 0$, as $\xi \rightarrow 0$.

The counterpart of Theorem 1.2 follows, case of $p = 1$.

Theorem 1.3. ([6]) *Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$. Then*

$$\|\Delta(x)\|_1 \leq \frac{2}{(r+1)(n-1)! \sqrt{\pi}} \left(\sum_{k=1}^{r+1} \binom{r+1}{k} \left(\prod_{r=1}^{\lfloor \frac{n+k}{2} \rfloor - 1} (n+k-2r) \right) \cdot M \cdot \left(\frac{1}{2} \right)^{\lfloor \frac{n+k}{2} \rfloor} \right) \omega_r \left(f^{(n)}, \sqrt{\xi} \right)_1 \xi^{\frac{n}{2}}, \quad (15)$$

where

$$M = \begin{cases} \frac{\sqrt{\pi}}{2}, & n+k \text{ is even} \\ 1, & n+k \text{ is odd} \end{cases}.$$

Hence as $\xi \rightarrow 0$ we obtain $\|\Delta(x)\|_1 \rightarrow 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R})$, $m = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ then $\|W_{r, \xi}(f) - f\|_1 \rightarrow 0$, as $\xi \rightarrow 0$.

The case $n = 0$ is mentioned next.

Proposition 1.1. ([6]) *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above. Then*

$$\|W_{r,\xi}(f) - f\|_p \leq \frac{2^{\frac{1}{2q} + \frac{1}{p}} \pi^{\frac{1}{2q} - \frac{1}{2}}}{q^{\frac{1}{2q}}} \theta^{\frac{1}{p}} \omega_r(f, \sqrt{\xi})_p, \quad (16)$$

where

$$\theta := \int_0^\infty (1+t)^{rp} e^{-\frac{pt^2}{2}} dt < \infty.$$

Hence as $\xi \rightarrow 0$ we obtain $W_{r,\xi}(f) \rightarrow$ unit operator I in the L_p norm, $p > 1$.

We also mention

Proposition 1.2. ([6]) *It holds*

$$\|W_{r,\xi}f - f\|_1 \leq \frac{2}{\sqrt{\pi}} \left(\int_0^\infty (1+t)^r e^{-t^2} dt \right) \omega_r(f, \sqrt{\xi})_1. \quad (17)$$

Hence as $\xi \rightarrow 0$ we get $W_{r,\xi} \rightarrow I$ in the L_1 norm.

In this work we study the approximation properties of multivariate smooth general Gauss-Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f; x_1, \dots, x_N) := \frac{1}{(\sqrt{\pi \xi_n})^N} \sum_{j=0}^r \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j) e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N. \quad (18)$$

Notice that

$$\frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N = 1, \quad (19)$$

see [5].

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, and

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \dots, r, \\ 1 - \sum_{i=1}^r (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \quad (20)$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^r \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \dots, m \in \mathbb{N}. \quad (21)$$

See that $\sum_{j=0}^r \alpha_{j,r}^{[m]} = 1$.

Also here $\xi_n \in (0, 1]$, $n \in \mathbb{N}$, and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is a Borel measurable function. The above operator $W_{r,n}^{[m]}$ is a special case of a more general operator $\theta_{r,n}^{[m]}$ studied in general in [3] by the author.

We mention next about $\theta_{r,n}^{[m]}$.

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$.

We define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]}(f; x_1, \dots, x_N) := \sum_{j=0}^r \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j) d\mu_{\xi_n}(s), \quad (22)$$

where $s := (s_1, \dots, s_N)$, $x := (x_1, \dots, x_N) \in \mathbb{R}^N$.

The operators $\theta_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi(u_1, \dots, u_N) = \sum_{i=1}^N u_i^2$ and also take $r = 2$, $m = 3$; $x_i = 0$, $i = 1, \dots, N$. See that $\varphi \geq 0$, however

$$\begin{aligned} \theta_{2,n}^{[3]}(\varphi; 0, 0, \dots, 0) &= \left(\sum_{j=1}^2 j^2 \alpha_{j,2}^{[3]} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^N s_i^2 \right) d\mu_{\xi_n}(s) = \\ &= \left(\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^N s_i^2 \right) d\mu_{\xi_n}(s) = \left(-2 + \frac{1}{2} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^N s_i^2 \right) d\mu_{\xi_n}(s) < 0. \end{aligned} \quad (23)$$

assuming that $\int_{\mathbb{R}^N} \left(\sum_{i=1}^N s_i^2 \right) d\mu_{\xi_n}(s) < \infty$.

Clearly in the case of $W_{r,n}^{[m]}$ we have that

$$d\mu_{\xi_n}(s) = \frac{1}{(\sqrt{\pi \xi_n})^N} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N, \quad s \in \mathbb{R}^N.$$

Lemma 1.1. *The operator $\theta_{r,n}^{[m]}$ preserve the constant functions in N variables.*

We need

Definition 1.1. *Let $f \in C_B(\mathbb{R}^N)$, the space of all bounded and continuous functions or uniformly continuous on $\mathbb{R}^N$. Then, the r th multivariate modulus of smoothness of f is given by (see, e.g. [4])*

$$\omega_r(f; h) := \sup_{\sqrt{u_1^2 + \dots + u_N^2} \leq h} \left\| \Delta_{u_1, u_2, \dots, u_N}^r(f) \right\|_{\infty} < \infty, \quad h > 0, \quad (24)$$

where $\|\cdot\|_{\infty}$ is the sup-norm and

$$\begin{aligned} \Delta_u^r f(x) &:= \Delta_{u_1, u_2, \dots, u_N}^r f(x_1, \dots, x_N) = \\ &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, \dots, x_N + ju_N). \end{aligned} \quad (25)$$

Let $m \in \mathbb{N}$ and let $f \in C^m(\mathbb{R}^N)$.

Suppose that all partial derivatives of f of order m are bounded, i.e.

$$\left\| \frac{\partial^m f(\cdot, \cdot, \dots, \cdot)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right\|_{\infty} < \infty, \quad (26)$$

for all $\alpha_j \in \mathbb{Z}^+, j = 1, \dots, N; \sum_{j=1}^N \alpha_j = m$.

In this work we apply the general theory developed in [3] about $\theta_{r,n}^{[m]}$ to the operators $W_{r,n}^{[m]}$, so we can obtain computationally specific results and show that the general theory has applications and it is a valid theory.

So for the very important in various branches of mathematics operators $W_{r,n}^{[m]}$ we prove the very essential properties of uniform approximation, L_p approximation, global smoothness preservation and simultaneously approximation, Voronovskaya asymptotic expansions and complex simultaneous approximation.

2. Auxilliary Essential Results

We need the following:

Theorem 2.1. Let $r, N, m \in \mathbb{N}$, with $m > r; \alpha_j \in \mathbb{Z}^+, j = 1, \dots, N : |\alpha| := \sum_{j=1}^N \alpha_j = m, \xi_n \in (0, 1], n \in \mathbb{N}$. Then

$$\begin{aligned} \tilde{u}_{\xi_n}(\alpha) &:= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \\ &\leq \left(\sqrt{\xi_n} \right)^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] \\ &\leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] < +\infty, \end{aligned} \quad (27)$$

are uniformly bounded.

Proof. We estimate

$$\begin{aligned} \tilde{u}_{\xi_n}(\alpha) &:= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N = \\ &\frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \end{aligned} \quad (28)$$

$$\begin{aligned}
& \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) \left(1 + \frac{\sum_{i=1}^N s_i}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\
& \frac{2^N (\sqrt{\xi_n})^m}{(\sqrt{\pi})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^{\alpha_i} \right) \left(\frac{1}{\sqrt{\xi_n}} + \frac{1}{\sqrt{\xi_n}} \sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right) \right)^r \\
& \quad e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{m-r} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \tag{29} \\
& (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \\
& \quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\
& (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(1 + \sum_{i=1}^N z_i \right)^r dz_1 \dots dz_N + \right. \\
& \quad \left. 2^r \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(\sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] \leq \\
& \quad (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N [(1+N)^r + \\
& \quad 2^r \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(\prod_{i=1}^N z_i^r \right) \left(\prod_{i=1}^N e^{-z_i} \right) \prod_{i=1}^N dz_i] = \tag{30} \\
& (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \prod_{i=1}^N \int_1^\infty z_i^{((\alpha_i+r)+1)-1} e^{-z_i} dz_i \right] =
\end{aligned}$$

(by [7], p. 348)

$$(\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \prod_{i=1}^N \Gamma((\alpha_i+r)+1, 1) \right],$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

We have proved that

$$\tilde{u}_{\xi_n}(\alpha) \leq (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \prod_{i=1}^N \Gamma((\alpha_i+r)+1, 1) \right] \leq$$

$$\begin{aligned} (\sqrt{\xi_n})^{m-r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] &\leq \\ \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] &< +\infty, \end{aligned} \quad (31)$$

therefore $\tilde{u}_{\xi_n}(\alpha)$ are uniformly bounded.

Above we used the formula

$$\Gamma(s+1, 1) = \frac{\lfloor es! \rfloor}{e}, \quad s \in \mathbb{N}.$$

Here $\alpha_i + r \in \mathbb{N}$, hence

$$\Gamma((\alpha_i + r) + 1, 1) = \frac{\lfloor e(\alpha_i + r)! \rfloor}{e} \leq \frac{\lfloor e(m+r)! \rfloor}{e}, \quad i = 1, \dots, N. \quad (32)$$

The claim is proved.

We continue with

Theorem 2.2. *Let $r, n, N \in \mathbb{N}$, $\xi_n \in (0, 1]$. Then*

$$\begin{aligned} \tilde{\Phi}_{\xi_n} &:= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\ &\frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right]. \end{aligned} \quad (33)$$

Proof. We estimate

$$\begin{aligned} \tilde{\Phi}_{\xi_n} &= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N = \\ &\frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \int_{\mathbb{R}_+^N} \left(\frac{1}{\sqrt{\xi_n}} + \frac{\sum_{i=1}^N s_i}{\sqrt{\xi_n}\sqrt{\xi_n}} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{(\sqrt{\xi_n})^r} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right) \right)^r e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{(\sqrt{\xi_n})^r} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \end{aligned} \quad (34)$$

$$\begin{aligned}
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \\
& \quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(1 + \sum_{i=1}^N z_i \right)^r dz_1 \dots dz_N + \right. \\
& \quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] \leq
\end{aligned} \tag{35}$$

$$\begin{aligned}
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right)^r e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] \leq \\
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^r \right) \prod_{i=1}^N e^{-z_i} \prod_{i=1}^N dz_i \right] = \\
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \prod_{i=1}^N \int_1^\infty z_i^r e^{-z_i} dz_i \right] = \\
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\int_1^\infty z^{(r+1)-1} e^{-z} dz \right)^N \right] =
\end{aligned} \tag{36}$$

(by [7], p. 348)

$$\begin{aligned}
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \Gamma^N(r+1, 1) \right] = \\
& \frac{1}{(\sqrt{\xi_n})^r} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right],
\end{aligned}$$

proving the claim.

We give

Theorem 2.3. *All as in Theorem 2.1 and $p > 1$. Then*

$$\begin{aligned}
\tilde{A}_{\xi_n}(\alpha) &:= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\
& \left(\sqrt{\xi_n} \right)^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right] \leq \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right] < +\infty,
\end{aligned} \tag{37}$$

are uniformly bounded, where $m > r$.

Proof. We estimate

$$\begin{aligned} \tilde{A}_{\xi_n}(\alpha) &= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N = \\ &= \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i p} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\ &= \left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{mp} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^{\alpha_i p} \right) \\ &\quad \left(\frac{1}{\sqrt{\xi_n}} + \frac{\sum_{i=1}^N s_i}{\sqrt{\xi_n} \sqrt{\xi_n}} \right)^{rp} e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \end{aligned} \quad (38)$$

(set $z_i = \frac{s_i}{\sqrt{\xi_n}}, i = 1, \dots, N$)

$$\begin{aligned} &\left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{p(m-r)} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N z_i \right)^{rp} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{p(m-r)} \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N z_i \right)^{rp} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \\ &\quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N z_i \right)^{rp} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{p(m-r)} \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N z_i \right)^{rp} dz_1 \dots dz_N + \right. \\ &\quad \left. 2^{rp} \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(\sum_{i=1}^N z_i \right)^{rp} e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] \leq \\ &\quad \left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^{p(m-r)} \\ &\quad \left[\frac{(1+N)^{rp}}{\prod_{i=1}^N (p\alpha_i + 1)} + 2^{rp} \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(\prod_{i=1}^N z_i^{rp} \right) \prod_{i=1}^N e^{-z_i} \prod_{i=1}^N dz_i \right] \end{aligned} \quad (39)$$

$$= \left(\sqrt{\xi_n} \right)^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\frac{(1+N)^{rp}}{\prod_{i=1}^N (p\alpha_i + 1)} + 2^{rp} \prod_{i=1}^N \int_1^\infty z_i^{(\alpha_i+r)p} e^{-z_i} dz_i \right] =$$

(by [7], p. 348)

$$\left(\sqrt{\xi_n} \right)^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\frac{(1+N)^{rp}}{\prod_{i=1}^N (p\alpha_i + 1)} + 2^{rp} \prod_{i=1}^N \Gamma((\alpha_i + r)p + 1, 1) \right] \leq$$

(by [7], p. 909)

$$\left(\sqrt{\xi_n} \right)^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p + 1, 1) \right]. \quad (40)$$

We have proved that

$$\begin{aligned} \tilde{A}_{\xi_n}(\alpha) &\leq \left(\sqrt{\xi_n} \right)^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p + 1, 1) \right] \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p + 1, 1) \right] < +\infty, \end{aligned} \quad (41)$$

are uniformly bounded, where $m > r$.

Above we used ([7], p. 909) the formula

$$\Gamma(\alpha, xy) = y^\alpha e^{-xy} \int_0^\infty e^{-ty} (t+x)^{\alpha-1} dt, \quad (42)$$

where $\operatorname{Re} y > 0$, $x > 0$, $\operatorname{Re} \alpha > 1$.

We notice that ($x = y = 1$, $\alpha_i \leq m$)

$$\begin{aligned} \Gamma((\alpha_i + r)p + 1, 1) &= e^{-1} \int_0^\infty e^{-t} (t+1)^{(\alpha_i+r)p} dt \leq \\ e^{-1} \int_0^\infty e^{-t} (t+1)^{(m+r)p} dt &= \Gamma((m+r)p + 1, 1). \end{aligned} \quad (43)$$

That is

$$\Gamma((\alpha_i + r)p + 1, 1) \leq \Gamma((m+r)p + 1, 1), \quad (44)$$

for all $i = 1, \dots, N$.

The theorem is proved.

We continue with

Theorem 2.4. Let $r, n, N \in \mathbb{N}$, $\xi_n \in (0, 1]$, $p > 1$. Then

$$B_{\xi_n} := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq$$

$$\frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{pr} + 2^{pr} \Gamma^N(rp+1, 1) \right]. \quad (45)$$

Proof. We estimate

$$\begin{aligned} B_{\xi_n} &= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{pr} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N = \\ &= \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{pr} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \leq \\ &= \left(\frac{2}{\sqrt{\pi}} \right)^N \int_{\mathbb{R}_+^N} \left(\frac{1}{\sqrt{\xi_n}} + \frac{\sum_{i=1}^N s_i}{\sqrt{\xi_n} \sqrt{\xi_n}} \right)^{pr} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \\ &= \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{(\sqrt{\xi_n})^{pr}} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^N \frac{s_i}{\sqrt{\xi_n}} \right)^{pr} e^{-\left(\sum_{i=1}^N \frac{s_i}{\sqrt{\xi_n}} \right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} = \\ &= \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{(\sqrt{\xi_n})^{pr}} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^N z_i \right)^{pr} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \\ &= \frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(1 + \sum_{i=1}^N z_i \right)^{pr} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \\ &\quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \sum_{i=1}^N z_i \right)^{pr} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\ &= \frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{pr} + 2^{pr} \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right)^{pr} e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\ &= \frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{pr} + 2^{pr} \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{pr} \right) \prod_{i=1}^N e^{-z_i} \prod_{i=1}^N dz_i \right] = \\ &= \frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{pr} + 2^{pr} \prod_{i=1}^N \int_1^\infty z_i^{pr} e^{-z_i} dz_i \right] = \end{aligned} \quad (46)$$

(by [7], p. 348)

$$\frac{1}{(\sqrt{\xi_n})^{pr}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{pr} + 2^{pr} \Gamma^N(rp+1, 1) \right],$$

proving the claim.

We finish this section with

Theorem 2.5. Let $n, N \in \mathbb{N}$, $\xi_n \in (0, 1]$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, N$: $|\alpha| := \sum_{j=1}^N \alpha_j = m \in \mathbb{N}$. Then

$$D_{\xi_n}(\alpha) := \left(\sqrt{\xi_n} \right)^{-m} \left(\frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \right) \leq \left(\frac{\mu}{\sqrt{\pi}} \right)^N, \quad (48)$$

where

$$\mu := \max \left\{ \sqrt{\pi}, \Gamma \left(\frac{m+1}{2} \right) \right\}, \quad (49)$$

where Γ is the gamma function.

Proof. We have that

$$\begin{aligned} D_{\xi_n}(\alpha) &\leq \left(\sqrt{\xi_n} \right)^{-m} \left(\frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \right) = \\ &\left(\sqrt{\xi_n} \right)^{-m} \left(\frac{2^N}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N \right) = \\ &\left(\sqrt{\xi_n} \right)^{-m} \left(\left(\frac{2}{\sqrt{\pi}} \right)^N \left(\sqrt{\xi_n} \right)^m \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^{\alpha_i} \right) e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} \right) = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i} \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_n = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \prod_{i=1}^N \int_0^\infty z_i^{\alpha_i} e^{-z_i^2} dz_i = \end{aligned} \quad (50)$$

(by [7], p. 349)

$$\left(\left(\frac{2}{\sqrt{\pi}} \right)^N \prod_{i=1}^N \left(\frac{\Gamma \left(\frac{\alpha_i+1}{2} \right)}{2} \right) \right) = \frac{1}{(\sqrt{\pi})^N} \left(\prod_{i=1}^N \Gamma \left(\frac{\alpha_i+1}{2} \right) \right) \leq (*). \quad (51)$$

Some important values of $\Gamma \left(\frac{\alpha_i+1}{2} \right)$ follow:

- if $\alpha_i = 0$, then $\Gamma \left(\frac{1}{2} \right) = \sqrt{\pi} \approx 1.77245\dots$,
- if $\alpha_i = 1$, then $\Gamma(1) = 1$,
- if $\alpha_i = 2$, then $\Gamma \left(\frac{3}{2} \right) = \frac{\sqrt{\pi}}{2} \approx 0.886226\dots$,
- if $\alpha_i = 3$, then $\Gamma(2) = 1$,

if $\alpha_i = 4$, then $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} \approx 1.32934$,

if $\alpha_i = 5$, then $\Gamma(3) = 2$.

The function Γ has minimum at $x_{\min} = 1.46\dots$, with $\Gamma(x_{\min}) = 0.885\dots$, and $\Gamma(x)$ is increasing for $x \geq x_{\min}$. To have $\frac{\alpha_i+1}{2} > 1.46\dots$, we need $\alpha_i > 1.92\dots$. Thus when $\alpha_i \geq 2$, then $\Gamma\left(\frac{\alpha_i+1}{2}\right)$ increases, and $\Gamma\left(\frac{\alpha_i+1}{2}\right) \leq \Gamma\left(\frac{m+1}{2}\right)$, when $m \geq 2$.

Hence it holds $\Gamma\left(\frac{\alpha_i+1}{2}\right) \leq \mu$, for $i = 1, \dots, N$, and

$$(*) \leq \left(\frac{\mu}{\sqrt{\pi}}\right)^N, \quad (52)$$

proving the claim.

3. Main Results for $W_{r,n}^{[m]}$

3.1. Uniform Approximation. We start with an application to $W_{r,n}^{[m]}$ of the following theorem.

Theorem 3.1. ([3], p. 11) Let $m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^N)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f(\cdot, \dots, \cdot)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right\|_{\infty} < \infty$, for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, N$: $|\alpha| := \sum_{i=1}^N \alpha_i = m$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence.

Suppose that for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ we have that

$$u_{\xi_n}(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \quad (53)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{j=1}^N \alpha_j = \tilde{j}$, call

$$c_{\alpha,n} := c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s_1, \dots, s_N). \quad (54)$$

Then

i)

$$\begin{aligned} E_{r,n}^{[m]}(x) &:= \left| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{j,r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n,\tilde{j}} f_{\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right| \\ &\leq \sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = m}} \frac{(\omega_r(f_{\alpha}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i \right)} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right). \end{aligned} \quad (55)$$

$\forall x \in \mathbb{R}^N$.
ii)

$$\left\| E_{r,n}^{[m]} \right\|_{\infty} \leq R.H.S.(55). \quad (56)$$

Given that $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, and u_{ξ_n} is uniformly bounded, then we derive that $\left\| E_{r,n}^{[m]} \right\| \rightarrow 0$ with rates.

iii) It holds also that

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_{\infty} \leq \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{|c_{\alpha,n,\tilde{j}}| \|f_{\alpha}\|_{\infty}}{\prod_{i=1}^N \alpha_i!} \right) + R.H.S.(55), \quad (57)$$

given that $\|f_{\alpha}\|_{\infty} < \infty$, for all $\alpha : |\alpha| = \tilde{j}, \tilde{j} = 1, \dots, m$. Furthermore, as $\xi_n \rightarrow 0$ when $n \rightarrow \infty$, assuming that $c_{\alpha,n,\tilde{j}} \rightarrow 0$, while u_{ξ_n} is uniformly bounded, we conclude that

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_{\infty} \rightarrow 0 \quad (58)$$

with rates.

A uniform approximation result for $W_{r,n}^{[m]}$ follows:

Theorem 3.2. Let $m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^N)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f(\cdot, \dots, \cdot)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right\|_{\infty} < \infty$, for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, N : |\alpha| := \sum_{j=1}^N \alpha_j = m$. Here $\xi_n \in (0, 1]$, $n \in \mathbb{N}$, $m > r \in \mathbb{N}$. For all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$, we denote

$$\tilde{u}_{\xi_n}(\alpha) := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N. \quad (59)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n} := \tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi \xi_{n,n}})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N. \quad (60)$$

Then
i)

$$\tilde{E}_{r,n}^{[m]}(x) := \left| W_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_{\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right|$$

$$\leq \sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = m}} \frac{(\omega_r(f_\alpha, \xi_n))}{\left(\prod_{i=1}^N \alpha_i!\right)} \tilde{u}_{\xi_n}(\alpha), \quad (61)$$

$\forall x \in \mathbb{R}^N$.

ii)

$$\left\| \tilde{E}_{r,n}^{[m]} \right\|_\infty \leq R.H.S.(61). \quad (62)$$

Given that $\xi_n \rightarrow 0$, as $n \rightarrow +\infty$, we have that $\tilde{u}_{\xi_n}(\alpha) \rightarrow 0$ and are uniformly bounded, and then we derive that $\left\| \tilde{E}_{r,n}^{[m]} \right\|_\infty \rightarrow 0$ with rates.

iii) It holds also that

$$\left\| W_{r,n}^{[m]}(f) - f \right\|_\infty \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{\left| \tilde{c}_{\alpha, n, \tilde{j}} \right| \|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) + R.H.S.(61), \quad (63)$$

given that $\|f_\alpha\|_\infty < +\infty$, for all $\alpha : |\alpha| = \tilde{j}, \tilde{j} = 1, \dots, m$. Furthermore, as $\xi_n \rightarrow 0$ when $n \rightarrow +\infty$, we have that $\tilde{c}_{\alpha, n, \tilde{j}} \rightarrow 0$ and $\tilde{u}_{\xi_n}(\alpha) \rightarrow 0$, and both are uniformly bounded, and we conclude that

$$\left\| W_{r,n}^{[m]}(f) - f \right\|_\infty \rightarrow 0 \quad (64)$$

with rates.

Proof. Mainly by applying Theorem 3.1. By Theorem 2.1 we get that $\tilde{u}_{\xi_n}(\alpha) \rightarrow 0$ and $\tilde{u}_{\xi_n}(\alpha)$ are uniformly bounded. By Theorem 2.5 we get that

$$\begin{aligned} |\tilde{c}_{\alpha, n}| &= |\tilde{c}_{\alpha, n, \tilde{j}}| \leq \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N \\ &\leq \left(\frac{\mu}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^m \leq \left(\frac{\mu}{\sqrt{\pi}} \right)^N. \end{aligned}$$

That is $\tilde{c}_{\alpha, n, \tilde{j}} \rightarrow 0$ and are uniformly bounded. The proof is complete.

We mention

Theorem 3.3. ([3], p. 14) Let $f \in C_B(\mathbb{R}^N)$, $N \geq 1$, $\xi_n \in (0, 1]$. Then

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r(f, \xi_n), \quad (65)$$

under the assumption

$$\Phi_{\xi_n} := \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \quad (66)$$

In case of f being uniformly, continuous and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, given that Φ_{ξ_n} are uniformly bounded, we derive

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_{\infty} \rightarrow 0 \quad (67)$$

with rates.

We give

Theorem 3.4. Let $f \in C_B(\mathbb{R}^N)$, $N \geq 1$, $\xi_n \in (0, 1]$. Then

1)

$$\left\| W_{r,n}^{[0]} f - f \right\|_{\infty} \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{\omega_r(f, \xi_n)}{(\sqrt{\xi_n})^r}, \quad (68)$$

where $r, n \in \mathbb{N}$.

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and consider $f \in C_B(\mathbb{R}^N)$ such that $\omega_r(f, t) \leq Kt^{r-1+\alpha}$, $\forall t > 0$, where $K > 0$ is independent of t .

In this case

$$\frac{\omega_r(f, \xi_n)}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (69)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we derive

$$\left\| W_{r,n}^{[0]} f - f \right\|_{\infty} \rightarrow 0. \quad (70)$$

Proof. By Theorems 2.2 and 3.3. For (69), since $r \geq 2$, we get that $\frac{r}{2} + \alpha - 1 > 0$.

3.2. L_p Approximation for $W_{r,n}^{[m]}$. We need

Definition 3.1. ([4], [8]) We call

$$\begin{aligned} \Delta_u^r f(x) &:= \Delta_{u_1, u_2, \dots, u_N}^r f(x_1, \dots, x_N) := \\ &\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, \dots, x_N + ju_N). \end{aligned} \quad (71)$$

Let $p \geq 1$, the modulus of smoothness of order r is given by

$$\omega_r(f; h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \quad (72)$$

$h > 0$.

We will apply

Theorem 3.5. ([3], p. 24) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_{\alpha} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ that we have

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \quad (73)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{j=1}^N \alpha_j = \tilde{j}$, call

$$c_{\alpha, n, \tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \quad (74)$$

Then

$$\begin{aligned} \left\| E_{r, n}^{[m]} \right\|_p &= \left\| \theta_{r, n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha, n, \tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p, x} \\ &\leq \left(\frac{m}{(q(m-1) + 1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ &\quad \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_\alpha, \xi_n)_p. \end{aligned} \quad (75)$$

As $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (75) we obtain that $\left\| E_{r, n}^{[m]} \right\| \rightarrow 0$ with rates.

One also finds by (75) that

$$\left\| \theta_{r, n}^{[m]}(f; x) - f(x) \right\|_{p, x} \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j}, r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|c_{\alpha, n, \tilde{j}}|}{\prod_{i=1}^N \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(75), \quad (76)$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

Assuming that $c_{\alpha, n, \tilde{j}} \rightarrow 0$, $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, we get $\left\| \theta_{r, n}^{[m]}(f) - f \right\|_p \rightarrow 0$, that is $\theta_{r, n}^{[m]} \rightarrow I$ the unit operator, in L_p norm, with rates.

We present

Theorem 3.6. Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $m > r$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi_n \in (0, 1]$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha, n, \tilde{j}} = \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N. \quad (77)$$

Then

$$\begin{aligned} \left\| \tilde{E}_{r,n}^{[m]} \right\|_p &= \left\| W_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p,x} \\ &\leq \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right]^{\frac{1}{p}} \\ &\quad \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right) (\sqrt{\xi_n})^{(m-r)} \omega_r(f_\alpha, \xi_n)_p. \end{aligned} \quad (78)$$

As $n \rightarrow +\infty$ and $\xi_n \rightarrow 0$, by (78) we obtain that $\left\| \tilde{E}_{r,n}^{[m]} \right\|_p \rightarrow 0$ with rates.

One also finds by (78) that

$$\left\| W_{r,n}^{[m]}(f; x) - f(x) \right\|_{p,x} \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|\tilde{c}_{\alpha,n,\tilde{j}}|}{\prod_{i=1}^N \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(78), \quad (79)$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

Assuming that $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, we get $\left\| W_{r,n}^{[m]}(f) - f \right\|_p \rightarrow 0$, that is $W_{r,n}^{[m]} \rightarrow I$ the unit operator, in L_p norm, with rates.

Proof. By Theorem 2.5 we get that $|\tilde{c}_{\alpha,n,\tilde{j}}| \leq \left(\frac{\mu}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^m \leq \left(\frac{\mu}{\sqrt{\pi}} \right)^N$ and $\tilde{c}_{\alpha,n,\tilde{j}} \rightarrow 0$. We use also Theorem 2.3 to uniformly bound

$\frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p e^{-\frac{\left(\sum_{i=1}^N s_i^2 \right)}{\xi_n}} ds_1 \dots ds_N$. Then we apply directly Theorem 3.5.

We continue with an application of

Theorem 3.7. ([3], p. 26) Let $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$; $N \geq 1$; $p, q > 1$; $\frac{1}{p} + \frac{1}{q} = 1$. Assume μ_{ξ_n} probability Borel measure on $\mathbb{R}^N$, $(\xi_n)_{n \in \mathbb{N}} > 0$ and bounded. Also suppose

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} d\mu_{\xi_n}(s) < \infty. \quad (80)$$

Then

$$\left\| \theta_{r,n}^{[0]}(f) - f \right\|_p \leq \quad (81)$$

$$\left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} d\mu_{\xi_n}(s) \right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p.$$

As $\xi_n \rightarrow 0$, when $n \rightarrow \infty$, we derive $\left\| \theta_{r,n}^{[0]}(f) - f \right\|_p \rightarrow 0$, i.e. $\theta_{r,n}^{[0]} \rightarrow I$, the unit operator, in L_p norm.

We give

Theorem 3.8. Let $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$; $N \geq 1$; $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Then

1)

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_p \leq \quad (82)$$

$$\left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left[(1+N)^{rp} + 2^{rp} \Gamma^N(rp+1, 1) \right]^{\frac{1}{p}} \frac{\omega_r(f, \xi_n)_p}{(\sqrt{\xi_n})^r}.$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and consider $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$: $\omega_r(f, t)_p \leq K t^{r-1+\alpha}$, $\forall t > 0$, where $K > 0$ is independent of t . In this case

$$\frac{\omega_r(f, \xi_n)_p}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (83)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we obtain

$$\left\| W_{r,n}^{[0]}f - f \right\|_p \rightarrow 0, \quad (84)$$

i.e. $W_{r,n}^{[0]} \rightarrow I$, the unit operator in L_p norm.

Proof. By Theorems 2.4 and 3.7. For $r \geq 2$ we get $\frac{r}{2} + \alpha - 1 > 0$.

We mention

Theorem 3.9. ([3], p. 27) Let $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$; $N \geq 1$. Assume μ_{ξ_n} probability Borel measure on $\mathbb{R}^N$, $(\xi_n)_{n \in \mathbb{N}} > 0$ and bounded. Also suppose

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \quad (85)$$

Then

$$\left\| \theta_{r,n}^{[0]}(f) - f \right\|_1 \leq \quad (86)$$

$$\left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r(f, \xi_n)_1.$$

As $\xi_n \rightarrow 0$, we get $\theta_{r,n}^{[0]} \rightarrow I$, in L_1 norm.

We give

Theorem 3.10. Let $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$; $N \geq 1$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Then
1)

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{\omega_r(f, \xi_n)_1}{(\sqrt{\xi_n})^r}. \quad (87)$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and consider $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N)) : \omega_r(f, t)_1 \leq Kt^{r-1+\alpha}$, $\forall t > 0$, where $K > 0$ is independent of t . In this case

$$\frac{\omega_r(f, \xi_n)_1}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (88)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we derive $\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \rightarrow 0$, i.e. $W_{r,n}^{[0]} \rightarrow I$ in L_1 norm.

Proof. By Theorems 2.2 and 3.9. Notice $\frac{r}{2} + \alpha - 1 > 0$.

We need

Theorem 3.11. ([3], p. 29) Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence. Suppose for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ that we have

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \quad (89)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$c_{\alpha, n, \tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \quad (90)$$

Then

$$\begin{aligned} \left\| E_{r,n}^{[m]} \right\|_1 &= \left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha, n, \tilde{j}} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{1,s} \\ &\leq \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \omega_r(f_\alpha, \xi_n)_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s). \end{aligned} \quad (91)$$

As $\xi_n \rightarrow 0$, we get $\left\| E_{r,n}^{[m]} \right\|_1 \rightarrow 0$ with rates.

From (91) we get

$$\left\| \theta_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|c_{\alpha,n,\tilde{j}}|}{\prod_{i=1}^N \alpha_i!} \|f_\alpha\|_1 \right) + R.H.S.(91), \quad (92)$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

As $n \rightarrow \infty$, assuming $\xi_n \rightarrow 0$ and $c_{\alpha,n,\tilde{j}} \rightarrow 0$, we obtain $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_1 \rightarrow 0$, that is $\theta_{r,n}^{[m]} \rightarrow I$ in L_1 norm, with rates.

We present

Theorem 3.12. Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, $m > r$, with $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N. \quad (93)$$

Then

$$\begin{aligned} \left\| \tilde{E}_{r,n}^{[m]} \right\|_1 &= \left\| W_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{1,x} \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) (\sqrt{\xi_n})^{m-r} \omega_r(f_\alpha, \xi_n)_1. \end{aligned} \quad (94)$$

As $\xi_n \rightarrow 0$, we get $\left\| \tilde{E}_{r,n}^{[m]} \right\|_1 \rightarrow 0$ with rates.

From (94) we get

$$\left\| W_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|\tilde{c}_{\alpha,n,\tilde{j}}|}{\prod_{i=1}^N \alpha_i!} \|f_\alpha\|_1 \right) + R.H.S.(94), \quad (95)$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

As $n \rightarrow \infty$, assuming $\xi_n \rightarrow 0$, we obtain $\left\| W_{r,n}^{[m]}(f) - f \right\|_1 \rightarrow 0$, that is $W_{r,n}^{[m]} \rightarrow I$ in L_1 norm, with rates.

Proof. We apply Theorems 2.1, 2.5 and 3.11.

3.3. Global Smoothness Preservation and Simultaneous Approximation of $W_{r,n}^{[m]}$. We need

Definition 3.2. ([3], p. 34) Let $f \in C(\mathbb{R}^N)$, $N \geq 1$, $m \in \mathbb{N}$, the m th modulus of smoothness for $1 \leq p \leq \infty$, is given by

$$\omega_m(f; h)_p := \sup_{\|t\|_2 \leq h} \|\Delta_t^m(f)\|_{p,x}, \quad (96)$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jt). \quad (97)$$

Denote

$$\omega_m(f; h)_\infty = \omega_m(f, h). \quad (98)$$

Above, $x, t \in \mathbb{R}^N$.

We present the related global smoothness preservation result

Theorem 3.13. We assume $W_{r,n}^{[\tilde{m}]}(f; x) \in \mathbb{R}$, $\tilde{m} \in \mathbb{Z}_+$, $\forall x \in \mathbb{R}$. Let $h > 0$, $f \in C(\mathbb{R}^N)$, $N \geq 1$.

i) Assume $\omega_m(f, h) < \infty$. Then

$$\omega_m(W_{r,n}^{[\tilde{m}]}f, h) \leq \left(\sum_{\tilde{j}=0}^r |\alpha_{\tilde{j},r}^{[\tilde{m}]}| \right) \omega_m(f, h). \quad (99)$$

ii) Assume $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$. Then

$$\omega_m(W_{r,n}^{[\tilde{m}]}f, h)_1 \leq \left(\sum_{\tilde{j}=0}^r |\alpha_{\tilde{j},r}^{[\tilde{m}]}| \right) \omega_m(f, h)_1. \quad (100)$$

iii) Assume $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$, $p > 1$. Then

$$\omega_m(W_{r,n}^{[\tilde{m}]}f, h)_p \leq \left(\sum_{\tilde{j}=0}^r |\alpha_{\tilde{j},r}^{[\tilde{m}]}| \right) \omega_m(f, h)_p. \quad (101)$$

Proof. Direct application of ([3]) Theorem 3.2, p. 35.

We make

Remark 3.1. Let $r = 1$, $\tilde{m} \in \mathbb{Z}_+$, then $\alpha_{0,1}^{[\tilde{m}]} = 0$, $\alpha_{1,1}^{[\tilde{m}]} = 1$. Hence

$$W_{1,n}^{[\tilde{m}]}(f; x) = \frac{1}{(\sqrt{\pi\tilde{\zeta}_n})^N} \int_{\mathbb{R}^N} f(x+s) e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\tilde{\zeta}_n}\right)} ds_1 \dots ds_N =: W_n(f; x). \quad (102)$$

By Theorem 3.13, we get

Theorem 3.14. We suppose $W_n(f; x) \in \mathbb{R}, \forall x \in \mathbb{R}$. Let $h > 0, f \in C(\mathbb{R}^N), N \geq 1$.

i) Assume $\omega_m(f, h) < \infty$. Then

$$\omega_m(W_n f, h) \leq \omega_m(f, h). \quad (103)$$

ii) Assume $f \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$. Then

$$\omega_m(W_n f, h)_1 \leq \omega_m(f, h)_1. \quad (104)$$

iii) Assume $f \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N)), p > 1$. Then

$$\omega_m(W_n f, h)_p \leq \omega_m(f, h)_p. \quad (105)$$

Next, we get an optimality result

Proposition 3.1. Above inequality (103):

$$\omega_m(W_n f, h) \leq \omega_m(f, h)$$

is sharp, namely it is attained by any

$$f_j^*(x) = x_j^m, \quad j = 1, \dots, N, \quad x = (x_1, \dots, x_j, \dots, x_N) \in \mathbb{R}^N. \quad (106)$$

Proof. Apply Proposition 3.5, p. 38, of [3].

We need

Theorem 3.15. ([3], p. 39) Let $f \in C^l(\mathbb{R}^N), l, N \in \mathbb{N}$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N, \xi_n > 0, (\xi_n)_{n \in \mathbb{N}}$ a bounded sequence. Let $\beta := (\beta_1, \dots, \beta_N), \beta_i \in \mathbb{Z}^+, i = 1, \dots, N; |\beta| := \sum_{i=1}^N \beta_i = l$. Here $f(x + sj), x, s \in \mathbb{R}^N$, is μ_{ξ_n} -integrable wrt s , for $j = 1, \dots, r$. There exist μ_{ξ_n} -integrable functions $h_{i_1, j}, h_{\beta_1, i_2, j}, h_{\beta_1, \beta_2, i_3, j}, \dots, h_{\beta_1, \beta_2, \dots, \beta_{N-1}, i_N, j} \geq 0 (j = 1, \dots, r)$ on $\mathbb{R}^N$ such that

$$\begin{aligned} \left| \frac{\partial^{i_1} f(x + sj)}{\partial x_1^{i_1}} \right| &\leq h_{i_1, j}(s), \quad i_1 = 1, \dots, \beta_1, \\ \left| \frac{\partial^{\beta_1 + i_2} f(x + sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}} \right| &\leq h_{\beta_1, i_2, j}(s), \quad i_2 = 1, \dots, \beta_2, \\ &\vdots \\ \left| \frac{\partial^{\beta_1 + \beta_2 + \dots + \beta_{N-1} + i_N} f(x + sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} \dots \partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| &\leq h_{\beta_1, \beta_2, \dots, \beta_{N-1}, i_N, j}(s), \quad i_N = 1, \dots, \beta_N, \end{aligned} \quad (107)$$

$\forall x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r, n}^{[\tilde{m}]}(f; x) \right)_\beta = \theta_{r, n}^{[\tilde{m}]}(f_\beta; x), \quad \tilde{m} \in \mathbb{Z}_+. \quad (108)$$

In particular it holds

$$\left(W_{r, n}^{[\tilde{m}]}(f; x) \right)_\beta = W_{r, n}^{[\tilde{m}]}(f_\beta; x), \quad (109)$$

when

$$d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N.$$

Corollary 3.1. (by Theorem 3.15, $r = 1$) It holds

$$(W_n(f; x))_\beta = W_n(f_\beta; x). \quad (110)$$

We present simultaneous global smoothness results.

Theorem 3.16. Let $h > 0$ and the assumptions of Theorem 3.15 are valid for $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$.

Here $\gamma = 0, \beta = (0, \dots, 0)$, $\tilde{m} \in \mathbb{Z}_+$.

i) Assume $\omega_m(f_\gamma, h) < \infty$. Then

$$\omega_m \left(\left(W_{r,n}^{[\tilde{m}]}(f) \right)_\gamma, h \right) \leq \left(\sum_{j=0}^r |\alpha_{j,r}^{[\tilde{m}]}| \right) \omega_m(f_\gamma, h). \quad (111)$$

ii) Additionally suppose $f_\gamma \in L_1(\mathbb{R}^N)$. Then

$$\omega_m \left(\left(W_{r,n}^{[\tilde{m}]}(f) \right)_\gamma, h \right)_1 \leq \left(\sum_{j=0}^r |\alpha_{j,r}^{[\tilde{m}]}| \right) \omega_m(f_\gamma, h)_1. \quad (112)$$

iii) Additionally suppose $f_\gamma \in L_p(\mathbb{R}^N)$, $p > 1$. Then

$$\omega_m \left(\left(W_{r,n}^{[\tilde{m}]}(f) \right)_\gamma, h \right)_p \leq \left(\sum_{j=0}^r |\alpha_{j,r}^{[\tilde{m}]}| \right) \omega_m(f_\gamma, h)_p. \quad (113)$$

It follows

Corollary 3.2. (to Theorem 3.16) Let $h > 0$, $r = 1$ and $\gamma = 0, \beta$.

i) Assume $\omega_m(f_\gamma, h) < \infty$. Then

$$\omega_m \left((W_n(f))_\gamma, h \right) \leq \omega_m(f_\gamma, h). \quad (114)$$

ii) Additionally suppose $f_\gamma \in L_1(\mathbb{R}^N)$. Then

$$\omega_m \left((W_n(f))_\gamma, h \right)_1 \leq \omega_m(f_\gamma, h)_1. \quad (115)$$

iii) Additionally suppose $f_\gamma \in L_p(\mathbb{R}^N)$, $p > 1$. Then

$$\omega_m \left((W_n(f))_\gamma, h \right)_p \leq \omega_m(f_\gamma, h)_p. \quad (116)$$

Next comes multi-simultaneous approximation.

Theorem 3.17. Let $f \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$, $m > r$. The assumptions of Theorem 3.15 are valid

for $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$. Call $\gamma = 0, \beta$. Assume $\|f_{\gamma+\alpha}\|_\infty < \infty$, and let

$$\tilde{u}_{\xi_n}(\alpha) := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N,$$

for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, N$, $|\alpha| := \sum_{j=1}^N \alpha_j = m$, $\xi_n \in (0, 1]$.

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha, n, \tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N. \quad (117)$$

Then

$$\begin{aligned} & \left\| \left(W_{r, n}^{[m]}(f; \cdot) \right)_\gamma - f_\gamma(\cdot) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{\tilde{c}_{\alpha, n, \tilde{j}} f_{\gamma+\alpha}(\cdot)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_\infty \\ & \leq \sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = m}} \frac{(\omega_r(f_{\gamma+\alpha}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i! \right)} \tilde{u}_{\xi_n}(\alpha). \end{aligned} \quad (118)$$

Proof. Based on Theorems 3.2, 3.15.

We continue with

Theorem 3.18. Let $f \in C_B^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$ (functions l -times continuously differentiable and

bounded). The assumptions of Theorem 3.15 are valid for $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$. Call

$\gamma = 0, \beta$, $\xi_n \in (0, 1]$. Then

1)

$$\left\| \left(W_{r, n}^{[0]} f \right)_\gamma - f_\gamma \right\|_\infty \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{\omega_r(f_\gamma, \xi_n)}{(\sqrt{\xi_n})^r}. \quad (119)$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and consider $f_\gamma : \omega_r(f_\gamma, t) \leq K t^{r-1+\alpha}$, $\forall t > 0$, where $K > 0$ is independent of t . We get

$$\frac{\omega_r(f_\gamma, \xi_n)}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (120)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we derive $\left(W_{r,n}^{[0]}f\right)_\gamma \rightarrow f_\gamma$ uniformly.

Proof. By Theorems 3.4 and 3.15.

We continue with

Theorem 3.19. Let $f \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$, $m > r$. The assumptions of Theorem 3.15 are valid

for $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$. Call $\gamma = 0, \beta$. Let $f_{(\gamma+\alpha)} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N. \quad (121)$$

Then

$$\begin{aligned} & \left\| \left(W_{r,n}^{[m]}(f; x) \right)_\gamma - f_\gamma(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{p,x} \\ & \leq \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right]^{\frac{1}{p}} \\ & \quad \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right) (\sqrt{\xi_n})^{(m-r)} \omega_r(f_{\gamma+\alpha}, \xi_n)_p. \end{aligned} \quad (122)$$

Proof. By Theorems 3.6, 3.15.

We give also

Theorem 3.20. Let $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 3.15 are valid for $d\mu_{\xi_n} =$

$\frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$. Call $\gamma = 0, \beta$. Let $f_\gamma \in L_p(\mathbb{R}^N)$, $x \in \mathbb{R}^N$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Then

$$\left\| \left(W_{r,n}^{[0]}f \right)_\gamma - f_\gamma \right\|_p \leq$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{N}{p}} \left[(1+N)^{rp} + 2^{rp} \Gamma^N(rp+1, 1)\right]^{\frac{1}{p}} \frac{\omega_r(f_\gamma, \xi_n)_p}{(\sqrt{\xi_n})^r}. \quad (123)$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and $f_\gamma : \omega_r(f_\gamma, t)_p \leq Kt^{r-1+\alpha}$, $\forall t > 0$, $K > 0$ independent of t . In this case

$$\frac{\omega_r(f_\gamma, \xi_n)_p}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}. \quad (124)$$

As $n \rightarrow +\infty$ and $\xi_n \rightarrow 0$, then $\left(W_{r,n}^{[0]}(f)\right)_\gamma \xrightarrow{\|\cdot\|_p} f_\gamma$.

Proof. By Theorems 3.8 and 3.15.

We continue with

Theorem 3.21. Let $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 3.15 are valid for $d\mu_{\xi_n} =$

$$\frac{1}{(\sqrt{\pi \xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N. \text{ Call } \gamma = 0, \beta. \text{ Let } f_\gamma \in L_1(\mathbb{R}^N), x \in \mathbb{R}^N, \xi_n \in (0, 1], n \in \mathbb{N}. \text{ Then}$$

$$\left\| \left(W_{r,n}^{[0]}(f)\right)_\gamma - f_\gamma \right\|_1 \leq \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+N)^r + 2^r \left(\frac{[er!]}{e}\right)^N\right] \frac{\omega_r(f_\gamma, \xi_n)_1}{(\sqrt{\xi_n})^r}. \quad (125)$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and $f_\gamma : \omega_r(f_\gamma, t)_1 \leq Kt^{r-1+\alpha}$, $\forall t > 0$, $K > 0$ independent of t . In this case

$$\frac{\omega_r(f_\gamma, \xi_n)_1}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (126)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we get $\left(W_{r,n}^{[0]}(f)\right)_\gamma \xrightarrow{\|\cdot\|_1} f_\gamma$.

Proof. By Theorems 3.10 and 3.15.

We continue with

Theorem 3.22. Let $f \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$, $m > r$. The assumptions of Theorem 3.15 for

$$d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi \xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N \text{ are valid. Call } \gamma = 0, \beta. \text{ Let } f_{(\gamma+\alpha)} \in L_1(\mathbb{R}^N), |\alpha| = m, x \in \mathbb{R}^N, \xi_n \in (0, 1], n \in \mathbb{N}. \text{ For } \tilde{j} = 1, \dots, m, \text{ and } \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}^+, i = 1, \dots, N,$$

$$|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}, \text{ call}$$

$$\tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi\tilde{\xi}_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\tilde{\xi}_n}} ds_1 \dots ds_N. \quad (127)$$

Then

$$\begin{aligned} & \left\| \left(W_{r,n}^{[m]}(f; x) \right)_\gamma - f_\gamma(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_{(\gamma+\alpha)}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{1,x} \\ & \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \omega_r(f_{\gamma+\alpha}, \tilde{\xi}_n)_1 \right) \\ & \quad \left(\sqrt{\tilde{\xi}_n} \right)^{m-r} \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right]. \end{aligned} \quad (128)$$

Proof. By Theorems 3.12, 3.15.

3.4. Voronovskaya Asymptotic Expansions for $W_{r,n}^{[m]}$. Here we will apply the following general theorem.

Theorem 3.23. Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with all $\|f_\alpha\|_\infty \leq M$, $M > 0$, all $\alpha : |\alpha| = m$. Let $\tilde{\xi}_n > 0$, $(\tilde{\xi}_n)_{n \in \mathbb{N}}$ bounded sequence, $\mu_{\tilde{\xi}_n}$ probability Borel measures on $\mathbb{R}^N$.

Call $c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) d\mu_{\tilde{\xi}_n}(s)$, all $|\alpha| = \tilde{j} = 1, \dots, m-1$. Suppose

$$\left(\sqrt{\tilde{\xi}_n} \right)^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) d\mu_{\tilde{\xi}_n}(s) \leq \rho,$$

all $\alpha : |\alpha| = m$, $\rho > 0$, for any such $(\tilde{\xi}_n)_{n \in \mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then

$$\theta_{r,n}^{[m]}(f; x) - f(x) = \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) + o \left(\left(\sqrt{\tilde{\xi}_n} \right)^{m-\gamma} \right). \quad (129)$$

When $m = 1$, the sum collapses.

Above we assume $\theta_{r,n}^{[m]}(f; x) \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$.

Proof. The proof of Theorem 3.23 is the same as the proof of Theorem 4.2, p. 53 of [3], just replace there ξ_n by $\sqrt{\tilde{\xi}_n}$.

We give

Theorem 3.24. Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with all $\|f_\alpha\|_\infty \leq M$, $M > 0$, all $\alpha : |\alpha| = m$,

$\xi_n \in (0, 1]$, and $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$ on $\mathbb{R}^N$. Assume $W_{r,n}^{[m]}(f; x) \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$.

Call $\tilde{c}_{\alpha,n,\tilde{j}} = \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$, all $|\alpha| = \tilde{j} = 1, \dots, m-1$. Let $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then

$$W_{r,n}^{[m]}(f; x) - f(x) = \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) + O\left(\left(\sqrt{\xi_n}\right)^{m-\gamma}\right). \quad (130)$$

When $m = 1$, the sum collapses.

Proof. By Theorems 2.5 and 3.23. Here $\rho = \left(\frac{\mu}{\sqrt{\pi}}\right)^N$.

Corollary 3.3. (to Theorem 3.24) Let $f \in C^1(\mathbb{R}^N)$, $N \geq 1$, with all $\left\| \frac{\partial f}{\partial x_i} \right\|_\infty \leq M$, $M > 0$, $i = 1, \dots, N$. Let $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then

$$W_{r,n}^{[1]}(f; x) - f(x) = O\left(\left(\sqrt{\xi_n}\right)^{1-\gamma}\right). \quad (131)$$

Above we assume $W_{r,n}^{[1]}(f; x) \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$.

Proof. By Theorems 2.5 and 3.23, here it is $\rho = 1$.

Corollary 3.4. (to Theorem 3.24) Let $f \in C^2(\mathbb{R}^2)$, with all $\left\| \frac{\partial^2 f}{\partial x_1^2} \right\|_\infty, \left\| \frac{\partial^2 f}{\partial x_2^2} \right\|_\infty, \left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|_\infty \leq M$, $M > 0$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Call

$$c_1 = \int_{\mathbb{R}^2} s_1 d\mu_{\xi_n}(s), \quad c_2 = \int_{\mathbb{R}^2} s_2 d\mu_{\xi_n}(s), \quad (132)$$

where

$$d\mu_{\xi_n} = \frac{1}{\sqrt{\pi\xi_n}} e^{-\frac{(s_1^2 + s_2^2)}{\xi_n}} ds_1 ds_2.$$

Let $0 < \gamma \leq 1$, $x \in \mathbb{R}^2$. Then

$$W_{r,n}^{[2]}(f; x) - f(x) = \left(\sum_{j=1}^r \alpha_{j,r}^{[2]} j \right) \left(c_1 \frac{\partial f}{\partial x_1}(x) + c_2 \frac{\partial f}{\partial x_2}(x) \right) + O\left(\left(\sqrt{\xi_n}\right)^{2-\gamma}\right). \quad (133)$$

Proof. By Theorems 2.5 and 3.23, here it is $\rho = 1$.

We also give

Theorem 3.25. Let $f \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$. Assumptions of Theorem 3.15 are valid for

$d\mu_{\xi_n}(s) = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$. Call $\gamma = 0, \beta$. Suppose $\|f_{\gamma+\alpha}\|_\infty \leq M$, $M > 0$, for all $\alpha : |\alpha| = m$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$.

Call $\tilde{c}_{\alpha, n, \tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$, all $|\alpha| = \tilde{j} = 1, \dots, m-1$; $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then

$$\left(W_{r,n}^{[m]}(f; x) \right)_\gamma - f_\gamma(x) = \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j}, r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha, n, \tilde{j}} f_{\gamma+\alpha}(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) + 0 \left(\left(\sqrt{\xi_n} \right)^{m-\gamma} \right). \quad (134)$$

When $m = 1$, the sum collapses.

Proof. Use of Theorem 2.5 and Theorem 4.6, p. 54 of [3], which is true also for $\sqrt{\xi_n}$ instead of ξ_n in the main assumption and conclusion. Here it is $\rho = \left(\frac{\mu}{\sqrt{\pi}} \right)^N$.

3.5. Simultaneous Approximation by multivariate complex $W_{r,n}^{[m]}$. We make

Remark 3.2. We consider here complex valued Borel measurable functions $f : \mathbb{R}^N \rightarrow \mathbb{C}$ such that $f = f_1 + if_2$, $i = \sqrt{-1}$, where $f_1, f_2 : \mathbb{R}^N \rightarrow \mathbb{R}$ are implied to be real valued Borel measurable functions.

We define the multivariate complex Gauss-Weierstrass singular operators

$$W_{r,n}^{[m]}(f; x) := W_{r,n}^{[m]}(f_1; x) + iW_{r,n}^{[m]}(f_2; x), \quad x \in \mathbb{R}^N. \quad (135)$$

We assume that $W_{r,n}^{[m]}(f_j; x) \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$, $j = 1, 2$.

One notices easily that

$$\left| W_{r,n}^{[m]}(f; x) - f(x) \right| \leq \left| W_{r,n}^{[m]}(f_1; x) - f_1(x) \right| + \left| W_{r,n}^{[m]}(f_2; x) - f_2(x) \right| \quad (136)$$

also

$$\left\| W_{r,n}^{[m]}(f; x) - f(x) \right\|_{\infty, x} \leq \left\| W_{r,n}^{[m]}(f_1; x) - f_1(x) \right\|_{\infty, x} + \left\| W_{r,n}^{[m]}(f_2; x) - f_2(x) \right\|_{\infty, x} \quad (137)$$

and

$$\left\| W_{r,n}^{[m]}(f) - f \right\|_p \leq \left\| W_{r,n}^{[m]}(f_1) - f_1 \right\|_p + \left\| W_{r,n}^{[m]}(f_2) - f_2 \right\|_p, \quad p \geq 1. \quad (138)$$

Furthermore, it holds

$$f_\alpha(x) = f_{1,\alpha}(x) + if_{2,\alpha}(x), \quad (139)$$

where α denotes a partial derivative of any order and arrangement.

We give

Theorem 3.26. Let $f : \mathbb{R}^N \rightarrow \mathbb{C}$, $N \geq 1$, such that $f = f_1 + if_2$, $j = 1, 2$. Here $m \in \mathbb{N}$, $f_j \in C^m(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f_j(\cdot, \dots, \cdot)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right\|_\infty < \infty$, for all $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$: $|\alpha| := \sum_{i=1}^N \alpha_i = m$, $m > r$. Here $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. For all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ we denote

$$\tilde{u}_{\xi_n}(\alpha) := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n} \right)} ds_1 \dots ds_N. \quad (140)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha, n} := \tilde{c}_{\alpha, n, \tilde{j}} := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n} \right)} ds_1 \dots ds_N. \quad (141)$$

Then

$$\begin{aligned} & \left\| W_{r, n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{\tilde{c}_{\alpha, n, \tilde{j}} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{\infty, x} \\ & \leq \sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = m}} \frac{(\omega_r(f_{1, \alpha}, \xi_n) + \omega_r(f_{2, \alpha}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i! \right)} \tilde{u}_{\xi_n}(\alpha). \end{aligned} \quad (142)$$

Proof. By Theorem 3.2.

We proceed with

Theorem 3.27. Let $f : \mathbb{R}^N \rightarrow \mathbb{C}$: $f = f_1 + if_2$, $N \geq 1$, $j = 1, 2$. Here $f_j \in C_B(\mathbb{R}^N)$ uniformly continuous, $\xi_n \in (0, 1]$. Then

1)

$$\left\| W_{r, n}^{[0]} f - f \right\|_\infty \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{(\omega_r(f_1, \xi_n) + \omega_r(f_2, \xi_n))}{(\sqrt{\xi_n})^r}, \quad (143)$$

where $r, n \in \mathbb{N}$.

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and $\omega_r(f_j, t) \leq K t^{r-1+\alpha}$, $\forall t > 0$, $K > 0$ is independent of t , $j = 1, 2$. We get

$$\frac{\omega_r(f_j, \xi_n)}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1 \right)}, \quad j = 1, 2 \quad (144)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we get $\|W_{r,n}^{[0]}(f) - f\|_\infty \rightarrow 0$.

Proof. Use of Theorem 3.4.

Next comes multi-simultaneous approximation

Theorem 3.28. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$, $m > r$. The assumptions of Theorem 3.15 are valid for

$$d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N \text{ and } f_j. \text{ Call } \gamma = 0, \beta. \text{ Assume } \|f_{j,\gamma+\alpha}\|_\infty < \infty, \text{ and let}$$

$$\tilde{u}_{\xi_n}(\alpha) := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N. \quad (145)$$

for all $\alpha_i \in \mathbb{Z}^+, i = 1, \dots, N : |\alpha| := \sum_{i=1}^N \alpha_i = m, \xi_n \in (0, 1]$.

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+, i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N. \quad (146)$$

Then

$$\begin{aligned} & \left\| \left(W_{r,n}^{[m]}(f; \cdot) \right)_\gamma - f_\gamma(\cdot) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = \tilde{j}}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(\cdot)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_\infty \\ & \leq \sum_{\substack{\alpha_1, \dots, \alpha_N \geq 0; \\ |\alpha| = m}} \frac{(\omega_r(f_{1,\gamma+\alpha}, \xi_n) + \omega_r(f_{2,\gamma+\alpha}, \xi_n))}{\left(\prod_{i=1}^N \alpha_i! \right)} \tilde{u}_{\xi_n}(\alpha). \end{aligned} \quad (147)$$

Proof. Based on Theorems 3.15, 3.17.

We continue with

Theorem 3.29. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in C_B^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$ (functions l -times continuously differentiable and bounded). The assumptions of Theorem 3.15 are valid

$$\text{for } d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N \text{ and } f_j, j = 1, 2. \text{ Call } \gamma = 0, \beta, \xi_n \in (0, 1]. \text{ Then}$$

1)

$$\left\| \left(W_{r,n}^{[0]} f \right)_\gamma - f_\gamma \right\|_\infty \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{(\omega_r(f_{1,\gamma}, \xi_n) + \omega_r(f_{2,\gamma}, \xi_n))}{(\sqrt{\xi_n})^r}. \quad (148)$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, and consider $f_{j,\gamma} : \omega_r(f_{j,\gamma}, t) \leq Kt^{r-1+\alpha}$, $\forall t > 0$, $j = 1, 2$, where $K > 0$ is independent of t . We get

$$\frac{\omega_r(f_{j,\gamma}, \xi_n)}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad (149)$$

$j = 1, 2$, and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we find $\left(W_{r,n}^{[0]} f \right)_\gamma \rightarrow f_\gamma$ uniformly.

Proof. By Theorems 3.15 and 3.18.

We proceed with L_p approximations

Theorem 3.30. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. Here $f_j \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $m > r$, $N \geq 1$, with $f_{j,\alpha} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi_n \in (0, 1]$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n,\tilde{j}} = \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N. \quad (150)$$

Then

$$\begin{aligned} & \left\| W_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i!\right)} \right) \right\|_{p,x} \\ & \leq \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right]^{\frac{1}{p}} \\ & \quad \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right) (\sqrt{\xi_n})^{(m-r)} \left[\omega_r(f_{1,\alpha}, \xi_n)_p + \omega_r(f_{2,\alpha}, \xi_n)_p \right]. \end{aligned} \quad (151)$$

Proof. By Theorem 3.6.

We continue with

Theorem 3.31. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in (C(\mathbb{R}^N) \cap L_p(\mathbb{R}^N))$; $N \geq 1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \xi_n \in (0, 1], n \in \mathbb{N}$. Then

1)

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_p \leq \quad (152)$$

$$\left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left[(1+N)^{rp} + 2^{rp} \Gamma^N(rp+1, 1) \right]^{\frac{1}{p}} \frac{\left(\omega_r(f_1, \xi_n)_p + \omega_r(f_2, \xi_n)_p \right)}{(\sqrt{\xi_n})^r}.$$

2) Let $r \in \mathbb{N} - \{1\}, 0 < \alpha \leq 1$, and assume that $\omega_r(f_j, t)_p \leq Kt^{r-1+\alpha}, \forall t > 0, j = 1, 2$, where $K > 0$ is independent of t . In this case

$$\frac{\omega_r(f_j, \xi_n)_p}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad j = 1, 2, \quad (153)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we get $\left\| W_{r,n}^{[0]}f - f \right\|_p \rightarrow 0$.

Proof. By Theorem 3.8.

We also give

Theorem 3.32. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in (C(\mathbb{R}^N) \cap L_1(\mathbb{R}^N))$; $N \geq 1$, $\xi_n \in (0, 1], n \in \mathbb{N}$. Then

1)

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \leq$$

$$\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e} \right)^N \right] \frac{[\omega_r(f_1, \xi_n)_1 + \omega_r(f_2, \xi_n)_1]}{(\sqrt{\xi_n})^r}. \quad (154)$$

2) Let $r \in \mathbb{N} - \{1\}, 0 < \alpha \leq 1$, and assume for $j = 1, 2$ that $\omega_r(f_j, t)_1 \leq Kt^{r-1+\alpha}, \forall t > 0, K > 0$ independent of t . We find that

$$\frac{\omega_r(f_j, \xi_n)_1}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad j = 1, 2, \quad (155)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we derive $\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \rightarrow 0$.

Proof. By Theorem 3.10.

We further present

Theorem 3.33. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in C^m(\mathbb{R}^N), m, N \in \mathbb{N}$, with $f_{j,\alpha} \in L_1(\mathbb{R}^N), |\alpha| = m, x \in \mathbb{R}^N, \xi_n \in (0, 1], n \in \mathbb{N}, m > r$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$,

$\alpha_i \in \mathbb{Z}^+, i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha, n, \tilde{j}} := \frac{1}{(\sqrt{\pi \tilde{\xi}_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\sum_{i=1}^N s_i^2}{\tilde{\xi}_n}} ds_1 \dots ds_N. \quad (156)$$

Then

$$\begin{aligned} & \left\| W_{r, n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha, n, \tilde{j}} f_{\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{1, x} \\ & \leq \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right] \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ & \quad \left(\sqrt{\tilde{\xi}_n} \right)^{m-r} [\omega_r(f_{1, \alpha}, \tilde{\xi}_n)_1 + \omega_r(f_{2, \alpha}, \tilde{\xi}_n)_1]. \end{aligned} \quad (157)$$

Proof. By Theorem 3.12.

We continue with simultaneous L_p approximations.

Theorem 3.34. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2, j = 1, 2$. Here $f_j \in C^{m+l}(\mathbb{R}^N), m, l, N \in \mathbb{N}, m > r$. The assumptions of Theorem 3.15 are valid for

$d\mu_{\tilde{\xi}_n} = \frac{1}{(\sqrt{\pi \tilde{\xi}_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\tilde{\xi}_n}\right)} ds_1 \dots ds_N$ and f_j . Call $\gamma = 0, \beta$. Let $f_{j, (\gamma+\alpha)} \in L_p(\mathbb{R}^N), |\alpha| = m, x \in \mathbb{R}^N$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \tilde{\xi}_n \in (0, 1], n \in \mathbb{N}$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}^+, i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha, n, \tilde{j}} := \frac{1}{(\sqrt{\pi \tilde{\xi}_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\sum_{i=1}^N s_i^2}{\tilde{\xi}_n}} ds_1 \dots ds_N. \quad (158)$$

Then

$$\begin{aligned} & \left\| \left(W_{r, n}^{[m]}(f; x) \right)_{\gamma} - f_{\gamma}(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha, n, \tilde{j}} f_{\gamma+\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{p, x} \\ & \leq \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left(\frac{m}{(q(m-1) + 1)^{\frac{1}{q}}} \right) \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p + 1, 1) \right]^{\frac{1}{p}} \end{aligned} \quad (159)$$

$$\left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\sqrt{\xi_n} \right)^{(m-r)} \left[\omega_r (f_{1,\gamma+\alpha}, \xi_n)_p + \omega_r (f_{2,\gamma+\alpha}, \xi_n)_p \right].$$

Proof. By Theorems 3.15, 3.19.

We give also

Theorem 3.35. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. Here $f_j \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 3.15 are valid for

$$d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N \text{ and } f_j. \text{ Call } \gamma = 0, \beta. \text{ Let } f_{j,\gamma} \in L_p(\mathbb{R}^N), x \in \mathbb{R}^N; p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \xi_n \in (0, 1], n \in \mathbb{N}. \text{ Then}$$

1)

$$\left\| \left(W_{r,n}^{[0]}(f) \right)_\gamma - f_\gamma \right\|_p \leq \quad (160)$$

$$\left(\frac{2}{\sqrt{\pi}} \right)^{\frac{N}{p}} \left[(1+N)^{rp} + 2^{rp} \Gamma^N(rp+1, 1) \right]^{\frac{1}{p}} \frac{\left[\omega_r (f_{1,\gamma}, \xi_n)_p + \omega_r (f_{2,\gamma}, \xi_n)_p \right]}{(\sqrt{\xi_n})^r}.$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, with $\omega_r (f_{j,\gamma}, t)_p \leq K t^{r-1+\alpha}$, $\forall t > 0$, $K > 0$ independent of t , $j = 1, 2$. In this case

$$\frac{\omega_r (f_{j,\gamma}, \xi_n)_p}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad j = 1, 2. \quad (161)$$

As $n \rightarrow +\infty$ and $\xi_n \rightarrow 0$, then $\left(W_{r,n}^{[0]}(f) \right)_\gamma \xrightarrow{\|\cdot\|_p} f_\gamma$.

Proof. By Theorems 3.15 and 3.20.

We continue with

Theorem 3.36. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. Here $f_j \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 3.15 are valid for

$$d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N \text{ and } f_j. \text{ Call } \gamma = 0, \beta. \text{ Let } f_{j,\gamma} \in L_1(\mathbb{R}^N), x \in \mathbb{R}^N, \xi_n \in (0, 1], n \in \mathbb{N}. \text{ Then}$$

1)

$$\left\| \left(W_{r,n}^{[0]}(f) \right)_\gamma - f_\gamma \right\|_1 \leq \quad (162)$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+N)^r + 2^r \left(\frac{\lfloor er! \rfloor}{e}\right)^N \right] \frac{[\omega_r(f_{1,\gamma}, \xi_n)_1 + \omega_r(f_{2,\gamma}, \xi_n)_1]}{(\sqrt{\xi_n})^r}.$$

2) Let $r \in \mathbb{N} - \{1\}$, $0 < \alpha \leq 1$, with $\omega_r(f_{j,\gamma}, t)_1 \leq Kt^{r-1+\alpha}$, $\forall t > 0$, $K > 0$ independent of t , $j = 1, 2$. In this case

$$\frac{\omega_r(f_{j,\gamma}, \xi_n)_1}{(\sqrt{\xi_n})^r} \leq K \xi_n^{\left(\frac{r}{2} + \alpha - 1\right)}, \quad j = 1, 2 \quad (163)$$

and as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, we get $\left(W_{r,n}^{[0]}(f)\right)_\gamma \xrightarrow{\|\cdot\|_1} f_\gamma$.

Proof. By Theorems 3.15 and 3.21.

We finish with

Theorem 3.37. Let $f : \mathbb{R}^N \rightarrow \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. Here $f_j \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$. The

assumptions of Theorem 3.15 for $d\mu_{\xi_n} = \frac{1}{(\sqrt{\pi\xi_n})^N} e^{-\left(\frac{\sum_{i=1}^N s_i^2}{\xi_n}\right)} ds_1 \dots ds_N$ and f_j are valid. Call $\gamma = 0, \beta$.

Let $f_{j,(\gamma+\alpha)} \in L_1(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$\tilde{c}_{\alpha,n,\tilde{j}} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N. \quad (164)$$

Then

$$\begin{aligned} & \left\| \left(W_{r,n}^{[m]}(f; x)\right)_\gamma - f_\gamma(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{\tilde{c}_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(x)}{\prod_{i=1}^N \alpha_i!} \right) \right\|_{1,x} \\ & \leq \left(\frac{2}{\sqrt{\pi}}\right)^N \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) [\omega_r(f_{1,\gamma+\alpha}, \xi_n)_1 + \omega_r(f_{2,\gamma+\alpha}, \xi_n)_1] \right) \\ & \quad \left(\sqrt{\xi_n} \right)^{m-r} \left[(1+N)^r + 2^r \left(\frac{\lfloor e(m+r)! \rfloor}{e} \right)^N \right]. \end{aligned} \quad (165)$$

Proof. By Theorems 3.15 and 3.22.

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