Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 7(1), 2021, Pages 173–181

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2021-0013](https://doi.org/10.2478/mjpaa-2021-0013)

A look at nonexpansive mappings in non-Archimedean vector spaces

SAMIH LAZAIZ¹

ABSTRACT. In a spherically complete ultrametric space every nonexpansive self-mapping T has a fixed point $\bar{x}$ or a minimal invariant ball $B(\bar{x}, d(\bar{x}, T(\bar{x})))$. We show how we can approximate this fixed center $\bar{x}$ in a non-Archimedean vector space. And, we give a synthetic study for increasing mapping in non-Archimedean local fields.

Mathematics Subject Classification (2020). 47H09, 32P05

Key words and phrases. Fixed centers, Increasing mappings, Nonexpansive mappings, Pseudo-order, Ultrametric spaces.

1. Introduction

In the last 40 years, the theory of non-Archimedean analysis has taken a large place in mathematical research, this is motivated by several aspects, among them, there is a problem in quantum theory of the rupture of the Archimedean axiom (i.e. the use of real numbers) at the Planck scale. So, to have a better understanding at the Planck scale, it is quite natural to use p-adic numbers (see [6]). In addition, there are several applications of non-Archimedean analysis whether in mathematical physics, gravity, cosmology, biological systems, cognitive science and much more, for details one can consult the survey [4].

Received : May 13, 2020 - Accepted: November 25, 2020.

©The Author(s) 2020. This article is published with open access by Sidi Mohamed Ben Abdallah University.

¹*LaSMA Laboratory, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdallah, Fes, Morocco.*

e-mail: samih.lazaiz@usmba.ac.ma .

Recall that a metric space (M, d) is said to be ultrametric space (or, non-Archimedean space) if and only if

$$d(x, z) \leq \max\{d(x, y); d(y, z)\},$$

for any $x, y, z \in M$. Let $x \in M$ and $r > 0$, we denote by $B(x; r)$ the closed ball and by $B(x; r^-)$ the open ball both of center x and radius r .

A very important notion in non-Archimedean analysis which allows to carry many results of the classical functional analysis in the ultrametric case is the notion of spherical completeness.

Definition 1.1 ([19, Definition 1.1.4]). *An ultrametric space (M, d) is said to be spherically complete if every chain of balls in M has nonempty intersection.*

A mapping $T : M \rightarrow M$ is said to be nonexpansive if

$$d(T(x), T(y)) \leq d(x, y) \quad \text{for each } x, y \in M.$$

Fixed point theory for nonexpansive mappings in ultrametric spaces find its root in the work of Petalas and Vidalis [10, Theorem 2], this theorem has been proved using Zorn's lemma. In 2012, Kirk and Shahzad [7, Theorem 4] revised the theorem by giving it a more illuminating form. Before we state this result recall that a ball $\hat{B} = B(x; r)$ in M is minimal T -invariant if $T(\hat{B}) \subseteq \hat{B}$ and $d(y, T(y)) = r$ for all $y \in \hat{B}$.

Theorem 1.1 ([7, 10]). *Suppose M is a spherically complete ultrametric space and $T : M \rightarrow M$ is a nonexpansive mapping. Then every ball of the form $B(x, d(x, T(x)))$ contains either a fixed point of T or a minimal T -invariant ball.*

The point $\bar{x} \in E$ such that $T(\bar{x}) = \bar{x}$ or $B(\bar{x}, d(\bar{x}, T(\bar{x})))$ is a minimal T -invariant ball will be called *fixed center*. Moreover, this theorem has been generalized and extended by many authors see for instance [1, 11] and the references therein.

Remark 1.1. *If T is a nonexpansive mapping, then every ball of the form $B(x, d(x, T(x)))$ is T -invariant. Indeed, let $y \in B(x, d(x, T(x)))$ then $d(x, y) \leq d(x, T(x))$ we have by nonexpansiveness*

$$d(T(y), T(x)) \leq d(x, T(x)).$$

Thus, $T(y) \in B(x, d(x, T(x)))$.

Many fixed point theorems for nonexpansive mappings are not constructive. There is no indication of how the fixed point by means of an iterative procedure can be approximated. From a practical point of view, it is important to be able to construct that fixed point. Berinde in [2], studied many of the approximate methods for nonexpansive mappings, namely Krasnoselskij iteration, the Mann iteration, the Ishikawa iteration and much more. The book is a good survey of the most used fixed point iteration procedures in classical Banach spaces. In a spherically complete ultrametric space, Prieß-Cramp and Ribenboim have used a transfinite Picard scheme to approximate the unique fixed point for strictly contractive mappings (i.e. mappings satisfying $d(T(x), T(y)) < d(x, y)$ for each $x \neq y$), for more details see [12, 13].

On the other hand, it is known that in the field of p -adic numbers $\mathbb{Q}_p$ there is no order which is compatible with both the group operation and the topology of $\mathbb{Q}_p$, see [14, Theorem

3.4]. This property, which is frequently used in real analysis, presents difficulties for the construction of a theory of non-Archimedean ordered fields. In 1969 Monna [8] introduced the concept of pseudo-order in non-Archimedean complete field and showed that this binary relational system behaves like an order (see also [9]). Later, Schikhof in [16, 17, 18] studied deeply this notion and introduced a several class of monotone mappings.

In section 2 of this paper, we introduce a transfinite Krasnoselskij method in order to approximate the fixed center for nonexpansive mappings. One can even consider this approach as another proof of Petalas-Vidalis' theorem [10, Theorem 2]. In section 3, we give a synthetic study of increasing mappings (i.e. mappings preserving pseudo-order) and we show that this class of mappings is fixed point free.

2. Approximation theorem for nonexpansive mappings

Let ω be the first transfinite ordinal and Ω be the first uncountable transfinite ordinal. Note that for each $\xi < \Omega$, ξ is countable. For the use of transfinite induction in fixed point theory, one can consult [3] and references therein.

The following result is needed throughout this work, for the proof see [5, Proposition A.6, pp. 284].

Proposition 2.1. *Suppose that a net $\{a_\alpha\}_{\alpha \in \Omega} \subseteq \mathbb{R}$ is bounded and either nonincreasing or nondecreasing. Then there exists $\alpha_0 \in \Omega$ such that $a_\alpha = a_{\alpha_0}$ for all $\alpha \geq \alpha_0$.*

Let $\mathbb{K}$ be a non-Archimedean local field (i.e. a field which is complete with respect to a non-trivial discrete non-Archimedean absolute value $|\cdot|$ and such that the residue class is finite). Let E be a non-Archimedean vector space over $\mathbb{K}$, that is a $\mathbb{K}$ -vector space endowed with a non-Archimedean norm $\|\cdot\|$:

- (1) $\|x\| = 0$ if and only if $x = 0$;
- (2) $\|ax\| = |a| \|x\|$ for any $x \in E$ and any $a \in \mathbb{K}$;
- (3) $\|x + y\| \leq \max\{\|x\|; \|y\|\}$ for any $x, y \in E$.

Let E be a non-Archimedean spherically complete normed space and $T : E \rightarrow E$ a nonexpansive mapping. Then, there exists $\bar{x} \in E$ such that $T(\bar{x}) = \bar{x}$ or $B(\bar{x}, \|\bar{x} - T(\bar{x})\|)$ is a minimal T -invariant ball. Our objective is to indicate how to approximate such a "fixed center" $\bar{x}$. Note that since this point $\bar{x}$ is not necessary unique, our approach only approximate one of these points. The set of fixed centers will be denoted by $Fic(T)$.

Let $k \in \mathbb{K}$ such that $0 < |k| \leq 1$. Let $\mathcal{P}$ be the set of all families $(x_\alpha)_{\alpha \in \Omega} \subset E$ which satisfy for every $\alpha \in \Omega$, the following two conditions:

- (i) $x_{\alpha+1} = (1 - k)T(x_\alpha) + kx_\alpha$;
- (ii) if μ is a limit ordinal, then $\|x_\alpha - x_\mu\| \leq \|x_\mu - T(x_\mu)\|$ for all $\mu < \alpha$.

Remark 2.1. *This construction will be used to approach some fixed center $\bar{x}$ of T . Note that the proof of Theorem 2.1 below will show that $\mathcal{P}$ is a nonempty set.*

Now, following [13] we introduce the following definition.

Definition 2.1. We say that $(x_\alpha)_{\alpha \in \Omega} \subset E$ is a Krasnoselskij approximation of $\bar{x}$ if $(x_\alpha)_{\alpha \in \Omega} \in \mathcal{P}$ and

$$\bigcap_{\alpha \in \Omega} B(x_\alpha, \|x_\alpha - T(x_\alpha)\|) = B(\bar{x}, \|\bar{x} - T(\bar{x})\|).$$

In fact, this definition captures the main idea of [13] and guarantees the extension of [13, Theorem 4] in the case of a non-Archimedean vector space. Note that if $\bar{x}$ is a fixed point of T , then $B(\bar{x}, \|\bar{x} - T(\bar{x})\|) = \{\bar{x}\}$.

Before we state the main theorem it is worth mentioning that we can reach this point $\bar{x}$ by a finite steps of iterations, in this case we say that $x = (x_\alpha)_{\alpha < \lambda}$ reaches $\bar{x}$ for some finite ordinal λ .

Theorem 2.1. Let E be a spherically complete non-Archimedean normed space. Assume that for any $\bar{x} \in \text{Fic}(T)$ cannot be reached by any $(x_\alpha)_{\alpha \in \Omega} \in \mathcal{P}$ such that $x_0 \neq \bar{x}$. Then for any $y \in E \setminus \{\bar{x}\}$ there exists a Krasnoselskij approximation $(x_\alpha)_{\alpha \in \Omega}$ of some $\bar{x} \in \text{Fic}(T)$ such that $x_0 = y$.

Proof. The idea of the proof is to construct a transfinite net $(x_\alpha)_{\alpha \in \Omega}$ where Ω is the first uncountable ordinal satisfying for each $\alpha \in \Omega$:

$$\begin{aligned} A_1(\alpha) : x_{\alpha+1} &= (1-k)T(x_\alpha) + kx_\alpha; \\ A_2(\alpha) : \|x_\alpha - x_\mu\| &\leq \|x_\mu - T(x_\mu)\| \text{ whenever } \mu < \alpha. \end{aligned}$$

Let $x_0 \in E$. For each n in $\mathbb{N}$, set

$$x_{n+1} = (1-k)T(x_n) + kx_n \quad \text{and} \quad B_n = B(x_n, \|x_n - T(x_n)\|).$$

Since B_n is T -invariant (see Remark 1.1) and

$$|1-k| \leq \max\{1, |k|\} \leq 1,$$

we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= |1-k| \|T(x_n) - x_n\| \\ &\leq \|T(x_n) - x_n\|. \end{aligned}$$

Then x_{n+1} and $T(x_{n+1})$ are in B_n . Thus

$$\begin{aligned} \|x_{n+1} - T(x_{n+1})\| &\leq \max\{\|x_{n+1} - x_n\|, \|x_n - T(x_{n+1})\|\} \\ &\leq \|x_n - T(x_n)\|. \end{aligned}$$

In addition, $B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$. Indeed, let $y \in B_{n+1}$ so we have

$$\|x_{n+1} - y\| \leq \|x_{n+1} - T(x_{n+1})\|$$

thus,

$$\begin{aligned} \|x_n - y\| &\leq \max\{\|x_n - x_{n+1}\|, \|x_{n+1} - y\|\} \\ &\leq \|x_n - T(x_n)\|. \end{aligned}$$

That is $B_{n+1} \subseteq B_n$ for all n . Then we get $B_m \subseteq B_n$ for all $n \leq m$.

This implies that $\{B_n\}_n$ is a descending chain of balls. By spherical completeness of E , we have $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$. Let $z \in \bigcap_{n \in \mathbb{N}} B_n$, since B_n is T -invariant we get

$$\|z - T(z)\| \leq \|x_n - T(x_n)\|, \quad \text{for all } n \in \mathbb{N}.$$

Now, if $z = T(z)$ or $B(z, \|z - T(z)\|)$ is minimal ball, we take $\bar{x} = z$. Otherwise choose

$$x_\omega = z \in \bigcap_{n \in \mathbb{N}} B_n.$$

Then $A_1(\omega)$ and $A_2(\omega)$ are satisfied.

Let $\beta \in \Omega$. Assume that the net $\{x_\alpha\}_{\alpha < \beta}$ has been defined. We need to define x_β and show that the properties $A_1(\beta)$ and $A_2(\beta)$ hold.

If β is not a limit ordinal, say $\beta = \alpha + 1$, take $x_\beta = (1 - k)T(x_\alpha) + kx_\alpha$. Then $A_1(\beta)$ and $A_2(\beta)$ hold.

Assume, β is a limit ordinal. Then for each $\mu \leq \alpha < \beta$, we have by $A_2(\alpha)$

$$\|x_\alpha - x_\mu\| \leq \|x_\mu - T(x_\mu)\|$$

that is $B_\alpha \subseteq B_\mu$. Again, using the spherical completeness of E , we get

$$\bigcap_{\alpha < \beta} B(x_\alpha, \|x_\alpha - T(x_\alpha)\|) \neq \emptyset.$$

As above, let $z \in \bigcap_{\alpha < \beta} B(x_\alpha, \|x_\alpha - T(x_\alpha)\|)$. If $z = T(z)$ or $B(z, \|z - T(z)\|)$ is a minimal ball take $\bar{x} = z$. Otherwise choose

$$x_\beta \in \bigcap_{\alpha < \beta} B(x_\alpha, \|x_\alpha - T(x_\alpha)\|)$$

then $A_1(\beta)$ and $A_2(\beta)$ are satisfied.

Thus, the net $(x_\alpha)_{\alpha \in \Omega}$ is well constructed. In addition we have for all $\mu < \alpha < \Omega$:

$$\|x_\alpha - x_\mu\| \leq \|x_\mu - T(x_\mu)\| \quad \text{and} \quad \|x_\alpha - T(x_\mu)\| \leq \|x_\mu - T(x_\mu)\|$$

by $A_2(\alpha)$ we have the first inequality and the second inequality is due to the fact that every point of a ball is a centre. Then,

$$\begin{aligned} \|x_\alpha - T(x_\alpha)\| &\leq \max\{\|x_\alpha - T(x_\mu)\|, \|T(x_\mu) - T(x_\alpha)\|\} \\ &\leq \|x_\mu - T(x_\mu)\| \end{aligned}$$

Hence $\{\|x_\alpha - T(x_\alpha)\|\}_{\alpha \in \Omega}$ is a nonincreasing net. Using Proposition 2.1 there exist $\alpha_0 \in \Omega$ such that $\|x_\alpha - T(x_\alpha)\|$ is constant for all $\alpha \geq \alpha_0$. From $A_2(\alpha_0 + 1)$, we have $B_{\alpha_0+1} \subseteq B_{\alpha_0}$ and since $\|x_{\alpha_0+1} - T(x_{\alpha_0+1})\| = \|x_{\alpha_0} - T(x_{\alpha_0})\|$ we get $B_{\alpha_0+1} \supseteq B_{\alpha_0}$, that is $B_{\alpha_0+1} = B_{\alpha_0}$. Let $r = \inf_{\alpha \leq \alpha_0} \|x_\alpha - T(x_\alpha)\|$ then $\bigcap_{\alpha \leq \alpha_0} B(x_\alpha, \|x_\alpha - T(x_\alpha)\|) = B(x_{\alpha_0}, r)$. Again, by $A_2(\alpha_0)$ we have :

$$\|x_{\alpha_0} - T(x_{\alpha_0})\| \leq \|x_\alpha - T(x_\alpha)\|$$

for all $\alpha \leq \alpha_0$, that is $\bar{x} = x_{\alpha_0}$.

3. Monotone mappings in non-Archimedean local fields

For a prime number p we denote by $\mathbb{Q}_p$ the non-Archimedean valued field of the p -adic numbers, by $\mathbb{Z}_p$ its valuation ring $\{x \in \mathbb{Q}_p : |x| \leq 1\}$, this is the typical example of a non-Archimedean local field.

Regarding a partially ordered relation on $\mathbb{Q}_p$ we know that we cannot introduce an order structure having natural properties. In fact, from [18, p. 67], see also [14, 15] it is not possible to define a partial ordering $\geq$ on $\mathbb{Q}_p$ satisfying :

- (i) $1 \geq 0 \geq -1$
- (ii) if $a \geq 0, b \geq 0$ then $a + b \geq 0$
- (iii) if $a_n \geq 0$ for all n , $\lim_{n \rightarrow \infty} a_n = a$ then $a \geq 0$,

that is to say $\mathbb{Q}_p$ does not admits an order which is compatible with both the group operation $+$ and the usual topology of $\mathbb{Q}_p$. To get an idea, we have the following surprising result.

Proposition 3.1 ([14, lemma 3.3]). *In $\mathbb{Q}_p$, the serie $\sum n.n!$ converges to -1 .*

To avoid this, A. F. Monna [8, 9] introduced the concept of sides of zero in a complete non-Archimedean field, which allowed him to define "positive element", "sign of element" and "increasing function".

We recall here the important properties that we will use in the sequel. For a full expository, see [17, section 4].

Let $\mathbb{K}$ be a non-Archimedean local field, $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ and $\mathbb{K}^+ = \{x \in \mathbb{K} : |1 - x| < 1\}$. In [17], $\mathbb{K}^+$ is called the group of positive elements of $\mathbb{K}$. The group of signs of elements of $\mathbb{K}$ is defined to be the group Σ whose elements are multiplicative cosets of $\mathbb{K}^+$, i.e.

$$\Sigma := \{a\mathbb{K}^+ : a \in \mathbb{K}^\times\}.$$

Remark 3.1.

- (i) Σ is an infinite group. Moreover, if we apply the above process to the archimedean local field $\mathbb{R}$, we find the two sides of zero, namely $(0, \infty)$ and $(-\infty, 0)$.
- (ii) Let $\sigma \in \Sigma$ and let $x, y \in \sigma$. Then, there exist $a \in \mathbb{K}^\times$ and $u, v \in \mathbb{K}^+$ such that

$$\sigma = a\mathbb{K}^+, x = au \text{ and } y = av.$$

Then,

$$\begin{aligned} |x - y| &= |au - av| \\ &= |a||u - v|. \end{aligned}$$

Moreover, since $u, v \in \mathbb{K}^+$, we have

$$\begin{aligned} |u - v| &= |(u - 1) + (1 - v)| \\ &\leq \max\{|u - 1|, |v - 1|\} \\ &< 1. \end{aligned}$$

Thus, $|x - y| < |a|$. In addition, since $u \in \mathbb{K}^+$ and $u = xa^{-1}$ we obtain

$$|xa^{-1} - 1| < 1 \Leftrightarrow |x - a| < |a|.$$

Then, $|a| = |x|$. Similarly $|a| = |y|$, so in particular we have $|x| = |y|$.

The following definition can be found in [8, 9, 17].

Definition 3.1 (Pseudo-order). Let $x, y \in \mathbb{K}$ and $\sigma \in \Sigma$. We say that $x <_\sigma y$ (y is greater than x in the sense of σ) if $y - x \in \sigma$.

The next property shows that this binary relation behaves like an order.

Proposition 3.2 ([18, Proposition 85.3]). Let $x, y, z \in \mathbb{K}$.

- (i) If $x \neq y$ then $x <_\sigma y$ for precisely one $\sigma \in \Sigma$. If $x = y$ then $x <_\sigma y$ for no $\sigma \in \Sigma$.
- (ii) If $x <_\sigma y$ for some σ then $x + z <_\sigma y + z$.
- (iii) If $x <_\sigma y$, $0 <_{\sigma'} z$ for some σ, σ' in Σ , then $xz <_{\sigma\sigma'} yz$.

With this in hand, Schikhof introduced the following definition.

Definition 3.2 ([18, Definition 24.6]). Let $X \subset \mathbb{K}$, $\sigma \in \Sigma$ and $T : X \rightarrow \mathbb{K}$. T is said to be increasing if for each $x, y \in X$ with $x \neq y$ we have

$$x <_\sigma y \Rightarrow T(x) <_\sigma T(y). \quad (3.1)$$

Note that if T is an increasing mapping, then it is an isometry. Indeed, for each $x, y \in X$ with $x \neq y$ the logical implication (3.1) can be written as

$$y - x \in \sigma \Rightarrow T(y) - T(x) \in \sigma,$$

so using Remark 3.1 (ii), we get $|T(x) - T(y)| = |x - y|$.

We have the following surprising result

Proposition 3.3. Let $X \subset \mathbb{K}$. Every increasing mapping $T : X \rightarrow X$ is an isomerty.

We use some idea of Prieß-Cramp example [11, Example 1] to prove the following remark.

Remark 3.2. The class of increasing mappings is a proper class of nonexpansive mappings. Indeed, let $X = \mathbb{Z}_3$ and define $T : X \rightarrow X$ by $T(x) = x^2 + x - 3$ for each $x \in X$, and let $\pi \in X$ such that

$$|1 - \pi| < |\pi|$$

then

$$\begin{aligned} |T(x) - T(y) - \pi| &= |(y^2 - x^2) + (y - x - \pi)| \\ &\leq \max\{|y^2 - x^2|, |y - x - \pi|\}. \end{aligned}$$

One can choose $y = 4$ and $x = 3$ then $y - x \in \sigma = B(\pi, |\pi|^-) = \{z \in \mathbb{Q}_3 : |z - \pi| < |\pi|\}$ (see [18] for more details), using the fact that all triangles are isosceles we get

$$|T(y) - T(x) - \pi| = |y^2 - x^2| = 1$$

that is $T(y) - T(x) \notin \sigma$. Furthermore it is shown that T is a nonexpansive mapping.

Since non-Archimedean local fields are spherically complete, the following question arise naturally :

Question. Let $X \subset \mathbb{K}$. Does an increasing mapping from X to itself has a fixed point ?

Unfortunately the answer to this question is negative. The following counter-example shows the deviation from "classical" analysis.

Example 3.1. Let $X = \mathbb{Q}_p$. Define $T : X \rightarrow X$ by $T(x) = x + 1$ for each $x \in X$. Let $\sigma \in \Sigma$, then there exists $\pi \in \mathbb{Q}_p^\times$ such that $\sigma = B(\pi, |\pi|^-)$.

We have to show that for each $\sigma \in \Sigma$, T is increasing mapping, i.e.

$$y - x \in B(\pi, |\pi|^-) \Rightarrow T(y) - T(x) \in B(\pi, |\pi|^-).$$

Indeed, we have obviously

$$\begin{aligned} |T(x) - T(y) - \pi| &= |y - 1 - x + 1 - \pi| \\ &< |\pi|. \end{aligned}$$

Thus, $|T(x) - T(y) - \pi| < |\pi|$. Then T is an increasing mapping from X to itself. Now, using Proposition 3.3, T is nonexpansive mapping and it is clear that T is fixed point free mapping

Acknowledgments

I would like to thank the anonymous referees for their careful reading of our manuscript and their many thoughtful comments and suggestions.

References

- [1] M. Babahmed, A. El Amrani, and S. Lazaiz. *Fixed points and diametral sets for sequentially bounded mappings in orbital ultrametric spaces*. *Proyecciones (Antofagasta, On line)*, 39(2):481–493, 2020.
- [2] V. Berinde. *Iterative approximation of fixed points*, volume 1912. Springer-Verlag Berlin Heidelberg, 2007.
- [3] K. Chaira, A. Eladraoui, M. Kabil, and S. Lazaiz. *Extension of kirk-saliga fixed point theorem in a metric space with a reflexive digraph*. *International Journal of Mathematics and Mathematical Sciences*, 2018, 2018.
- [4] B. Dragovich, A. Y. Khrennikov, S. V. Kozyrev, I. V. Volovich, and E. I. Zelenov. *p-adic mathematical physics: the first 30 years*. *p-Adic Numbers, Ultrametric Analysis and Applications*, 9(2):87–121, 2017.
- [5] M. A. Khamsi and W. A. Kirk. *An introduction to metric spaces and fixed point theory*, volume 53. John Wiley & Sons, 2011.
- [6] A. Y. Khrennikov. *non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models*, volume 427. Springer Science & Business Media, 2013.
- [7] W. A. Kirk and N. Shahzad. *Some fixed point results in ultrametric spaces*. *Topology and its Applications*, 159(15):3327–3334, 2012.
- [8] A. F. Monna. *Sur une classe de corps topologiques munis d'un pseudo-ordre*. In *Indagationes Mathematicae (Proceedings)*, volume 72, pages 364–375. North-Holland, 1969.
- [9] A. F. Monna. *Ordre, pseudo-ordre, primes et convexité*. In *Indagationes Mathematicae (Proceedings)*, volume 74, pages 181–190. Elsevier, 1971.
- [10] C. Petalas and T. Vidalis. *A fixed point theorem in non-Archimedean vector spaces*. *Proceedings of the American Mathematical Society*, 118(3):819–821, 1993.
- [11] S. Priess-Crampe. *Fixed points and stable balls in ultrametric spaces*. *Results in Mathematics*, 43(1-2):163–167, 2003.
- [12] S. Priess-Crampe. *Reaching fixed points as limits in subspaces*. *Journal of Geometry*, 107(2):387–402, 2016.
- [13] S. Priess-Crampe and P. Ribenboim. *The approximation to a fixed point*. *J. Fixed Point Theory Appl.*, 14(1), 41–53 (2013).
- [14] G. Rangan. *On orderability of topological groups*. *International Journal of Mathematics and Mathematical Sciences*, 8(4):747–754, 1985.

- [15] P. Ribenboim. *The theory of classical valuations*. Springer Science & Business Media, 2012.
- [16] W. H. Schikhof. *non-Archimedean monotone functions*. Groupe de travail d'analyse ultramétrique, http://www.numdam.org/item/GAU_1978-1979__6__A7_0/, Volume 6 (1978-1979), Talk no. 13, 8 p.
- [17] W. H. Schikhof. *p-adic monotone functions*. <https://hdl.handle.net/2066/57035>, 1980.
- [18] W. H. Schikhof. *Ultrametric Calculus: an introduction to p-adic analysis*, volume 4. Cambridge University Press, 2007.
- [19] W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 4. Cambridge University Press, 2010.