Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 7(2), 2021, Pages 214–226

ISSN: Online 2351-8227 - Print 2605-6364

DOI: [10.2478/mjpaa-2021-0015](https://doi.org/10.2478/mjpaa-2021-0015)

Further generalized refinement of Young's inequalities for τ -mesurable operators

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ABSTRACT. In this paper, we prove that if $a, b > 0$ and $0 \leq \nu \leq 1$. Then for all positive integer m

(1) - For $\nu \in [0, \frac{1}{2^n}]$, we have

$$\left(a^\nu b^{1-\nu}\right)^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt[m]{b^m} - \sqrt[m]{(ab^{2^{k-1}-1})^m}\right)^2 \leq \left(\nu a + (1-\nu)b\right)^m.$$

(2) - For $\nu \in [\frac{2^n-1}{2^n}, 1]$, we have

$$\left(a^\nu b^{1-\nu}\right)^m + \sum_{k=1}^n 2^{k-1} (1-\nu)^m \left(\sqrt[m]{a^m} - \sqrt[m]{(ba^{2^{k-1}-1})^m}\right)^2 \leq \left(\nu a + (1-\nu)b\right)^m,$$

we also prove two similar inequalities for the cases $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$ and $\nu \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$. These inequalities provides a generalization of an important refinements of the Young inequality obtained in 2017 by S. Furuchi. As applications we shall give some refined Young type inequalities for the traces, determinants, and p -norms of positive τ -mesurable operators.

Mathematics Subject Classification (2020). 26D07, 47A63.

Key words and phrases. AM-GM inequality, Young's inequality, Determinants, Trace, p -norms, τ -mesurable operators.

Received : September 16, 2020 - Accepted: December 02, 2020.

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1. Introduction

The famous Young inequality for scalars says that if $a, b > 0$ and $0 \leq \nu \leq 1$, then

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b. \quad (1.1)$$

The first refinements of Young inequality is the squared version proved in [7] as follows

$$(a^\nu b^{1-\nu})^2 + r_0^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2, \quad (1.2)$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Kittaneh and Manasrah [9] refined Young's inequality so that

$$a^\nu b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, \quad (1.3)$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Recently, Furuichi in [3], gave the following generalized refinement of Young's inequality as follows:

Theorem 1.1. *Let a and b be two positive numbers and $0 \leq \nu \leq 1$. Then*

(1) *For $\nu \in [0, \frac{1}{2^n}]$, we have*

$$a^\nu b^{1-\nu} + \sum_{k=1}^n 2^{k-1} \nu (\sqrt{b} - \sqrt[2^k]{ab^{2^{k-1}-1}})^2 \leq \nu a + (1-\nu)b.$$

(2) *For $\nu \in [\frac{2^n-1}{2^n}, 1]$, we have*

$$a^\nu b^{1-\nu} + \sum_{k=1}^n 2^{k-1} (1-\nu) (\sqrt{a} - \sqrt[2^k]{ba^{2^{k-1}-1}})^2 \leq \nu a + (1-\nu)b.$$

Manasrah and Kittaneh [2] gave a generalization refinements of (1.2) and (1.3), as follows

Theorem 1.2. *Let a and b be two positive numbers and $0 \leq \nu \leq 1$. Then for $m = 1, 2, 3, \dots$, we have*

$$(a^\nu b^{1-\nu})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\nu a + (1-\nu)b)^m, \quad (1.4)$$

where $r_0 = \min\{\nu, 1-\nu\}$.

For another generalized refinement of inequalities (1.2) and (1.3) one can see [8].

One of the aims of this paper is to extend the Theorem 1.1, to the following one :

(1) *For $\nu \in [0, \frac{1}{2^n}]$, we have*

$$\left(a^\nu b^{1-\nu}\right)^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{b^m} - \sqrt[2^k]{(ab^{2^{k-1}-1})^m}\right)^2 \leq \left(\nu a + (1-\nu)b\right)^m.$$

(2) *For $\nu \in [\frac{2^n-1}{2^n}, 1]$, we have*

$$\left(a^\nu b^{1-\nu}\right)^m + \sum_{k=1}^n 2^{k-1} (1-\nu)^m \left(\sqrt{a^m} - \sqrt[2^k]{(ba^{2^{k-1}-1})^m}\right)^2 \leq \left(\nu a + (1-\nu)b\right)^m.$$

As applications we shall give some refined Young type inequalities for the traces, determinants, and p -norms of positive τ -measurable operators.

2. Further generalized refinements of Young's inequality

In this section, we show the main results of this paper. To do this, we need the following theorem concerning the arithmetic-geometric mean inequality

Theorem 2.1. *Let n be a positive integer. For $k = 1, 2, \dots, n$, let $x_k > 0$, and let $v_k \geq 0$ satisfy $\sum_{k=1}^n v_k = 1$. Then, we have*

$$\prod_{k=1}^n x_k^{v_k} \leq \sum_{k=1}^n v_k x_k. \quad (2.1)$$

We need also the following lemma

Lemma 2.1. *Let m be a positive integer and let v a positive number, such that $0 \leq v \leq 1$. Then we have*

$$\sum_{k=1}^m \binom{m}{k} k v^k (1-v)^{m-k} = m v \quad (2.2)$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k) v^k (1-v)^{m-k} = m(1-v), \quad (2.3)$$

where $\binom{m}{k}$ is the binomial coefficient.

For a proof of Lemma 2.1, one can see [1].

The first main result to be proved in this paper is:

Theorem 2.2. *Let a and b be two positive numbers and $0 \leq v \leq 1$. Then for all positive integer m*

(1) *For $v \in [0, \frac{1}{2^n}]$, we have*

$$\left(a^v b^{1-v}\right)^m + \sum_{k=1}^n 2^{k-1} v^m \left(\sqrt{b^m} - \sqrt[2^k]{(ab^{2^{k-1}-1})^m}\right)^2 \leq \left(va + (1-v)b\right)^m. \quad (2.4)$$

(2) *For $v \in [\frac{2^n-1}{2^n}, 1]$, we have*

$$\left(a^v b^{1-v}\right)^m + \sum_{k=1}^n 2^{k-1} (1-v)^m \left(\sqrt{a^m} - \sqrt[2^k]{(ba^{2^{k-1}-1})^m}\right)^2 \leq \left(va + (1-v)b\right)^m. \quad (2.5)$$

Proof. Suppose that $0 \leq v \leq \frac{1}{2^n}$. We claim that

$$\left(va + (1-v)b\right)^m - \sum_{k=1}^n 2^{k-1} v^m \left(\sqrt{b^m} - \sqrt[2^k]{(ab^{2^{k-1}-1})^m}\right)^2 \geq \left(a^v b^{1-v}\right)^m.$$

We have, the following identities

$$\begin{aligned} & \left(va + (1-v)b\right)^m - \sum_{k=1}^n 2^{k-1} v^m \left(\sqrt{b^m} - \sqrt[2^k]{(ab^{2^{k-1}-1})^m}\right)^2 \\ &= \sum_{k=0}^m \binom{m}{k} v^k (1-v)^{m-k} a^k b^{m-k} - \sum_{k=1}^n 2^{k-1} v^m \left(b^m + \sqrt[2^{k-1}]{(ab^{2^{k-1}-1})^m} - 2 \sqrt[2^k]{(ab^{2^k-1})^m}\right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m-1} \binom{m}{k} v^k (1-v)^{m-k} a^k b^{m-k} + \left((1-v)^m - \sum_{k=1}^n 2^{k-1} v^m \right) b^m + 2^n v^m \sqrt[n]{(ab^{2^n-1})^m} \\
&= \sum_{k=1}^{m-1} \binom{m}{k} v^k (1-v)^{m-k} a^k b^{m-k} + \left((1-v)^m - (2^n - 1)v^m \right) b^m + 2^n v^m \sqrt[n]{(ab^{2^n-1})^m} \\
&= \sum_{k=0}^m v_k x_k,
\end{aligned}$$

where x_k is given by:

$$x_0 := b^m, \quad \text{with } v_0 := \left((1-v)^m - (2^n - 1)v^m \right),$$

and for $1 \leq k \leq m-1$,

$$x_k := a^k b^{m-k}, \quad \text{with } v_k := \binom{m}{k} v^k (1-v)^{m-k}$$

and

$$x_m := \sqrt[n]{(ab^{2^n-1})^m} \quad \text{with } v_m := 2^n v^m.$$

We have

- (1) $x_k > 0$ for all $k \in \{0, 1, \dots, m\}$
- (2) $v_k \geq 0$ for all $k \in \{0, 1, \dots, m\}$, with $\sum_{k=0}^m v_k = 1$.

Hence by the arithmetic-geometric mean inequality,

$$\begin{aligned}
(va + (1-v)b)^m &= \sum_{k=1}^n 2^{k-1} v^m \left(\sqrt{b^m} - \sqrt[n]{(ab^{2^{k-1}-1})^m} \right)^2 \\
&\geq \prod_{k=0}^m x_k^{v_k} = a^{\alpha(m)} b^{\beta(m)},
\end{aligned}$$

where

$$\begin{aligned}
\alpha(m) &= \sum_{k=1}^{m-1} k \binom{m}{k} v^k (1-v)^{m-k} + 2^n v^m \frac{m}{2^n} \\
&= mv, \quad (\text{by Lemma 2.1})
\end{aligned}$$

and

$$\begin{aligned}
\beta(m) &= \sum_{k=1}^{m-1} (m-k) \binom{m}{k} v^k (1-v)^{m-k} + m \left((1-v)^m - (2^n - 1)v^m \right) + 2^n mv^m \frac{2^n - 1}{2^n} \\
&= m(1-v) \quad (\text{by Lemma 2.1}).
\end{aligned}$$

If $v \in [\frac{2^n-1}{2^n}, 1]$, then $1-v \in [0, \frac{1}{2^n}]$. So by changing two elements a, b and two weights $v, 1-v$ in inequality (2.4), the desired inequality is obtained.

To prove the second main result of this section we need the following lemma.

Lemma 2.2. *Let n be a positive integer such that $n \geq 3$, let ν be a positive number such that $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$. Then we have*

$$\left((1-\nu)^m - \nu^m - (1-2\nu)^m \right) \geq 0$$

and

$$\left(2\nu^m - (2^{n-1} - 1)(1-2\nu)^m \right) \geq 0.$$

Proof. we have

$$\begin{aligned} \left((1-\nu)^m - \nu^m - (1-2\nu)^m \right) &= (1-2\nu)((1-\nu)^{m-1} + \dots + \nu^{m-1}) - (1-2\nu)^m \\ &= (1-2\nu)\left((1-\nu)^{m-1} + \dots + \nu^{m-1} - (1-2\nu)^{m-1}\right) \\ &= (1-2\nu)\left((1-\nu)^{m-1} - (1-2\nu)^{m-1} + \dots + \nu^{m-1}\right) \\ &\geq 0. \end{aligned}$$

Putting $f(\nu) = 2\nu^m - (2^{n-1} - 1)(1-2\nu)^m$, then we have $f'(\nu) = 2m\nu^{m-1} + 2m(2^{n-1} - 1)(1-2\nu)^{m-1} \geq 0$. So $f(\nu)$ is increasing, then

$$f(\nu) \geq f\left(\frac{2^{n-1}-1}{2^n}\right) = \frac{\left(\frac{2^{n-1}-1}{2}\right)^m - \frac{2^{n-1}-1}{2}}{2^{nm-(m+1)}} \geq 0.$$

The second main result to be proved in this paper is the following theorem.

Theorem 2.3. *Let n be a positive integer such that $n \geq 3$, let a and b be two positive numbers and $0 \leq \nu \leq 1$. Then for all positive integer m*

(1) *For $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$, we have*

$$\begin{aligned} \left(a^\nu b^{1-\nu} \right)^m + \nu^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + (1-2\nu)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{b^m}{a^m}} - 1 \right)^2 \\ \leq \left(\nu a + (1-\nu)b \right)^m. \end{aligned} \quad (2.6)$$

(2) *For $\nu \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$, we have*

$$\begin{aligned} \left(a^\nu b^{1-\nu} \right)^m + (1-\nu)^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + (2\nu-1)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{a^m}{b^m}} - 1 \right)^2 \\ \leq \left(\nu a + (1-\nu)b \right)^m. \end{aligned} \quad (2.7)$$

Proof. Suppose that $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$. We claim that

$$\left(a^\nu b^{1-\nu} \right)^m + \nu^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + (1-2\nu)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{b^m}{a^m}} - 1 \right)^2$$

$$\leq \left(\nu a + (1 - \nu)b \right)^m.$$

We have, the following identities

$$\begin{aligned} & \left(\nu a + (1 - \nu)b \right)^m - \nu^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 - (1 - 2\nu)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{b^m}{a^m}} - 1 \right)^2 \\ &= \sum_{k=0}^m \binom{m}{k} \nu^k (1 - \nu)^{m-k} a^k b^{m-k} - \nu^m (a^m + b^m - 2(ab)^{\frac{m}{2}}) \\ & \quad - (1 - 2\nu)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^{k-1}]{\frac{b^m}{a^m}} - 2 \sqrt[2^k]{\frac{b^m}{a^m}} + 1 \right) \\ &= \sum_{k=1}^{m-1} \binom{m}{k} \nu^k (1 - \nu)^{m-k} a^k b^{m-k} + \left((1 - \nu)^m - \nu^m - (1 - 2\nu)^m \right) b^m \\ & \quad + \left(2\nu^m - (2^{n-1} - 1)(1 - 2\nu)^m \right) (ab)^{\frac{m}{2}} + 2^{n-1} (1 - 2\nu)^m a^{\frac{m}{2} - \frac{m}{2^n}} b^{\frac{m}{2} + \frac{m}{2^n}} \\ &= \sum_{k=0}^{m+1} \nu_k x_k, \end{aligned}$$

where x_k is given by:

$$x_0 := b^m, \quad \text{with } \nu_0 := \left((1 - \nu)^m - \nu^m - (1 - 2\nu)^m \right),$$

and for $1 \leq k \leq m - 1$,

$$x_k := a^k b^{m-k}, \quad \text{with } \nu_k := \binom{m}{k} \nu^k (1 - \nu)^{m-k},$$

$$x_m := (ab)^{\frac{m}{2}} \quad \text{with } \nu_m := \left(2\nu^m - (2^{n-1} - 1)(1 - 2\nu)^m \right)$$

and

$$x_{m+1} := a^{\frac{m}{2} - \frac{m}{2^n}} b^{\frac{m}{2} + \frac{m}{2^n}} \quad \text{with } \nu_{m+1} := 2^{n-1} (1 - 2\nu)^m.$$

We have

- (1) $x_k > 0$ for all $k \in \{0, 1, \dots, m + 1\}$
- (2) $\nu_k \geq 0$ for all $k \in \{0, 1, \dots, m + 1\}$, with $\sum_{k=0}^{m+1} \nu_k = 1$.

Hence by the arithmetic-geometric mean inequality,

$$\begin{aligned} & \left(\nu a + (1 - \nu)b \right)^m - \nu^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 - (1 - 2\nu)^m \sqrt{(ab)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{b^m}{a^m}} - 1 \right)^2 \\ & \geq \prod_{k=0}^{m+1} x_k^{\nu_k} = a^{\alpha(m)} b^{\beta(m)}, \end{aligned}$$

where

$$\begin{aligned}\alpha(m) &= \sum_{k=1}^{m-1} k \binom{m}{k} v^k (1-v)^{m-k} \\ &+ \frac{m}{2} (2v^m - (2^{n-1} - 1)(1-2v)^m) + (\frac{m}{2} - \frac{m}{2^n}) 2^{n-1} (1-2v)^m \\ &= mv, \text{ (by Lemma 2.1)}\end{aligned}$$

and

$$\begin{aligned}\beta(m) &= \sum_{k=1}^{m-1} (m-k) \binom{m}{k} v^k (1-v)^{m-k} + m((1-v)^m - v^m - (1-2v)^m) \\ &+ \frac{m}{2} (2v^m - (2^{n-1} - 1)(1-2v)^m) + (\frac{m}{2} + \frac{m}{2^n}) 2^{n-1} (1-2v)^m \\ &= m(1-v) \text{ (by Lemma 2.1).}\end{aligned}$$

If $v \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$, then $1-v \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$. So by changing two elements a, b and two weights $v, 1-v$ in inequality (2.6), the desired inequality is obtained.

Remark 2.1. If we set $m = 1$ in Theorem 2.3, then we recapture the Proposition 2.2 in [3].

3. Applications

In this section we provide some improvements to some inequalities due to J. Shao [10], for the traces, determinants, and p -norms of positive τ -measurable operators.

Let $\mathcal{M} \subset B(\mathcal{H})$ be a finite von Neumann algebra on the separable Hilbert space $\mathcal{H}$, namely, $\mathcal{M}$ is a $*$ -subalgebra of $B(\mathcal{H})$ containing the identity 1, which is closed for the weak operator topology. A trace τ on the von Neumann algebra $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \mapsto [0, +\infty)$ which is additive, positively homogeneous and unitarily invariant, that is, $\tau(x) = \tau(u^*xu)$ for all $x \in \mathcal{M}^+$ and unitary $u \in \mathcal{M}$, where $\mathcal{M}^+ = \{x \in \mathcal{M}, x \geq 0\}$. A trace τ is called

- (1) faithful is for all $x \in \mathcal{M}^+$, $\tau(x) = 0$ implies that $x = 0$,
- (2) semi-finite is for every $x \in \mathcal{M}^+$, with $\tau(x) > 0$, there exists $0 \leq y \leq x$, such that $0 < \tau(y) < \infty$,
- (3) normal if $x_i \uparrow_i x \in \mathcal{M}^+$, implies that $\tau(x_i) \uparrow_i \tau(x)$.

A trace is called finite if $\tau(1) < \infty$.

For $0 < p < +\infty$, $L_p(\mathcal{M}, \tau)$ is defined as the set of all τ -mesurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < +\infty.$$

$L_p(\mathcal{M}, \tau)$ is a Banach space under $\|\cdot\|_p$ for $1 \leq p < +\infty$, see [11] for more information.

Definition 3.1 ([6]). Let $\mathcal{M}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, with a normal faithful finite trace τ . For $x \in \mathcal{M}$, we define the determinant of x by $\Delta_\tau(x) = \exp \tau(\log |x|)$ if $|x|$ is invertible, and otherwise we define $\Delta_\tau(x) = \inf \Delta_\tau(|x| + \varepsilon 1)$, the infimum takes over all scalars $\varepsilon > 0$.

Next we state for easy reference the following properties of τ -mesurable operators (see [4],[5]) which we shall need later

- (1) $\Delta_\tau(\lambda x) = |\lambda| \Delta_\tau(x), \quad \forall \lambda \in \mathbb{R},$
- (2) $\Delta_\tau(1) = 1, \quad \Delta_\tau(xy) = \Delta_\tau(x) \Delta_\tau(y),$
- (3) $\Delta_\tau(x) = \Delta_\tau(x^*) = \Delta_\tau(|x|), \quad \Delta_\tau(|x|^\alpha) = \Delta_\tau(|x|)^\alpha, \quad \alpha \in \mathbb{R}^+,$
- (4) $\Delta_\tau(x^{-1}) = (\Delta_\tau(x))^{-1},$ if x is invertible in $\mathcal{M},$
- (5) $\Delta_\tau(x) \leq \Delta_\tau(y),$ if $0 \leq x \leq y,$
- (6) $\lim_{\varepsilon \rightarrow 0^+} \Delta_\tau(x + \varepsilon 1) = \Delta_\tau(x),$ if $0 \leq x.$

The version Young's inequalities for the trace, determinants and p -norm, states as follows: for any $x, y, z \in \mathcal{M}^+$ and positive integer $m,$ we have

$$\left((\tau(x^\nu y^{1-\nu})) \right)^m \leq \left(\tau(\nu x + (1-\nu)y) \right)^m, \quad (3.1)$$

$$\left(\Delta_\tau(x^\nu y^{1-\nu}) \right)^m \leq \left(\Delta_\tau(\nu x + (1-\nu)y) \right)^m, \quad (3.2)$$

$$\|x^\nu z y^{1-\nu}\|_p^m \leq \left[\nu \|xz\|_p + (1-\nu) \|zy\|_p \right]^m. \quad (3.3)$$

By using inequality (1.4), J. Shao [10] proved the next inequalities:

$$\left(\Delta_\tau(x^\nu y^{1-\nu}) \right)^m + r_0^m \left(\left((\Delta_\tau(x))^{\frac{m}{2}} - (\Delta_\tau(y))^{\frac{m}{2}} \right)^2 \right) \leq \left(\Delta_\tau(\nu x + (1-\nu)y) \right)^m,$$

$$\left((\tau(x^\nu y^{1-\nu})) \right)^m + r_0^m \left(\left((\tau(x))^{\frac{m}{2}} - (\tau(y))^{\frac{m}{2}} \right)^2 \right) \leq \left(\tau(\nu x + (1-\nu)y) \right)^m,$$

$$\|x^\nu z y^{1-\nu}\|_p^m + r_0^m \left(\left(\|xz\|_p^{\frac{m}{2}} - \|zy\|_p^{\frac{m}{2}} \right)^2 \right) \leq \left[\nu \|xz\|_p + (1-\nu) \|zy\|_p \right]^m.$$

By using Theorems 2.2 and 2.3 we obtain a new refinements of the inequalities (3.1), (3.2) and (3.3).

3.1. Refinement of Young's type inequality for determinants. Before giving our result, we need the following lemma.

Lemma 3.1 ([6]). *Let $x, y \in \mathcal{M}^+.$ Then we have*

$$\Delta_\tau(x) + \Delta_\tau(y) \leq \Delta_\tau(x + y).$$

The first result of this subsection concerns the determinants of τ -mesurable operators which can be reads as follows:

Theorem 3.1. *Let $x, y \in \mathcal{M}^+,$ and $0 \leq \nu \leq 1.$ Then for all positive integer m*

(1) For $\nu \in [0, \frac{1}{2^n}]$, we have

$$\begin{aligned} \left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m &+ \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\Delta_\tau(y)^m} - \sqrt[2^k]{(\Delta_\tau(x)\Delta_\tau(y)^{2^{k-1}-1})^m}\right)^2 \\ &\leq \Delta_\tau(\nu x + (1-\nu)y)^m. \end{aligned} \quad (3.4)$$

(2) For $\nu \in [\frac{2^n-1}{2^n}, 1]$, we have

$$\begin{aligned} \left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m &+ \sum_{k=1}^n 2^{k-1} (1-\nu)^m \left(\sqrt{\Delta_\tau(x)^m} - \sqrt[2^k]{(\Delta_\tau(y)\Delta_\tau(x)^{2^{k-1}-1})^m}\right)^2 \\ &\leq \Delta_\tau(\nu x + (1-\nu)y)^m. \end{aligned} \quad (3.5)$$

Proof. Applying Theorem 2.2 and Lemma 3.1, we have

$$\begin{aligned} \Delta_\tau(\nu x + (1-\nu)y)^m &\geq \left[\nu \Delta_\tau(x) + (1-\nu) \Delta_\tau(y)\right]^m \text{ (by Lemma 3.1)} \\ &\geq \left[\left(\Delta_\tau(x)\right)^\nu \left(\Delta_\tau(y)\right)^{1-\nu}\right]^m \\ &+ \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\Delta_\tau(y)^m} - \sqrt[2^k]{(\Delta_\tau(x)\Delta_\tau(y)^{2^{k-1}-1})^m}\right)^2 \\ &\quad \text{(by Theorem 2.2)} \\ &= \left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m \\ &+ \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\Delta_\tau(y)^m} - \sqrt[2^k]{(\Delta_\tau(x)\Delta_\tau(y)^{2^{k-1}-1})^m}\right)^2. \end{aligned}$$

By the same process we can show the inequality (3.5).

The second result of this subsection concerns the determinants of τ -mesurable operators which can be reads as follows:

Theorem 3.2. Let n be a positive integer such that $n \geq 3$, let $x, y \in \mathcal{M}^+$, and $0 \leq \nu \leq 1$. Then for all positive integer m

(1) For $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$, we have

$$\begin{aligned} \left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m &+ \nu^m \left(\Delta_\tau(x)^{\frac{m}{2}} - \Delta_\tau(y)^{\frac{m}{2}}\right)^2 \\ &+ (1-2\nu)^m \sqrt{(\Delta_\tau(xy))^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\Delta_\tau(y)^m}{\Delta_\tau(x)^m}} - 1\right)^2 \\ &\leq \Delta_\tau(\nu x + (1-\nu)y)^m. \end{aligned} \quad (3.6)$$

(2) For $\nu \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$, we have

$$\begin{aligned} & \left(\Delta_\tau(x^\nu y^{1-\nu}) \right)^m + (1-\nu)^m \left(\Delta_\tau(x)^{\frac{m}{2}} - \Delta_\tau(y)^{\frac{m}{2}} \right)^2 \\ & + (2\nu-1)^m \sqrt{(\Delta_\tau(xy))^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\Delta_\tau(x)^m}{\Delta_\tau(y)^m}} - 1 \right)^2 \\ & \leq \Delta_\tau(\nu x + (1-\nu)y)^m. \end{aligned} \quad (3.7)$$

3.2. Refinement of Young's type inequality for p -norms. Before giving our result, we need the following lemma.

Lemma 3.2 ([12]). Let $x, y \in L_p(\mathcal{M}, \tau)$ be a positive operators, where $1 \leq p < +\infty$, $z \in \mathcal{M}$, and $0 \leq \nu \leq 1$. Then we have

$$\|x^\nu z y^{1-\nu}\|_p \leq \|xz\|_p^\nu \|zy\|_p^{1-\nu}.$$

In particular,

$$\tau(x^\nu y^{1-\nu}) \leq \tau(x)^\nu \tau(y)^{1-\nu}.$$

The first result of this subsection concerns the p -norms of τ -mesurable operators which can be reads as follows:

Theorem 3.3. Let $x, y \in L_p(\mathcal{M}, \tau)$ be a positive operators, where $1 \leq p < +\infty$, $z \in \mathcal{M}$, and $0 \leq \nu \leq 1$. Then for all positive integer m

(1) For $\nu \in [0, \frac{1}{2^n}]$, we have

$$\begin{aligned} & \|x^\nu z y^{1-\nu}\|_p^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt[2^k]{\|zy\|_p^m} - \sqrt[2^k]{(\|xz\|_p \|zy\|_p^{2^{k-1}-1})^m} \right)^2 \\ & \leq \left[\nu \|xz\|_p + (1-\nu) \|zy\|_p \right]^m. \end{aligned} \quad (3.8)$$

(2) For $\nu \in [\frac{2^{n-1}-1}{2^n}, 1]$, we have

$$\begin{aligned} & \|x^\nu z y^{1-\nu}\|_p^m + \sum_{k=1}^n 2^{k-1} (1-\nu)^m \left(\sqrt[2^k]{\|xz\|_p^m} - \sqrt[2^k]{(\|zy\|_p \|xz\|_p^{2^{k-1}-1})^m} \right)^2 \\ & \leq \left[\nu \|xz\|_p + (1-\nu) \|zy\|_p \right]^m. \end{aligned} \quad (3.9)$$

Proof. Applying Theorem 2.2 and Lemma 3.2, we have

$$\begin{aligned} & \|x^\nu z y^{1-\nu}\|_p^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt[2^k]{\|zy\|_p^m} - \sqrt[2^k]{(\|xz\|_p \|zy\|_p^{2^{k-1}-1})^m} \right)^2 \\ & \leq \left[\|xz\|_p^\nu \|zy\|_p^{1-\nu} \right]^m \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\|zy\|_p^m} - \sqrt[2^k]{(\|xz\|_p \|zy\|_p^{2^{k-1}-1})^m} \right)^2 \quad (\text{by Lemma 3.2}) \\
& \leq \left[\nu \|xz\|_p + (1-\nu) \|zy\|_p \right]^m \quad (\text{by Theorem 2.2}).
\end{aligned}$$

By the same process we can show the inequality (3.9).

The second result of this subsection concerns the p -norms of τ -mesurable operators which can be reads as follows:

Theorem 3.4. *Let n be a positive integer such that $n \geq 3$, let $x, y \in L_p(\mathcal{M}, \tau)$ be a positive operators, where $1 \leq p < +\infty$, $z \in \mathcal{M}$, and $0 \leq \nu \leq 1$. Then for all positive integer m*

(1) *For $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$, we have*

$$\begin{aligned}
& \|x^\nu zy^{1-\nu}\|_p^m + \nu^m \left(\|xz\|_p^{\frac{m}{2}} - \|zy\|_p^{\frac{m}{2}} \right)^2 \\
& + (1-2\nu)^m \sqrt{(\|xz\|_p \|zy\|_p)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\|zy\|_p^m}{\|xz\|_p^m}} - 1 \right)^2 \\
& \leq \left(\nu \|xz\|_p + (1-\nu) \|zy\|_p \right)^m. \tag{3.10}
\end{aligned}$$

(2) *For $\nu \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$, we have*

$$\begin{aligned}
& \|x^\nu zy^{1-\nu}\|_p^m + (1-\nu)^m \left(\|xz\|_p^{\frac{m}{2}} - \|zy\|_p^{\frac{m}{2}} \right)^2 \\
& + (2\nu-1)^m \sqrt{(\|xz\|_p \|zy\|_p)^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\|xz\|_p^m}{\|zy\|_p^m}} - 1 \right)^2 \\
& \leq \left(\nu \|xz\|_p + (1-\nu) \|zy\|_p \right)^m. \tag{3.11}
\end{aligned}$$

3.3. Refinement of Young's type inequality for traces. We end this paper by giving an inequality for traces of positive τ -mesurable operators. The first result of this subsection concerns the trace of τ -mesurable operators which can be reads as follows:

Theorem 3.5. *Let $x, y \in \mathcal{M}^+$, and $0 \leq \nu \leq 1$. Then for all positive integer m*

(1) *For $\nu \in [0, \frac{1}{2^n}]$, we have*

$$\begin{aligned}
& \left(\tau(x^\nu y^{1-\nu}) \right)^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\tau(y)^m} - \sqrt[2^k]{(\tau(x)\tau(y)^{2^{k-1}-1})^m} \right)^2 \\
& \leq \left[\tau(\nu x + (1-\nu)y) \right]^m. \tag{3.12}
\end{aligned}$$

(2) *For $\nu \in [\frac{2^n-1}{2^n}, 1]$, we have*

$$\left(\tau(x^\nu y^{1-\nu}) \right)^m + \sum_{k=1}^n 2^{k-1} (1-\nu)^m \left(\sqrt{\tau(x)^m} - \sqrt[2^k]{(\tau(y)\tau(x)^{2^{k-1}-1})^m} \right)^2$$

$$\leq \left[\tau(\nu x + (1 - \nu)y) \right]^m. \quad (3.13)$$

Proof. Applying Theorem 2.2 and Lemma 3.2, we have

$$\begin{aligned} & \left(\tau(x^\nu y^{1-\nu}) \right)^m + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\tau(y)^m} - \sqrt[2^k]{(\tau(x)\tau(y)^{2^{k-1}-1})^m} \right)^2 \\ & \leq \left[(\tau(x))^\nu (\tau(y))^{1-\nu} \right]^m \\ & + \sum_{k=1}^n 2^{k-1} \nu^m \left(\sqrt{\tau(y)^m} - \sqrt[2^k]{(\tau(x)\tau(y)^{2^{k-1}-1})^m} \right)^2 \text{ (by Lemma 3.2)} \\ & \leq \left[\tau(\nu x + (1 - \nu)y) \right]^m \text{ (by Theorem 2.2).} \end{aligned}$$

By the same process we can show the inequality (3.13).

The second result of this subsection concerns the trace of τ -mesurable operators which can be reads as follows:

Theorem 3.6. *Let n be a positive integer such that $n \geq 3$, let $x, y \in \mathcal{M}^+$, and $0 \leq \nu \leq 1$. Then for all positive integer m*

(1) *For $\nu \in [\frac{2^{n-1}-1}{2^n}, \frac{1}{2}]$, we have*

$$\begin{aligned} & \left(\tau(x^\nu y^{1-\nu}) \right)^m + \nu^m \left(\tau(x)^{\frac{m}{2}} - \tau(y)^{\frac{m}{2}} \right)^2 \\ & + (1 - 2\nu)^m \sqrt{(\tau(x)\tau(y))^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\tau(y)^m}{\tau(x)^m}} - 1 \right)^2 \\ & \leq \tau(\nu x + (1 - \nu)y)^m. \end{aligned} \quad (3.14)$$

(2) *For $\nu \in [\frac{1}{2}, \frac{2^{n-1}+1}{2^n}]$, we have*

$$\begin{aligned} & \left(\tau(x^\nu y^{1-\nu}) \right)^m + (1 - \nu)^m \left(\tau(x)^{\frac{m}{2}} - \tau(y)^{\frac{m}{2}} \right)^2 \\ & + (2\nu - 1)^m \sqrt{(\tau(x)\tau(y))^m} \sum_{k=2}^n 2^{k-2} \left(\sqrt[2^k]{\frac{\tau(x)^m}{\tau(y)^m}} - 1 \right)^2 \\ & \leq \tau(\nu x + (1 - \nu)y)^m. \end{aligned} \quad (3.15)$$

Acknowledgements. The authors would like to express their deep thanks to the anonymous referees for their helpful comments and suggestions on the initial version of the manuscript which lead to the improvement of this paper.

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