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Entropy and renormalized solutions for some nonlinear anisotropic elliptic equations with variable exponents and L^1 -data

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ABSTRACT. We prove in this paper some existence and unicity results of entropy and renormalized solutions for some nonlinear elliptic equations with general anisotropic diffusivities and variable exponents. The data are assumed to be merely integrable.

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1. Introduction

This paper is motivated by the study whose model, including the stationary equation, is

$$\begin{cases} -\sum_{l=1}^N \frac{\partial}{\partial x_l} \left(\left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u}{\partial x_l} \right) + |u|^{s(x)-1} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

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where $N \geq 2$, the right-hand side $f \in L^1(\Omega)$ and $p_l : \bar{\Omega} \rightarrow (1, +\infty)$ is a continuous function for $l = 1, \dots, N$.

For (1.1) we assume that the exponents functions $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as follows:

$$\begin{cases} \bar{p}(\cdot) < N, & \frac{1}{\bar{p}(\cdot)} = \frac{1}{N} \sum_{l=1}^N \frac{1}{p_l(\cdot)}, \\ p_l(\cdot) > 1 \quad \text{and} \quad \frac{\bar{p}(\cdot)(N-1)}{N(\bar{p}(\cdot)-1)} < p_l(\cdot) < \frac{\bar{p}(\cdot)(N-1)}{N-\bar{p}(\cdot)}, & l = 1, \dots, N, \\ s(\cdot) > \max_{x \in \bar{\Omega}} p_l(x), & l = 1, \dots, N. \end{cases} \quad (1.2)$$

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field. We use the framework of renormalized solutions. This notion was introduced by DiPerna and Lions [28] in their study of the Boltzmann equation, Boccardo et al [23] and Lions and Murat (see Lions book on the Navier-Stokes equations [37]). For the corresponding isotropic parabolic equations with L^1 -data, existence and uniqueness of renormalized solutions is established in Blanchard and Murat [22], see also Rakotoson [38, 39] and Lions [37] for some time dependent problems motivated by the Navier-Stokes equations. Finally, let us mention that an equivalent notion of solutions, called entropy solutions, was introduced independently by Benilan et al. [15], see also [11], several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [42, 32, 43, 6, 8, 19, 1, 7, 5, 2, 16, 9, 17, 20, 3, 10, 4, 18, 30, 31, 29, 25, 21]).

In the isotropic case with $p_l(\cdot) = p(\cdot) > 2 - \frac{1}{N}$ for $l = 1, \dots, N$, existence and regularity results are proved in [14, 40]. The existence and regularity results for distributional solutions with L^1 -data have been obtained in [24, 36] for a class of anisotropic elliptic and parabolic equations with constant variable components. For an anisotropic parabolic reaction-diffusion-advection system with a zero-flux boundary condition, similar results are established in [13]. Observe that (1.2) implies $\bar{p}(\cdot) > 2 - \frac{1}{N}$ and thus $\frac{N(\bar{p}(\cdot)-1)}{\bar{p}(\cdot)(N-1)} p_l(\cdot) > 1$. This is in accordance with the isotropic case [14]. On the other hand, the condition $s(\cdot) > p_l(\cdot)$ in (1.2) is stronger than in [14]. This is a consequence of the anisotropic Sobolev inequality [41] (applied to our anisotropic equation)

The scope of the present paper is to solve the well-posedness (existence and uniqueness) and regularity of entropy and renormalized solutions in an appropriate function space for nonlinear anisotropic elliptic equations with variable exponents. There are two difficulties associated with this strategy. In view of the assumption that f is only integrable on Ω , the first difficulty is to obtain suitable a priori estimates on u_ε and the partial derivatives $\frac{\partial u_\varepsilon}{\partial x_l}$, $l = 1, \dots, N$, that are independent of ε . The second difficulty lies in passing to the limit in the nonlinear anisotropic vector field (with variable exponents) $A(x, \nabla u_\varepsilon)$ and the nonlinear term $|u_\varepsilon|^{s(\cdot)-1} u_\varepsilon$.

The paper is structured as follows: In Section 2 we recall some basic notations and a Sobolev inequality for anisotropic Sobolev spaces. Our main results are stated in Section 3, while the

proofs are given in Section 4. In Section 5 we show the existing uniqueness renormalized solutions for elliptic equations. Finally, we prove equivalence between entropy and renormalized solutions to elliptic equations in Section 6.

2. Sobolev spaces with variable exponents

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to [34] and [33] for further properties of variable exponent Lebesgue-Sobolev spaces.

The continuous real-valued function $p : \overline{\Omega} \rightarrow [1, +\infty)$ satisfies the log-continuity if

$$\forall x_1, x_2 \in \overline{\Omega}, \quad |x_1 - x_2| < 1, \quad |p(x_1) - p(x_2)| < w(|x_1 - x_2|), \quad (2.1)$$

where $\limsup_{\alpha \rightarrow 0^+} w(\alpha) \ln(\frac{1}{\alpha}) < +\infty$. Let $p^- = \min_{x \in \overline{\Omega}} p(x)$ and $p^+ = \max_{x \in \overline{\Omega}} p(x)$. We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R}; u \text{ is measurable with } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

The following inequality will be used later

$$\min\{\|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}\} \leq \int_{\Omega} |u(x)|^{p(x)} dx \leq \max\{\|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}\}. \quad (2.2)$$

If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\int_{\Omega} |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \quad (2.3)$$

holds true.

The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(\cdot) \leq r_2(\cdot)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

In this paper we will most exclusively work with Lebesgue and Lebesgue-Sobolev spaces with continuous variable exponents.

Now, we define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We define also $W_0^{1,p(\cdot)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Assuming $p^- > 1$ the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces. The space $(W_0^{1,p(\cdot)}(\Omega))^*$ denotes the dual of $W_0^{1,p(\cdot)}(\Omega)$.

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $p^- \geq 1$, the Poincaré inequality holds (cf. [35])

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \quad (2.4)$$

for some universal constant C which depends on Ω and the function p .

For $p \in C(\overline{\Omega}, \mathbb{R}^+)$ with $1 < p^- \leq p^+ < N$ the Sobolev embedding holds (see e.g. [33])

$$W^{1,p(\cdot)}(\Omega) \subset L^{r(\cdot)}(\Omega), \quad (2.5)$$

for any measurable function $r : \Omega \rightarrow [1, +\infty)$ such that

$$\operatorname{ess\,inf}_{x \in \Omega} \left(\frac{Np(x)}{N - p(x)} - r(x) \right) > 0.$$

Next, we recall the notion of anisotropic Sobolev spaces. Let $p_1, \dots, p_N$ be N real numbers with $p_l \geq 1$, $l = 1, \dots, N$. With a slight abuse of the notation, we introduce the anisotropic Sobolev space

$$W^{1,p_l(\cdot)}(\Omega) = \left\{ u \in L^{p_l(\cdot)}(\Omega) : \frac{\partial u}{\partial x_l} \in L^{p_l(\cdot)}(\Omega) \right\},$$

and

$$W_0^{1,p_l(\cdot)}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_l} \in L^{p_l(\cdot)}(\Omega) \right\}.$$

which are Banach spaces under the norm

$$\|u\| = \|u\|_{L^{p_l(\cdot)}(\Omega)} + \left\| \frac{\partial u}{\partial x_l} \right\|_{L^{p_l(\cdot)}(\Omega)},$$

for $l = 1, \dots, N$.

Let us recall the anisotropic Sobolev imbedding theorems (with constant variable exponents) due to Troisi [41].

Theorem 2.1. Suppose $g \in \bigcap_{l=1}^N W_0^{1,p_l}(\Omega)$, and let

$$\begin{cases} q = \bar{p}^*, & \text{if } \bar{p}^* < N, \\ q \in [1, \infty), & \text{if } \bar{p}^* \geq N. \end{cases}$$

Then there exists a constant C , depending on $N, p_1, \dots, p_N$ if $\bar{p} < N$ and also on q and $|\Omega|$ if $\bar{p} \geq N$, such that

$$\|g\|_{L^q(\Omega)} \leq C \prod_{l=1}^N \left\| \frac{\partial g}{\partial x_l} \right\|_{L^{p_l}(\Omega)}^{1/N}. \quad (2.6)$$

Theorem 2.2. Let Q be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes. Suppose $u \in \bigcap_{l=1}^N W^{1,p_l}(Q)$, and set

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{l=1}^N \frac{1}{p_l}, \quad r = \begin{cases} \bar{p}^* := \frac{N\bar{p}}{N-\bar{p}}, & \text{if } \bar{p}^* < N, \\ \text{any number from } [1, \infty), & \text{if } \bar{p}^* \geq N. \end{cases}$$

Then there exists a constant C , depending on $N, p_1, \dots, p_N$ if $\bar{p} < N$ and also on r and $\text{meas}(Q)$ if $\bar{p} \geq N$, such that

$$\|u\|_{L^r(Q)} \leq C \prod_{l=1}^N \left[\left\| \frac{\partial u}{\partial x_l} \right\|_{L^{p_l}(Q)} + \|u\|_{L^{p_l}(Q)} \right]^{\frac{1}{N}}. \quad (2.7)$$

Theorems 2.1 and 2.2 will be used to prove the “interpolation” lemma below, which is a technical result we will use later to obtain a priori estimates.

3. Statements of results

Instead of (1.1) we will consider more general nonlinear anisotropic elliptic equations with variable exponents of the form

$$-\text{div} A(x, \nabla u) + g(x, u) = f(x) \quad \text{in } \Omega. \quad (3.1)$$

The vector field $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ has components $a_l : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$, $l = 1, \dots, N$, and we assume that there exist two constants C_A and C'_A such that for all $\xi_1, \xi_2 \in \mathbb{R}^N$ and for a.e. x

$$A(x, \xi) \cdot \xi \geq C_A \sum_{l=1}^N |\xi_l|^{p_l(x)}, \quad (3.2)$$

$$|a_l(x, \xi)| \leq C'_A \left(1 + \sum_{l=1}^N |\xi_l|^{p_l(x)-1} \right), \quad l = 1, \dots, N, \quad (3.3)$$

$$[A(x, \xi_1) - A(x, \xi_2)] [\xi_1 - \xi_2] > 0, \quad \xi_1 \neq \xi_2. \quad (3.4)$$

The nonlinear function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be measurable in $x \in \Omega$ for all $\sigma \in \mathbb{R}$ and continuous in $\sigma \in \mathbb{R}$ for a.e. $x \in \Omega$. Furthermore,

$$g(x, \sigma)\sigma \geq 0, \quad \text{for all } \sigma \in \mathbb{R} \text{ and a.e. } x \in \Omega, \quad (3.5)$$

$$\sup \{ |g(x, \sigma)| : |\sigma| \leq \tau \} \in L^1(\Omega), \quad \forall \tau \in \mathbb{R}. \quad (3.6)$$

Finally, there should exist $s > p_l$, $l = 1, \dots, N$, such that

$$g(x, \sigma)\text{sign}(\sigma) \geq |\sigma|^{s(x)}, \quad \text{for all } \sigma \in \mathbb{R} \text{ and a.e. } x \in \Omega. \quad (3.7)$$

For any given $\gamma > 0$, we define the truncation function $T_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_\gamma(z) = \begin{cases} -\gamma, & \text{if } z \leq -\gamma, \\ z, & \text{if } |z| < \gamma, \\ \gamma, & \text{if } z \geq \gamma. \end{cases}$$

Moreover, we will need the following associated function (renormalization)

$$\phi_\gamma(r) = T_{\gamma+1}(r) - T_\gamma(r),$$

for any $\gamma > 0$. Notice that T_γ and ϕ_γ are Lipschitz functions satisfying $|T_\gamma(r)| \leq \gamma$ and $|\phi_\gamma(r)| \leq 1$.

Pick any positive $C^\infty(\mathbb{R})$ function $s(\cdot)$ such that $s(z) = 1$ if $|z| \leq 1$, $s(z) = 0$ if $|z| \geq 2$, and $0 \leq s(z) \leq 1$ for all $z \in \mathbb{R}$. For any $n \geq 2$, define the function $S_n(r)$ by $S_n(r) = \int_0^r s_n(z) dz$, where

$$s_n(z) = \begin{cases} 1, & \text{if } |z| \leq n-1, \\ s(z - (n-1)\text{sign}(z)), & \text{if } |z| \geq n-1, \end{cases}$$

where $\text{sign}(z)$ denotes the sign of z . For each integer $n \geq 2$, the function S_n satisfies

$$\begin{cases} S_n(r) = S_n(T_{n+1}(r)), & \|S'_n\|_{L^\infty(\mathbb{R})} \leq \|s\|_{L^\infty(\mathbb{R})}, \\ \text{supp } S'_n \subset [-(n+1), n+1], & \text{supp } S''_n \subset [-(n+1), -n] \cup [n, n+1]. \end{cases}$$

We shall use the following definitions of entropy solutions and renormalized solutions for problem (3.1) in the following sense:

Definition 3.1. An entropy solution of (3.1) is a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such

$$u \in W_0^{1,1}(\Omega) \cap L^{s(\cdot)}(\Omega), \quad A(x, \nabla u) \in \left(L^1(\Omega)\right)^N,$$

and $\forall \varphi \in C_c^1(\Omega)$

$$\int_\Omega A(x, \nabla u) \cdot \nabla T_\gamma(u - \varphi) dx + \int_\Omega g(x, u) T_\gamma(u - \varphi) dx = \int_\Omega f T_\gamma(u - \varphi) dx. \quad (3.8)$$

Definition 3.2. A renormalized solution of (3.1) is a function $u \in L^1(\Omega)$, satisfying the following conditions:

$$T_\gamma(u) \in W_0^{1,p(\cdot)}(\Omega) \text{ for any } \gamma > 0, \quad (3.9)$$

and

$$\lim_{\gamma \rightarrow \infty} \sum_{l=1}^N \int_{\{\gamma \leq |u| \leq \gamma+1\}} \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)} dx = 0. \quad (3.10)$$

For any renormalization $S \in C^\infty(\mathbb{R})$ such that $\text{supp } S' \subset [-M, M]$ for some $M > 0$,

$$-\sum_{l=1}^N \frac{\partial}{\partial x_l} \left(S'(u) \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u}{\partial x_l} \right) + \sum_{l=1}^N S''(u) \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)} + g(x, u) S'(u) = f S'(u) \quad \text{in } \mathcal{D}'(\Omega). \quad (3.11)$$

Our main results are collected in the following theorem:

Theorem 3.1. Assume (3.2)-(3.7) hold and the corresponding exponents functions $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as in (1.2). Let $f \in L^1(\Omega)$. Then (3.1) has at least one entropy and renormalized solution u .

Remark 3.1. The notion of entropy and renormalized solution for problem (3.1) are equivalent.

4. Proof of Theorem 3.1

Let $(f_\varepsilon)_{0 < \varepsilon < 1} \subset C_c^\infty(\Omega)$ be a sequence of smooth approximations of f such that

$$\begin{cases} |f_\varepsilon| \leq \frac{1}{\varepsilon} & \text{and} & |f_\varepsilon| \leq |f|; \\ f_\varepsilon \rightarrow f & \text{in } L^1(\Omega) & \text{as } \varepsilon \rightarrow 0. \end{cases} \quad (4.1)$$

Then the result in [14], provide us with the existence of a sequence of functions

$$(u_\varepsilon)_{0 < \varepsilon \leq 1} \subset \bigcap_{l=1}^N W_0^{1,p_l(\cdot)}(\Omega) \cap L^{s(\cdot)}(\Omega),$$

each of them satisfying the weak formulation

$$\int_{\Omega} A(x, \nabla u_\varepsilon) \cdot \nabla \varphi \, dx + \int_{\Omega} g(x, u_\varepsilon) \varphi = \int_{\Omega} f_\varepsilon \varphi \, dx, \quad (4.2)$$

for all $\varphi \in \bigcap_{l=1}^N W_0^{1,p_l(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We prove Theorem 3.1 in three main steps. We prove first ε -uniform a priori estimates for u_ε , so we obtain a.e. convergence of u_ε . In the second step, we prove strong L^1 convergence of the nonlinear term g in (4.2). Finally, we complete the proof of Theorem 3.1 by passing to the limit in (4.1) as $\varepsilon \rightarrow 0$.

In the remaining part of this paper, we use C, C_1, C_2 , etc. to denote constants that are independent of ε .

4.1. A priori estimates.

Proposition 4.1. *Assume (3.2)-(3.7) hold, and that the exponents $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as in (1.2). Then, there exist a constant C , not depending on ε , such that*

$$\sum_{l=1}^N \int_{\{|u_\varepsilon| \leq \gamma\}} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)} dx \leq C\gamma, \quad \forall \gamma > 0, \quad (4.3)$$

$$\|g(x, u_\varepsilon)\|_{L^1(\Omega)} \leq C, \quad (4.4)$$

and

$$\sum_{l=1}^N \int_{\Omega} \frac{\left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)}}{(1 + |u_\varepsilon|)^\gamma} dx \leq C, \quad \forall \gamma > 1. \quad (4.5)$$

Proof. Inserting $\varphi = T_\gamma(u_\varepsilon)$ into (4.2) gives

$$\int_{\Omega} A(x, \nabla u_\varepsilon) \cdot \nabla T_\gamma(u_\varepsilon) \, dx + \int_{\Omega} g(x, u_\varepsilon) T_\gamma(u_\varepsilon) \, dx = \int_{\Omega} f_\varepsilon T_\gamma(u_\varepsilon) \, dx.$$

From the definition of T_γ and g the second integral is nonnegative, and since $|f_\varepsilon| \leq |f|$, we get (4.3).

To prove (4.5), we introduce for $\gamma > 1$ the test function

$$\varphi_\gamma(\sigma) = \begin{cases} (\gamma - 1) \int_0^\sigma \frac{1}{(1+t)^\gamma} dt = 1 - \frac{1}{(1+\sigma)^{\gamma-1}}, & \sigma \geq 0, \\ -\varphi_\gamma(-\sigma), & \sigma < 0, \end{cases} \quad (4.6)$$

Substituting $\varphi = \varphi_\gamma(u_\varepsilon)$ in (4.2) gives

$$\int_\Omega A(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \varphi'_\gamma(u_\varepsilon) dx + \int_\Omega g(x, u_\varepsilon) \varphi_\gamma(u_\varepsilon) dx = \int_\Omega f_\varepsilon \varphi_\gamma(u_\varepsilon) dx. \quad (4.7)$$

Using the structure condition (3.2) in (4.7), we get

$$C_A \int_\Omega \sum_{l=1}^N \frac{\left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)}}{(1 + |u_\varepsilon|)^\gamma} dx \leq \|f\|_{L^1(\Omega)}. \quad (4.8)$$

This gives (4.5).

Next, we take $\varphi = \frac{1}{\gamma} T_\gamma(u_\varepsilon)$ in (4.2). After letting $\gamma \rightarrow \infty$ we find

$$\int_\Omega |g(x, u_\varepsilon)| dx \leq \|f\|_{L^1(\Omega)}, \quad (4.9)$$

which yields (4.4).

To continue we need the following lemma.

Lemma 4.1. *Let $p_l(\cdot)$ be a continuous function on $\overline{\Omega}$ for $l = 1, \dots, N$, and u_ε a function in $\bigcap_{l=1}^N W_0^{1, p_l(\cdot)}(\Omega)$. Suppose that the exponents $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as in (1.2), and that there exists a constant c such that*

$$\|u_\varepsilon\|_{L^{s(\cdot)}(\Omega)} \leq c, \quad (4.10)$$

and

$$\sum_{l=1}^N \int_\Omega \frac{\left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)}}{(1 + |u_\varepsilon|)^\gamma} dx \leq c. \quad (4.11)$$

Then for all continuous functions $q_l(\cdot)$ on $\overline{\Omega}$ such that

$$1 \leq q_l(x) < \frac{N(\overline{p}(x) - 1)}{\overline{p}(x)(N - 1)} p_l(x), \quad l = 1, \dots, N, \quad (4.12)$$

there exists a constant C , depending on Ω , N , $p_1(\cdot), \dots, p_N(\cdot)$, $q_1(\cdot), \dots, q_N(\cdot)$, and c , such that

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l(\cdot)}(\Omega)} \leq C, \quad l = 1, \dots, N, \quad (4.13)$$

and

$$\|u_\varepsilon\|_{L^{\overline{q}(\cdot)}(\Omega)} \leq C, \quad \frac{1}{\overline{q}(\cdot)} = \frac{1}{N} \sum_{l=1}^N \frac{1}{q_l(\cdot)}. \quad (4.14)$$

Proof. In a first step let q_l^+ be a constant satisfying (recall that $q_l^+ = \max_{x \in \bar{\Omega}} q_l(x)$ and $p_l^- = \min_{x \in \bar{\Omega}} p_l(x)$)

$$1 \leq q_l^+ < \frac{N(\bar{p}^- - 1)}{\bar{p}^-(N-1)} p_l^-. \quad (4.15)$$

Observe that (4.15) implies $q_l^+ < p_l^-$. Now let $q_l^+ = \kappa p_l^-$, $l = 1, \dots, N$, for any $\kappa \in \left(0, \frac{N(\bar{p}^- - 1)}{\bar{p}^-(N-1)}\right)$ where $\frac{1}{\bar{p}^-} = \frac{1}{N} \sum_{l=1}^N \frac{1}{p_l^-}$. Let $\gamma = \frac{1-\kappa}{\kappa} \bar{q}^{+\star}$, so that $\gamma \frac{q_l^+}{p_l^- - q_l^+} = \bar{q}^{+\star}$, where

$$\bar{q}^{+\star} = \frac{N\bar{q}^+}{N - \bar{q}^+} \text{ and } \frac{1}{\bar{q}^+} = \frac{1}{N} \sum_{l=1}^N \frac{1}{q_l^+}.$$

Recalling $\frac{p_l^- - q_l^+}{q_l^+} \bar{q}^{+\star} > 1$, we see that $\gamma > 1$. Using (4.11) and Hölder inequality, we get

$$\begin{aligned} \int_{\Omega} \left| \frac{\partial u_{\varepsilon}}{\partial x_l} \right|^{q_l^+} dx &\leq \left(\int_{\Omega} \frac{\left| \frac{\partial u_{\varepsilon}}{\partial x_l} \right|^{p_l^-}}{(1 + |u_{\varepsilon}|)^{\gamma}} dx \right)^{\frac{q_l^+}{p_l^-}} \left(\int_{\Omega} (1 + |u_{\varepsilon}|)^{\gamma \frac{q_l^+}{p_l^- - q_l^+}} dx \right)^{\frac{p_l^- - q_l^+}{p_l^-}} \\ &\leq C_1 \left(\int_{\Omega} (1 + |u_{\varepsilon}|)^{\gamma \frac{q_l^+}{p_l^- - q_l^+}} dx \right)^{\frac{p_l^- - q_l^+}{p_l^-}} \\ &\leq C_2 \left(\int_{\Omega} |u_{\varepsilon}|^{\bar{q}^{+\star}} dx \right)^{\frac{p_l^- - q_l^+}{p_l^-}} + C_3. \end{aligned} \quad (4.16)$$

Using (4.16) and the anisotropic Sobolev inequality (2.6), we deduce

$$\begin{aligned} \left(\int_{\Omega} |u_{\varepsilon}|^{\bar{q}^{+\star}} dx \right)^{\frac{1}{\bar{q}^{+\star}}} &\leq c_3 + c_4 \prod_{l=1}^N \left(\int_{\Omega} |u_{\varepsilon}|^{\bar{q}^{+\star}} dx \right)^{\frac{p_l^- - q_l^+}{q_l^+ p_l^- N}} \\ &= c_3 + c_4 \left(\int_{\Omega} |u_{\varepsilon}|^{\bar{q}^{+\star}} dx \right)^{\sum_{l=1}^N \frac{p_l^- - q_l^+}{q_l^+ p_l^- N}} \\ &= c_3 + c_4 \left(\int_{\Omega} |u_{\varepsilon}|^{\bar{q}^{+\star}} dx \right)^{\frac{1}{\bar{q}^+} - \frac{1}{\bar{p}^-}}. \end{aligned} \quad (4.17)$$

for some constants $c_3, c_4 > 0$. In other words,

$$\|u_{\varepsilon}\|_{L^{\bar{q}^{+\star}}(\Omega)} \leq c_5 + c_6 \|u_{\varepsilon}\|_{L^{\bar{q}^{+\star}}(\Omega)}^a, \quad a := \frac{\bar{p}^- - \bar{q}^+}{\bar{q}^+ \bar{p}^-} \bar{q}^{+\star}.$$

for some constants $c_5, c_6 > 0$. One checks easily that the assumption $\bar{p} < N$ implies $a < 1$, and we can therefore conclude that for all $1 \leq q_l^+ < \frac{N(\bar{p}-1)}{\bar{p}^+(N-1)} p_l^-$,

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l^+}(\Omega)} \leq C, \quad l = 1, \dots, N,$$

and

$$\|u_\varepsilon\|_{L^{\bar{q}^+(\cdot)}(\Omega)} \leq C, \quad \frac{1}{\bar{q}^+(\cdot)} = \frac{1}{N} \sum_{l=1}^N \frac{1}{q_l^+(\cdot)}.$$

for some constant $C > 0$.

Now let us consider a continuous variable exponent q_l on $\bar{\Omega}$ satisfying the pointwise estimate (4.12) for $l = 1, \dots, N$. By the continuity of $p_l(\cdot)$ and $q_l(\cdot)$ on $\bar{\Omega}$ there exists a constant $\delta > 0$ such that

$$\max_{y \in Q(x, \delta) \cap \bar{\Omega}} q(y) < \min_{y \in Q(x, \delta) \cap \bar{\Omega}} \frac{N(\bar{p}(y) - 1)}{\bar{p}(y)(N - 1)} p_l(y), \quad l = 1, \dots, N. \quad (4.18)$$

Herein, $Q(x, \delta)$ is a cube with center x and diameter δ . Observe that $\bar{\Omega}$ is compact and therefore we can cover it with a finite number of cubes $(Q_i)_{i=1, \dots, k}$ with edges parallel to the coordinate axes, such that there exists a constant $\alpha > 0$ such that

$$\alpha < \text{meas}(Q_i \cap \Omega) < \delta \text{ for all } i = 1, \dots, k. \quad (4.19)$$

We denote by $q_{i,l}^-$ and $q_{i,l}^+$ (respectively $p_{i,l}^-$ and $p_{i,l}^+$) the local minimum and the local maximum of q_l on $\bar{Q}_i \cap \bar{\Omega}$ (respectively the local minimum and the local maximum of p_l on $\bar{Q}_i \cap \bar{\Omega}$) for $l = 1, \dots, N$.

Using now the same arguments as before locally, we see that the inequality (4.16) holds on $Q_i \cap \Omega$, respectively. The anisotropic Sobolev inequality (2.7) gives

$$\left(\int_{Q_i \cap \Omega} |u_\varepsilon|^{\bar{q}_i^{+*}} dx \right)^{\frac{1}{\bar{q}_i^{+*}}} \leq c_7 \prod_{l=1}^N \left[\left(\int_{Q_i \cap \Omega} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_{i,l}^+} dx \right)^{\frac{1}{q_{i,l}^+}} + \left(\int_{Q_i \cap \Omega} |u_\varepsilon|^{q_{i,l}^+} dx \right)^{\frac{1}{q_{i,l}^+}} \right]^{\frac{1}{N}}, \quad (4.20)$$

for some constant $c_7 > 0$. Since $q_l(\cdot) < s(\cdot)$ for $l = 1, \dots, N$, it follows from (4.20) and (4.10) that

$$\left(\int_{Q_i \cap \Omega} |u_\varepsilon|^{\bar{q}_i^{+*}} dx \right)^{\frac{1}{\bar{q}_i^{+*}}} \leq c_7 \prod_{l=1}^N \left(\int_{Q_i \cap \Omega} \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{q_{i,l}^+} dx \right)^{\frac{1}{q_{i,l}^+ N}} + c_8. \quad (4.21)$$

for some constant $c_8 > 0$.

Keeping in mind (4.18), we deduce from (4.17) (with $\bar{q} = \bar{q}_i$, $p_l = p_{i,l}$ and $q_l = q_{i,l}$ for $i = 1, \dots, k$ and $l = 1, \dots, N$) and (4.21)

$$\|u_\varepsilon\|_{L^{\bar{q}_i^{+*}}(Q_i \cap \Omega)} \leq c_9 + c_{10} \|u_\varepsilon\|_{L^{q_i^{+*}}(Q_i \cap \Omega)}^a, \quad a := \frac{\bar{p}_{i,l}^- - \bar{q}_{i,l}^+}{\bar{q}_{i,l}^+ \bar{p}_{i,l}^-} \bar{q}_i^{+*},$$

for some constants $c_9, c_{10} > 0$. Obviously, this implies that, for some constant C , depending on $p_{i,l}, q_{i,l}$ and Ω ,

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_{i,l}^+}(Q_i \cap \Omega)} \leq C, \quad l = 1, \dots, N, \quad i = 1, \dots, k, \quad (4.22)$$

and

$$\|u_\varepsilon\|_{L^{\bar{q}_i^{+*}}(Q_i \cap \Omega)} \leq C, \quad \frac{1}{\bar{q}_i^{+*}} = \frac{1}{N} \sum_{l=1}^N \frac{1}{q_{i,l}^+}, \quad i = 1, \dots, k. \quad (4.23)$$

Finally, since $\bar{q}_i^{+*} = \frac{Nq_i^+}{N-q_i^+} \geq \bar{q}^*(x) \geq \bar{q}(x)$ and $q_{i,l}^+ \geq q_l(x)$ for all $x \in Q_i \cap \Omega$ and for all $i = 1, \dots, k$, we conclude from (4.22) and (4.23) that for $l = 1, \dots, N$,

$$\|u_\varepsilon\|_{L^{\bar{q}^*(\cdot)}(\Omega)} + \left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l(\cdot)}(\Omega)} \leq c_{11},$$

for some constant c_{11} depending on $p_l(\cdot), q_l(\cdot)$ and Ω . This concludes the proof of the lemma for (4.11).

Observe that the consequence of Proposition 4.1 and Lemma 4.1: there exists a constant C , depending on $\Omega, N, p_1(\cdot), \dots, p_N(\cdot), q_1(\cdot), \dots, q_N(\cdot), \|f\|_{L^1(\Omega)}$ but not ε , such that

$$\left\| \frac{\partial u_\varepsilon}{\partial x_l} \right\|_{L^{q_l(\cdot)}(\Omega)} \leq C, \quad l = 1, \dots, N, \quad (4.24)$$

and

$$\|u_\varepsilon\|_{L^{\bar{q}(\cdot)}(\Omega)} \leq C, \quad \frac{1}{\bar{q}(\cdot)} := \frac{1}{N} \sum_{l=1}^N \frac{1}{q_l(\cdot)}, \quad (4.25)$$

for every continuous function $q_l(\cdot)$ for $l = 1, \dots, N$, such that $1 \leq q_l(\cdot) < \frac{N(\bar{p}(\cdot) - 1)}{\bar{p}(\cdot)(N-1)} p_l(\cdot)$.

4.2. Strong convergence. In view of Proposition 4.1, u_ε is uniformly (in ε) bounded in $W^{1,q_0}(\Omega)$, where

$$q_0^- := \min_{x \in \bar{\Omega}} q_l(\cdot) \text{ for } l = 1, \dots, N, \quad (4.26)$$

and $q_1(\cdot), \dots, q_N(\cdot)$ are restricted as in Lemma 4.1. Without loss of generality, we can therefore assume that

$$\begin{cases} u_\varepsilon \rightarrow u & \text{strongly in } L^{q_0^-}(\Omega) \text{ and a.e. in } \Omega, \\ g(x, u_\varepsilon) \rightarrow g(x, u) & \text{a.e. in } \Omega, \\ T_\gamma(u_\varepsilon) \rightarrow T_\gamma(u) & \text{weakly in } \bigcap_{l=1}^N W_0^{1,p_l(\cdot)}(\Omega). \end{cases} \quad (4.27)$$

Next, we need to prove the strong convergence in $L^1(\Omega)$ of the sequence $(g(x, u_\varepsilon))_{0 < \varepsilon \leq 1}$ to $g(x, u)$.

Proposition 4.2. Assume (3.2)-(3.7) hold, and that the corresponding exponents $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as in (1.2). Then the sequence $(g(x, u_\varepsilon))_{0 < \varepsilon \leq 1}$ converges to $g(x, u)$ a.e. in Ω and strongly in $L^1(\Omega)$.

The prove of Proposition 4.2 is similar to this in [12], so we omit the proof.

Lemma 4.2. The sequence $(u_\varepsilon)_{0 \leq \varepsilon < 1}$ satisfies the following energy estimate

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\{n \leq |u_\varepsilon| \leq n+1\}} \sum_{l=1}^N \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)} dx = 0.$$

Proof. Let ϕ_n be as defined in Subsection 2. Substituting $\varphi = \phi_n(u_\varepsilon)$ into (4.2) yields

$$\int_{\Omega} A(x, \nabla u_\varepsilon) \cdot \nabla \phi_n(u_\varepsilon) dx + \int_{\Omega} g(x, u_\varepsilon) \phi_n(u_\varepsilon) dx = \int_{\Omega} f_\varepsilon \phi_n(u_\varepsilon) dx.$$

Since $\frac{\partial u_\varepsilon}{\partial x_l} \phi_n(u_\varepsilon) = \mathbf{1}_{\{n \leq |u_\varepsilon| \leq n+1\}} \frac{\partial u_\varepsilon}{\partial x_l}$ a.e. in Ω , we deduce from this that

$$\limsup_{\varepsilon \rightarrow 0} \int_{\{n \leq |u_\varepsilon| \leq n+1\}} \sum_{l=1}^N \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)} dx \leq \frac{1}{C_A} \int_{\Omega} |f| \phi_n(u) dx. \quad (4.28)$$

Since $u \in L^1(\Omega)$, we obtain the a.e. convergence of $\phi_n(u)$ to 0 as $n \rightarrow \infty$. Consequently as $|\phi_n| \leq 1$, by Lebesgue Theorem we deduce $\int_{\Omega} |f| \phi_n(u) dx \rightarrow 0$ as $n \rightarrow \infty$. Passing to the limit $n \rightarrow \infty$ in (4.28) yields the desired result. ■

4.3. Strong convergence of truncations. To continue our proof of Theorem 3.1 we need the following proposition.

Proposition 4.3. Fix any truncation level $\gamma > 0$. As $\varepsilon \rightarrow 0$, we have

$$\limsup_{\varepsilon \rightarrow 0} \sum_{l=1}^N \int_{\Omega} \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)} dx \leq \sum_{l=1}^N \int_{\Omega} \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)} dx \quad (4.29)$$

and

$$T_\gamma(u_\varepsilon) \rightarrow T_\gamma(u) \text{ strongly in } \bigcap_{l=1}^N W_0^{1, p_l(\cdot)}(\Omega). \quad (4.30)$$

Proof. We start by showing how (4.30) can be deduced from (4.29).

Proof of (4.30). We recall the following well-known inequalities, which hold for any two real vectors a, b and a real p :

$$(a|a|^{p-2} - b|b|^{p-2})(a - b) \geq c(p) \begin{cases} |a - b|^p, & \text{if } p \geq 2 \\ \frac{|a - b|^2}{(|a| + |b|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \quad (4.31)$$

where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p - 1$ when $1 < p < 2$.

Therefore we have for $p_l(\cdot) \geq 2$ for $l = 1, \dots, N$,

$$\begin{aligned}
& 2^{2-p_l^+} \int_{\{x \in \Omega: p_l(x) \geq 2\}} \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)} dx \\
& \leq \int_{\Omega} \left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial T_\gamma(u)}{\partial x_l} \right) \\
& \quad \times \left(\frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right) dx =: E(\varepsilon).
\end{aligned} \tag{4.32}$$

On the set where $1 < p_l(\cdot) < 2$ for $l = 1, \dots, N$, we employ (4.31) as follows:

$$\begin{aligned}
& \int_{\{x \in \Omega: 1 < p_l(x) < 2\}} \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)} dx \\
& \leq \int_{\{x \in \Omega: 1 < p_l(x) < 2\}} \frac{\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)}}{\left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{\frac{p_l(x)(2-p_l(x))}{2}}} \\
& \quad \times \left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{\frac{p_l(x)(2-p_l(x))}{2}} dx \\
& \leq 2 \left\| \frac{\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^{p_l(x)}}{\left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{\frac{p_l(x)(2-p_l(x))}{2}}} \right\|_{L^{2/p_l(x)}(\Omega)} \\
& \quad \times \left\| \left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{\frac{p_l(x)(2-p_l(x))}{2}} \right\|_{L^{2/(2-p_l(x))}(\Omega)} \\
& \leq 2 \max \left\{ \left(\int_{\Omega} \frac{\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^2}{\left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{2-p_l(x)}} dx \right)^{p_l^-/2} \right. \\
& \quad \left. , \left(\int_{\Omega} \frac{\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} - \frac{\partial T_\gamma(u)}{\partial x_l} \right|^2}{\left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{2-p_l(x)}} dx \right)^{p_l^+/2} \right\} \\
& \quad \times \max \left\{ \left(\int_{\Omega} \left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{p_l(x)} dx \right)^{(2-p_l^+)/2} \right. \\
& \quad \left. , \left(\int_{\Omega} \left(\left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right| + \left| \frac{\partial T_\gamma(u)}{\partial x_l} \right| \right)^{p_l(x)} dx \right)^{(2-p_l^-)/2} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \max \left\{ (p_l^- - 1)^{-p_l^-/2} (E(\varepsilon))^{p_l^-/2}, (p_l^- - 1)^{-p_l^+/2} (E(\varepsilon))^{p_l^+/2} \right\} \\
&\quad \times \max \left\{ \left(\int_{\Omega} \left(\left| \frac{\partial T_{\gamma}(u_{\varepsilon})}{\partial x_l} \right| + \left| \frac{\partial T_{\gamma}(u)}{\partial x_l} \right| \right)^{p_l(x)} dx \right)^{(2-p_l^+)/2} \right. \\
&\quad \left. , \left(\int_{\Omega} \left(\left| \frac{\partial T_{\gamma}(u_{\varepsilon})}{\partial x_l} \right| + \left| \frac{\partial T_{\gamma}(u)}{\partial x_l} \right| \right)^{p_l(x)} dx \right)^{(2-p_l^-)/2} \right\}.
\end{aligned} \tag{4.33}$$

Since $T_{\gamma}(u_{\varepsilon})$ is bounded in $\bigcap_{l=1}^N W_0^{1,p_l(\cdot)}(\Omega)$ and using (4.27) and (4.29), it follows that $E(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, sending $\varepsilon \rightarrow 0$ in (4.32) and (4.33) yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \frac{\partial T_{\gamma}(u_{\varepsilon})}{\partial x_l} - \frac{\partial T_{\gamma}(u)}{\partial x_l} \right|^{p_l(x)} dx = 0, \tag{4.34}$$

which proves (4.30).

Proof of (4.29). We substitute $\varphi = S'_n(u_{\varepsilon})V_{\varepsilon}$ in (4.2), where

$$V_{\varepsilon} = T_{\gamma}(u_{\varepsilon}) - T_{\gamma}(u).$$

The result is

$$J_{\varepsilon,n}^1 + J_{\varepsilon,n}^2 + J_{\varepsilon,n}^3 = J_{\varepsilon,n}^4, \tag{4.35}$$

where

$$\begin{aligned}
J_{\varepsilon,n}^1 &= \int_{\Omega} S'_n(u_{\varepsilon}) A(x, \nabla u_{\varepsilon}) \cdot \nabla V_{\varepsilon} dx, \\
J_{\varepsilon,n}^2 &= \int_{\Omega} S''_n(u_{\varepsilon}) A(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} V_{\varepsilon} dx, \\
J_{\varepsilon,n}^3 &= \int_{\Omega} g(x, u_{\varepsilon}) S'_n(u_{\varepsilon}) V_{\varepsilon} dx, \\
J_{\varepsilon,n}^4 &= \int_{\Omega} f_{\varepsilon} S'_n(u_{\varepsilon}) V_{\varepsilon} dx.
\end{aligned} \tag{4.36}$$

Our goal is to pass to the limit in (4.35)-(4.36) as ε and n goes to 0 and ∞ , respectively, with the truncation level γ kept fixed!

Limit of $J_{\varepsilon,n}^2$. Using Lemma 4.2 and as $\text{supp } S'_n \subset [-(n+1), -n] \cup [n, n+1]$, we obtain for any $n \geq 2$

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |J_{\varepsilon,n}^2| \leq C \limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\{|n \leq |u_{\varepsilon}| \leq n+1\}} \left(1 + \left| \frac{\partial u_{\varepsilon}}{\partial x_l} \right|^{p_l(x)} \right) dx = 0, \tag{4.37}$$

where C is a constant independent of n .

Limit of $J_{\varepsilon,n}^3$ and $J_{\varepsilon,n}^4$. In view of Lebesgue's convergence theorem implies that for any $n \geq 2$

$$\lim_{\varepsilon \rightarrow 0} J_{\varepsilon,n}^3 = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} J_{\varepsilon,n}^4 = 0. \tag{4.38}$$

Now we can pass to the limsup in (4.35) when $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$, respectively. The result is that for any $\gamma \geq 0$

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} \frac{\partial V_\varepsilon}{\partial x_l} dx \leq 0. \quad (4.39)$$

By the definition of S_n (see Subsection 3), we have for any $n \geq \gamma$

$$S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} = \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)}.$$

Using this and the definition of V_ε , we deduce from (4.39)

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)} dx \leq \limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} dx. \quad (4.40)$$

By the definition of S_n , we obtain

$$S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} = S'_n(u_\varepsilon) \left| \frac{\partial T_{n+1}(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial T_{n+1}(u_\varepsilon)}{\partial x_l} \quad \text{a.e. in } \Omega.$$

Using the a.e. convergence of $S'_n(u_\varepsilon)$ to $S'_n(u)$ as $\varepsilon \rightarrow 0$, the boundedness of S'_n , and the weak convergence of $T_{n+1}(u_\varepsilon)$ to $T_{n+1}(u)$ in $\bigcap_{l=1}^N W_0^{1,p_l(\cdot)}(\Omega)$ as $\varepsilon \rightarrow 0$, together with an application of the usual Minty argument, we obtain

$$S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} \rightarrow S'_n(u) \left| \frac{\partial T_{n+1}(u)}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial T_{n+1}(u)}{\partial x_l}$$

weakly in $L^{p'(\cdot)}(\Omega)$.

Sending $\varepsilon \rightarrow 0$ in (4.40), we deduce for any $n \geq \gamma$

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} \right|^{p_l(x)} dx &\leq \limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} S'_n(u_\varepsilon) \left| \frac{\partial u_\varepsilon}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u_\varepsilon}{\partial x_l} \frac{\partial T_\gamma(u_\varepsilon)}{\partial x_l} dx \\ &= \limsup_{n \rightarrow \infty} \int_{\Omega} S'_n(u) \left| \frac{\partial T_{n+1}(u)}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial T_{n+1}(u)}{\partial x_l} \frac{\partial T_{n+1}(u)}{\partial x_l} dx \\ &= \int_{\Omega} \left| \frac{\partial T_{n+1}(u)}{\partial x_l} \right|^{p_l(x)} dx. \end{aligned} \quad (4.41)$$

The proof of Proposition 4.3 is completed. $\blacksquare$

4.4. Concluding the proof of Theorem 3.1. Let $S \in C^\infty(\mathbb{R})$ be such that $\text{supp } S' \subset [-M, M]$ for some $M > 0$. Multiplying (3.1) by $S'(u_\varepsilon)$ (where u and f are replaced by u_ε and f_ε respectively) yields

$$-\text{div} \left(S'(u_\varepsilon) A(x, \nabla u_\varepsilon) \right) + S''(u_\varepsilon) A(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon = -S'(u_\varepsilon) g(x, u_\varepsilon) + f_\varepsilon S'(u_\varepsilon) \quad \text{in } \mathcal{D}'(\Omega). \quad (4.42)$$

In the following we pass to the limit $\varepsilon \rightarrow 0$ (in the sense of distributions) in each of the terms in (4.42). Thanks to (4.27) and Proposition 4.2, it is easy to pass to the limit in the right hand side term.

Let us consider the first term.

$$S'(u_\varepsilon)A(x, \nabla u_\varepsilon) = S'(u_\varepsilon)A(x, \nabla T_M(u_\varepsilon)),$$

and, because of (4.30),

$$S'(u_\varepsilon)A(x, \nabla T_M(u_\varepsilon)) \rightarrow S'(u)A(x, \nabla T_M(u)) \text{ strongly in } L^{p(\cdot)'}(\Omega).$$

Since

$$S'(u)A(x, \nabla T_M(u)) = S'(u)A(x, \nabla u),$$

this concludes passing to the limit in the first term in (4.42).

Let us now study the second term. Clearly, since $\text{supp } S'' \subset [-M, M]$, we have

$$S''(u_\varepsilon)A(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon = S''(u_\varepsilon)A(x, \nabla T_M(u_\varepsilon)) \cdot \nabla T_M(u_\varepsilon) \quad \text{a.e. in } \Omega.$$

Thanks to (4.30), the fact that $S''(u_\varepsilon)$ converges to $S''(u)$ a.e. in Ω , and the boundedness of S'' , the following convergence is true:

$$S''(u_\varepsilon)A(x, \nabla T_M(u_\varepsilon)) \cdot \nabla T_M(u_\varepsilon) \rightarrow S''(u)A(x, \nabla T_M(u)) \cdot \nabla T_M(u) \quad \text{strongly in } L^1(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

This concludes the treatment of the third term in (4.42), as

$$S''(u)A(x, \nabla T_M(u)) \cdot \nabla T_M(u) = S''(u)A(x, \nabla u) \cdot \nabla u.$$

Let us derive the entropy formulation for the regularized sequence u_ε . Let $\phi \in C_c(\Omega)$ and consider $\varphi = T_\gamma(u_\varepsilon - \phi)$ as test function in (4.2). The result is

$$\int_{\Omega} A(x, \nabla u_\varepsilon) \cdot \nabla T_\gamma(u_\varepsilon - \phi) dx + \int_{\Omega} g(x, u_\varepsilon) T_\gamma(u_\varepsilon - \phi) = \int_{\Omega} f_\varepsilon T_\gamma(u_\varepsilon - \phi) dx. \quad (4.43)$$

Using the strong convergence of f_ε and $g(\cdot, u_\varepsilon)$, (4.27) and Proposition 4.3, we can pass to the limits as $\varepsilon \rightarrow 0$ in (4.43) to deduce

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla T_\gamma(u - \phi) dx + \int_{\Omega} g(x, u) T_\gamma(u - \phi) = \int_{\Omega} f T_\gamma(u - \phi) dx,$$

for all $\phi \in C_c(\Omega)$.

5. Uniqueness of renormalized solutions

The purpose of this section is to adapt some of the existing uniqueness results for elliptic equations with variable exponents and L^1 data (see e.g. [27]).

We have the following theorem.

Theorem 5.1. *The renormalized solution of problem (3.1) is unique and the comparison principle holds.*

Proof. In what follows, we let u, v be two renormalized solutions in the sense of Definition 3.2 with data f, g , respectively.

For $\mu > 0$, define the function S_μ in $W^{2,\infty}(\mathbb{R})$ defined by $S_\mu(0) = 0$ and

$$S'_\mu(r) = \begin{cases} 1, & \text{for } |r| \leq \mu, \\ \mu + 1 - |r|, & \text{for } \mu \leq |r| \leq \mu + 1, \\ 0, & \text{for } |r| \geq \mu + 1. \end{cases} \quad (5.1)$$

Note that $\text{supp } S'_\mu \subset [-\mu - 1, \mu + 1]$ and S_μ in $W^{2,\infty}(\mathbb{R})$, thus we can take $S = S_\mu$ in (3.11).

The function $T_\gamma((S_\mu(u) - S_\mu(v))^+)$ belongs to $\bigcap_{l=1}^N W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and thus it can be used as a test function in (3.11).

Now, we use $T_\gamma((S_\mu(u) - S_\mu(v))^+)$ as a test function in (3.11) for u and for v with data f and g , respectively, and then subtract the resulting equations. The result is

$$A_1^{\mu,\gamma} + A_2^{\mu,\gamma} + A_3^{\mu,\gamma} + A_4^{\mu,\gamma} = \int_{\Omega} (fS'_\mu(u) - gS'_\mu(v))T_\gamma((S_\mu(u) - S_\mu(v))^+) dx, \quad (5.2)$$

where

$$\begin{aligned} A_1^{\mu,\gamma} &= \int_{\Omega} [S'_\mu(u)A(x, \nabla u) - S'_\mu(v)A(x, \nabla v)] \cdot \nabla T_\gamma((S_\mu(u) - S_\mu(v))^+) dx, \\ A_2^{\mu,\gamma} &= \int_{\Omega} S''_\mu(u)T_\gamma((S_\mu(u) - S_\mu(v))^+)A(x, \nabla u) \cdot \nabla u dx, \\ A_3^{\mu,\gamma} &= - \int_{\Omega} S''_\mu(v)T_\gamma((S_\mu(u) - S_\mu(v))^+)A(x, \nabla v) \cdot \nabla v dx, \\ A_4^{\mu,\gamma} &= - \int_{\Omega} (S'_\mu(u)g(x, u) - S'_\mu(v)g(x, v))T_\gamma((S_\mu(u) - S_\mu(v))^+) dx. \end{aligned}$$

Observe that

$$\begin{aligned} A_1^{\mu,\gamma} &= \int_{\Omega} (A(x, \nabla S_\mu(u)) - A(x, \nabla S_\mu(v))) \cdot \nabla T_\gamma((S_\mu(u) - S_\mu(v))^+) dx \\ &\quad + \int_{\Omega} (S'_\mu(u)A(x, \nabla u) - A(x, \nabla S_\mu(u))) \cdot \nabla T_\gamma((S_\mu(u) - S_\mu(v))^+) dx \\ &\quad - \int_{\Omega} (S'_\mu(v)A(x, \nabla v) - A(x, \nabla S_\mu(v))) \cdot \nabla T_\gamma((S_\mu(u) - S_\mu(v))^+) dx \\ &:= A_{1,1}^{\mu,\gamma} + A_{1,2}^{\mu,\gamma} + A_{1,3}^{\mu,\gamma}. \end{aligned}$$

By the definition of S_μ , we obtain from (3.3)

$$\begin{aligned} |A_{1,2}^{\mu,\gamma}| &\leq C \sum_{l=1}^N \left(\int_{\{\mu \leq |u| \leq \mu+1\}} \left(1 + \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)} \right) dx + \int_{\{\mu-\gamma \leq |u| \leq \mu+1\}} \left(1 + \left| \frac{\partial v}{\partial x_l} \right|^{p_l(x)} \right) dx \right) \\ &\leq \varepsilon_1(\mu), \end{aligned} \quad (5.3)$$

for any $\mu > \gamma > 0$, where $\varepsilon_1(\mu)$ tends to zero as $\mu \rightarrow \infty$.

Along the same lines we obtain

$$|A_{1,3}^{\mu,\gamma}| \leq \varepsilon_2(\mu), \quad (5.4)$$

where $\varepsilon_2(\mu)$ tends to zero as $\mu \rightarrow \infty$. Next, observe that $|S''_\mu(r)| = \mathbf{1}_{\{\mu \leq |r| \leq \mu+1\}}$. Using this to deduce

$$\lim_{\mu \rightarrow \infty} |A_2^{\mu, \gamma}| = \lim_{\mu \rightarrow \infty} |A_3^{\mu, \gamma}| = 0.$$

We let $\mu_0 \geq 0$ such that $\mu \geq \mu_0$. Observe that

$$A_{1,1}^{\mu, \gamma} \geq \int_{\{0 \leq u-v \leq \gamma\} \cap \{|u|, |v| \leq \mu_0\}} (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla(u-v) dx.$$

Using all the previous calculations, together with the facts that, as $\mu \rightarrow \infty$,

$$\begin{aligned} (S'_\mu(u)g(x, u) - S'_\mu(v)g(x, v))T_\gamma((S_\mu(u) - S_\mu(v))^+) &\rightarrow (g(x, u) - g(x, v))T_\gamma((u-v)^+), \\ (fS'_\mu(u) - gS'_\mu(v))T_\gamma((S_\mu(u) - S_\mu(v))^+) &\rightarrow (f - g)T_\gamma((u-v)^+), \end{aligned}$$

strongly in $L^1(\Omega)$, it follows, by sending $\mu \rightarrow \infty$ in (5.2) that

$$\int_{\{0 \leq u-v \leq \gamma\} \cap \{|u|, |v| \leq \mu_0\}} (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla(u-v) dx \leq \int_{\Omega} (f - g)T_\gamma((u-v)^+) dx. \quad (5.5)$$

Supposing now $f \leq g$ a.e. in Ω in (5.5), we deduce

$$\sum_{l=1}^N \int_{\{0 \leq u-v \leq \gamma\} \cap \{|u|, |v| \leq \mu_0\}} \left(\left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial u}{\partial x_l} - \left| \frac{\partial v}{\partial x_l} \right|^{p_l(x)-2} \frac{\partial v}{\partial x_l} \right) \frac{\partial u_\varepsilon}{\partial x_l} (u-v) dx = 0. \quad (5.6)$$

Sending $\gamma \rightarrow \infty$ in (5.6) and using the inequality (4.31), we obtain

$$\frac{\partial T_{\mu_0}(u)}{\partial x_l} = \frac{\partial T_{\mu_0}(v)}{\partial x_l} \text{ a.e. on the set } \{u > v\} \cap \{|u|, |v| \leq \mu_0\}.$$

As μ_0 is arbitrary, we get

$$\frac{\partial u}{\partial x_l} = \frac{\partial v}{\partial x_l} \text{ a.e. on the set } \{u > v\}, \text{ i.e. } \nabla(u-v)^+ = 0 \text{ a.e. on } \Omega.$$

Since $(u-v)^+ \in W_0^{1,1}(\Omega)$, we deduce

$$(u-v)^+ = 0 \text{ a.e. on the set } \Omega,$$

i.e. $u \leq v$ a.e. on Ω . The uniqueness assertion follows from this comparison principle. $\blacksquare$

6. Comparison between entropy and renormalized solutions

In this section, we extend a previous notion of entropy solutions and prove that they coincide with the renormalized solutions we prove the following theorem.

Theorem 6.1. *Under hypotheses (3.2)-(3.7) hold and the corresponding exponents functions $p_1(\cdot), \dots, p_N(\cdot)$ and $s(\cdot)$ are restricted as in (1.2). Then, a function is an entropy solution of (3.1) if and only if it is a renormalized solution of (3.1).*

Proof. We prove now that the entropy solution u satisfies all the other properties of the renormalized solution. First we observe that the entropy solution u in Theorem 6.1 satisfies the requirements that $T_\gamma(u_\varepsilon) \in W_0^{1,p(x)}(\Omega)$ for given $\gamma > 0$. Inserting $\varphi = T_1(u_\varepsilon - T_\gamma(u_\varepsilon))$ into (4.2) gives

$$\int_{\{\gamma \leq |u_\varepsilon| \leq \gamma+1\}} A(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx + \int_{\{\gamma \leq |u_\varepsilon| \leq \gamma+1\}} g(x, u_\varepsilon) u_\varepsilon dx = \int_{\Omega} f_\varepsilon T_1(u_\varepsilon - T_\gamma(u_\varepsilon)) dx.$$

By (3.2), (3.5) and letting $\varepsilon \rightarrow 0$, we have

$$\sum_{l=1}^N \int_{\{\gamma \leq |u| \leq \gamma+1\}} \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)} dx \leq \int_{\{\gamma \leq |u| \leq \gamma+1\}} |f| dx,$$

then, we deduce the renormalized condition, that is,

$$\lim_{\gamma \rightarrow \infty} \sum_{l=1}^N \int_{\{\gamma \leq |u| \leq \gamma+1\}} \left| \frac{\partial u}{\partial x_l} \right|^{p_l(x)} dx = 0. \quad (6.1)$$

Let $(u_\varepsilon)_\varepsilon$ be a sequence constructed in (4.2), which satisfies the requirement that $T_\gamma(u_\varepsilon)$ strongly converges to $T_\gamma(u)$ in $W_0^{1,p(x)}(\Omega)$ for every $\gamma > 0$.

Let $S \in C^\infty(\mathbb{R})$ be such that $\text{supp } S' \in [-M, M]$ for some $M > 0$. For every $\varphi \in C_0^\infty(\Omega)$, taking $S'(u_\varepsilon)\varphi \in W_0^{1,p_l(x)}(\Omega)$ as a test function in (4.2). We get

$$\int_{\Omega} A(x, \nabla u_\varepsilon) (S''(u_\varepsilon)\varphi \nabla u_\varepsilon + S'(u_\varepsilon) \nabla \varphi) dx + \int_{\Omega} g(x, \nabla u_\varepsilon) S'(u_\varepsilon) \varphi dx = \int_{\Omega} f_\varepsilon S'(u_\varepsilon) \varphi dx. \quad (6.2)$$

We start with the first integral in (6.2), we have

$$\int_{\Omega} A(x, \nabla u_\varepsilon) (S''(u_\varepsilon)\varphi \nabla u_\varepsilon + S'(u_\varepsilon) \nabla \varphi) dx = \int_{\Omega} A(x, \nabla T_\gamma(u_\varepsilon)) (S''(u_\varepsilon)\varphi \nabla T_\gamma(u_\varepsilon) + S'(u_\varepsilon) \nabla \varphi) dx,$$

from (3.3), we have $A(x, \nabla T_\gamma(u_\varepsilon))$ is bounded in $(L^{p'(x)}(\Omega))^N$, then

$$A(x, \nabla T_\gamma(u_\varepsilon)) \rightarrow A(x, \nabla T_\gamma(u)) \text{ a.e. in } \Omega.$$

Hence, we get

$$A(x, \nabla T_\gamma(u_\varepsilon)) \rightharpoonup A(x, \nabla T_\gamma(u)) \quad \text{in } (L^{p'(x)}(\Omega))^N,$$

and since

$$S''(u_\varepsilon)\varphi \nabla T_\gamma(u_\varepsilon) + S'(u_\varepsilon) \nabla \varphi \rightarrow S''(u)\varphi \nabla T_\gamma(u) + S'(u) \nabla \varphi \quad \text{in } (L^{p(x)}(\Omega))^N,$$

we deduce that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A(x, \nabla T_\gamma(u_\varepsilon)) (S''(u_\varepsilon)\varphi \nabla T_\gamma(u_\varepsilon) + S'(u_\varepsilon) \nabla \varphi) dx \\ = \int_{\Omega} A(x, \nabla T_\gamma(u)) (S''(u)\varphi \nabla T_\gamma(u) + S'(u) \nabla \varphi) dx \\ = \int_{\Omega} A(x, \nabla u) (S''(u)\varphi \nabla u + S'(u) \nabla \varphi) dx. \end{aligned}$$

We next examine the second term on the right-hand side of (6.2), we have $S(u_\varepsilon) \varphi \rightharpoonup S(u_\varepsilon) \varphi$ weak-* in $L^\infty(\Omega)$, then

$$\int_{\Omega} g(x, u_\varepsilon) S'(u_\varepsilon) \varphi \, dx \longrightarrow \int_{\Omega} g(x, u) S'(u) \varphi \, dx, \quad (6.3)$$

and

$$\int_{\Omega} f_\varepsilon S'(u_\varepsilon) \varphi \, dx \longrightarrow \int_{\Omega} f S'(u) \varphi \, dx. \quad (6.4)$$

By combining (6.2)-(6.4), we deduce that

$$\int_{\Omega} A(x, \nabla u) (S''(u) \varphi \nabla u + S'(u) \nabla \varphi) \, dx + \int_{\Omega} g(x, \nabla u) S'(u) \varphi \, dx = \int_{\Omega} f S'(u) \varphi \, dx, \quad (6.5)$$

which is (3.11) in Definition 3.2. Then, we conclude that u is a renormalized solution to problem (3.1). Thanks to uniqueness of renormalized solutions see Section 5, which concludes the proof of the Theorem 6.1. ■

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