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Some results about Lipschitz p -Nuclear Operators

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ABSTRACT. The aim of this paper is to study the onto isometries of the space of strongly Lipschitz p -nuclear operators, introduced by D. Chen and B. Zheng (Nonlinear Anal., 75, 2012). We give some new results about such isometries and we focus, in particular, on the case $F = \ell_{p^*}$.

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1. Introduction

In [11], Farmer and Johnson introduced a natural non linear version of p -summing operator, where the domain of such operators is a metric space that need not be a normed space. Since then, several authors have developed many works related to the classe of Lipschitz operator ideal [1], in the sense of creating a theory analogous to the classe of linear operator ideal [16]. For example see [1, 3, 6, 7, 8, 9, 12] and the references therein, with a linear case that has already been introduced and studied by others.

In this paper, we contribute to the theory of non-linear of p -nuclear operators by extending the notion of the isometries of p -nuclear operators to Lipschitz case whose linear analogue has studied by Yousef and Khalil in [15]. This paper is organized as follows: In the second section, we gave some notations and Preliminaries interesting for our work. In the third one,

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we studied the Lipschitz version the isometric onto operators of $\mathcal{N}_p(E, F)$ [15]. In the last one, we continued in same direction with $F = \ell_{p^*}$, where we have given results, among others, concerning rank and atoms.

2. Notation and Preliminaries

For $1 \leq p \leq \infty$, p^* denotes the exponent conjugate to p , that is $\frac{1}{p} + \frac{1}{p^*} = 1$. E, F are (real or complex) Banach spaces and $\mathcal{L}(E, F)$ denote the Banach space of all bounded linear operators from E into F with the operator norms. T belong to $\mathcal{L}(E, F)$ is called an isometry if $\|Tx\| = \|x\|$ for all $x \in E$.

The space $Lip_0(X, F)$ is a Banach space of all Lipschitz operators T from a pointed metric space X , metric space with a distinguished point denoted 0 , into F such that $T(0) = 0$, under the Lipschitz norm given by

$$Lip(T) = \sup \left\{ \frac{\|T(x) - T(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

We use the shorthand $X^\sharp := Lip_0(X, \mathbb{K}) = Lip_0(X)$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A molecule on X is a real-valued function $m : X \rightarrow \mathbb{R}$ with finite support that satisfies $\sum_{x \in X} m(x) = 0$, the real linear space of all molecules on X is denoted by $\mathcal{M}(X)$. For $x, y \in X$, the molecule m_{xy} is defined by $m_{xy} = \chi_{\{x\}} - \chi_{\{y\}}$, where χ_A is the characteristic function of the set A . For $m \in \mathcal{M}(X)$, we can write $m = \sum_{j=1}^n \alpha_j m_{x_j y_j}$ and α_j are scalars. It's well known that $Lip_0(X)$ has a predual, namely the space of Arens-Eells of X , denoted $\mathcal{A}E(X)$ [2], which is the completion of the space of molecules with the norm

$$\|m\|_{\mathcal{A}E(X)} = \inf \left\{ \sum_{j=1}^n |\alpha_j| d(x_j, y_j) : m = \sum_{j=1}^n \alpha_j m_{x_j y_j} \right\}.$$

Sawashima [17] defined the Lipschitz adjoint (or dual) between a pointed metric spaces X and Y with $T(0) = 0$; i.e., $T \in Lip_0(X, Y)$, as the continuous linear operator

$$\begin{aligned} T^\sharp : Y^\sharp &\longrightarrow X^\sharp \\ f &\longmapsto T^\sharp(f) = f \circ T, \end{aligned}$$

where $f \in Y^\sharp$. The norm of $T^\sharp$ is given by $\|T^\sharp\| = Lip(T)$. If $Y = F$ is a Banach space, the restriction of $T^\sharp$ to Y^* is called the Lipschitz transpose operator of T and is denoted by T^t .

The notion of strongly Lipschitz p -nuclear operators was introduced by Chen and Zheng in [9], such that a Lipschitz operator $T : X \rightarrow F$ is strongly Lipschitz p -nuclear operator if, and only if, T has the form:

$$T = \sum_j f_j \otimes y_j,$$

where $(f_j)_j$ in $X^\sharp$ and $(y_j)_j$ in F . The collection of all strongly Lipschitz p -nuclear operators from X to F is denoted by $\mathcal{SN}_p^L(X, F)$, and the strongly Lipschitz p -nuclear norm was defined as following

$$sv_p^L(T) = \inf \left[\left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \right].$$

For $p = 1$,

$$sv_p^L(T) = \left(\sum_j (Lip(f_j)) \right) \left(\sup_j \|y_j\| \right).$$

For $p = \infty$,

$$sv_p^L(T) = \left(\sup_j (Lip(f_j)) \right) \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, y^* \rangle| \right), \lim_j (Lip(f_j)) = 0.$$

Furthermore, Chen and Zheng in [9, Theorem 2.2] prove that, T is strongly Lipschitz p -nuclear operator if, and only if, T has a factorization $T = AM_\lambda B$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & F \\ B \downarrow & & \uparrow A \\ \ell_\infty & \xrightarrow{M_\lambda} & \ell_p(c_0, p = \infty) \end{array}$$

FIGURE 1. Factorization diagram of a strongly Lipschitz p -nuclear operator

where $B \in Lip_0(X, \ell_\infty)$, $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)(\mathcal{L}(\ell_\infty, c_0), p = \infty)$ is the diagonal operator defined as follows (see [10, p.111]): $M_\lambda(\xi_n) = (\xi_n \lambda_n)_n$, $(\xi_n)_n \in \ell_\infty$, with $\|M_\lambda\| = \|\lambda\|_p$, and $A \in \mathcal{L}(\ell_p, F)$. Moreover

$$sv_p^L(T) = \inf \|A\| \|M_\lambda\| Lip(B),$$

where the infimum being extended over all factorizations as above.

The concept of strongly Lipschitz (p, r, s) -nuclear operators introduced and investigated by Belacel and Chen in [3] which is an extension to the notion of linear (p, r, s) -nuclear operators [16]. As in the linear case, the strongly Lipschitz p -nuclear operators coincide with the strongly Lipschitz $(p, 1, p)$ -nuclear operators. The set of all strongly Lipschitz p -nuclear operators $\mathcal{SN}_p^L(X, F)$ is a Banach Lipschitz operator ideal [1].

In [12, Definition 2.1], Vargas and al. introduced the notion of Lipschitz compact operators, such that a Lipschitz operator $T : X \rightarrow F$, $T(0) = 0$ is Lipschitz compact if

$$\left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y \right\}$$

is a relatively compact subset of F .

3. Onto isometries

In this section, we study the Lipschitz version of the isometric onto operators of $\mathcal{N}_p(E, F)$ [15, Sect. 3].

Theorem 3.1. *Let $(S_j^\sharp)_j$ be a sequence of an isometric onto operators on $X^\sharp$, and R be an isometric onto operator on F . Let T be a strongly Lipschitz p -nuclear operator from X into F such that*

$$J(T) = J\left(\sum_j f_j \otimes y_j\right) = \sum_j \alpha_j S_j^\sharp f_{\varphi(j)} \otimes Ry_j,$$

where $(\alpha_j)_j$ in $\mathbb{R}$, $|\alpha_j| = 1$ and φ is a permutation on $\mathbb{N}$. Then J is an isometric operator of $\mathcal{SN}_p^L(X, F)$.

Proof. Assume that $T \in \mathcal{SN}_p^L(X, F)$, then T has the following form

$$T = \sum_j f_j \otimes y_j$$

where $(f_j)_j$ in $X^\sharp$ and $(y_j)_j$ in F , we have

$$\begin{aligned} J(T) &= \sum_j \alpha_j S_j^\sharp f_{\varphi(j)} \otimes Ry_j \\ &= \sum_j \tilde{f}_j \otimes \tilde{y}_j. \end{aligned}$$

Now,

$$sv_p^L(T) = \inf \left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}.$$

Since $S_j^\sharp$ is an isometry, $|\alpha_j| = 1$ and φ is a permutation,

$$\begin{aligned} \sum_j (Lip(\alpha_j S_j^\sharp f_{\varphi(j)}))^p &= \sum_j |\alpha_j|^p (Lip(S_j^\sharp f_{\varphi(j)}))^p \\ &= \sum_j (Lip(f_{\varphi(j)}))^p \\ &= \sum_j (Lip(f_j))^p \end{aligned}$$

and also R is an isometric onto operator on F , so

$$\begin{aligned} \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle Ry_j, y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} &= \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \sup_{\|R^* y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned}$$

$$= \sup_{\|\tilde{y}^*\| \leq 1} \left(\sum_j |\langle y_j, \tilde{y}^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}.$$

Thus, $J(T)$ is strongly Lipschitz p -nuclear operator of $\mathcal{SN}_p^L(X, F)$ and

$$sv_p^L(J(T)) = sv_p^L(T).$$

Theorem 3.2. *Let S be an isometric onto operator of X into X and R be an isometric onto operator of F into F . Assume T be a strongly Lipschitz p -nuclear operator from X into F such that*

$$J(T) = RTS.$$

Then $J(T)$ is an isometric onto operator of $\mathcal{SN}_p^L(X, F)$.

Proof. Let $T \in \mathcal{SN}_p^L(X, F)$, then T has the following representation

$$T = \sum_j f_j \otimes y_j$$

so

$$\begin{aligned} J(T)x &= RTSx \\ &= R\left(\sum_j (f_j(Sx))y_j\right) \\ &= \sum_j (S^\sharp(f_jx))Ry_j \end{aligned}$$

Thus,

$$J(T) = \sum_j S^\sharp f_j \otimes Ry_j.$$

Since S is an isometric onto operator of X , then $S^\sharp$ is an isometric onto operator of $X^\sharp$, and R is an isometric onto operator of F . Now, we show that J preserves norm,

$$\begin{aligned} sv_p^L(J(T)) &= \inf \left(\sum_j (Lip(S^\sharp f_j))^p \right)^{\frac{1}{p}} \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \inf \left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|R^* y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \inf \left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|\tilde{y}^*\| \leq 1} \left(\sum_j |\langle y_j, \tilde{y}^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= sv_p^L(T). \end{aligned}$$

Thus,

$$sv_p^L(J(T)) = sv_p^L(T).$$

To show that J is onto, let $T = \sum_j f_j \otimes y_j$ be a strongly Lipschitz p -nuclear operator, and $\tilde{T} = \sum_j f_j S^{-1} \otimes R^{-1} y_j$, so

$$J(\tilde{T}) = \sum_j f_j \otimes y_j = T,$$

this implies that J is onto.

Hence, J is an isometric onto operator of $\mathcal{SN}_p^L(X, F)$.

4. Lipschitz operators, where $F = \ell_{p^*}$

In this section, we characterize the Lipschitz operators that takes values in ℓ_{p^*} whose linear analogue has found in [15, Sect. 4].

Theorem 4.1. *Let $2 \leq p < \infty$ and T be a Lipschitz operator from X into ℓ_{p^*} . Then T is a strongly Lipschitz p -nuclear operator from X into ℓ_{p^*} , with*

$$sv_p^L(T) = \left(\sum_j |\lambda_j|^p \right)^{\frac{1}{p}},$$

where $(\lambda_j)_j \in \ell_{p^*}$.

Proof. Given the fact that each Lipschitz operator can be factored as [18, Theorem 2.2.4]

FIGURE 2. Factorization diagram of Lipschitz operator

where $\delta_X(x) = m_{x0}$ is an isometric embending from X into $\mathcal{AE}(X)$ and T_L is unique linear operator from $\mathcal{AE}(X)$ to F (for more details see [18, p. 38–43]). Since T_L is p -nuclear operator [15, Sect.4] and $T = T_L \circ \delta_X$, then by [1, Proposition 3.2] T is a strongly Lipschitz p -nuclear operator with $sv_p^L(T) = v_p(T_L)$, by means of [15, Theorem 6] we get

$$sv_p^L(T) = \left(\sum_j |\lambda_j|^p \right)^{\frac{1}{p}}.$$

Using Theorem 4.1, we obtain the following results:

Corollary 4.1. *Let $2 \leq p < \infty$ and T be a strongly Lipschitz p -nuclear operator from X into ℓ_{p^*} . Then T has representation for which the infimum is attained.*

We will give in the next corollary the same result as [9, Theorem 2.1] with E only a Banach space.

Corollary 4.2. *Let $2 \leq p < \infty$ and T be a linear operator from E into ℓ_{p^*} . Then*

$$sv_p^L(T) = v_p(T).$$

Proof. From [15, Theorem 6] and by Theorem 4.1 we get

$$sv_p^L(T) = v_p(T).$$

Corollary 4.3. *The space $\mathcal{SN}_p^L(X, \ell_{p^*})$ and $\ell_{p^*}^{L, w^*}(X^\sharp)$ are isometrically isomorphic.*

Proof. There is an isomorphism isometry between $Lip_0(X, \ell_{p^*})$ and $\ell_{p^*}^{L, w^*}(X^\sharp)$ [3, Lemma 2.4], and $Lip_0(X, \ell_{p^*}) = \mathcal{SN}_p^L(X, \ell_{p^*})$ Theorem 4.1. Hence $\mathcal{SN}_p^L(X, \ell_{p^*})$ and $\ell_{p^*}^{L, w^*}(X^\sharp)$ are isometrically isomorphic.

Corollary 4.4. *Let J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J is Lipschitz compact operator.*

Proof. It's known that $Lip_{0K}(X, \ell_{p^*}) \subset Lip_0(X, \ell_{p^*})$ [12] and Then by Theorem 4.1 $Lip_{0K}(X, \ell_{p^*}) = \mathcal{SN}_p^L(X, \ell_{p^*})$.

The next theorem we show that an isometric onto operators of a Lipschitz operator ideal preserves the rank.

Theorem 4.2. *Let $2 \leq p < \infty$ and J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J preserves the rank.*

Proof. It's known that an isometry on ℓ_p preserve the support. We will use this in our proof. If $T \in \mathcal{SN}_p^L(X, \ell_{p^*})$, then by Theorem 4.1

$$T = \sum_j \lambda_j f_j \otimes \delta_j.$$

Therefore $J(T) \in \mathcal{SN}_p^L(X, \ell_{p^*})$,

$$J(T) = \sum_j \xi_j g_j \otimes \delta_j,$$

where $\|(\lambda_j)\|_p = \|(\xi_j)\|_p$ and $Lip(f_j) = Lip(g_j)$. This produces

$$\hat{J} : \ell_p \longrightarrow \ell_p,$$

where $\hat{J}(\lambda_j) = (\xi_j)_j$. Clearly $\hat{J}$ is an isometric onto operator on ℓ_p . So it preserve the support. That is $supp(\lambda_j) \cap supp(\xi_j) = \emptyset$, then $supp(\hat{J}(\lambda_j)) \cap supp(\hat{J}(\xi_j)) = \emptyset$. Indeed

$$\begin{aligned} \|\hat{J}(\lambda_j) + \hat{J}(\xi_j)\|_p^p &= \|\hat{J}(\lambda_j + \xi_j)\|_p^p \\ &= \|\lambda_j + \xi_j\|_p^p \end{aligned}$$

$$\begin{aligned}
&= \|(\lambda_j)\|_p^p + \|(\xi_j)\|_p^p \\
&= \|(\hat{J}(\lambda_j))\|_p^p + \|(\hat{J}(\xi_j))\|_p^p.
\end{aligned}$$

Since $\text{rank}(J(T)) = |\text{supp}(\hat{J}(\xi_j))| = |\text{supp}(\lambda_j)| = \text{rank}(T)$. So it's preserves rank.

For $f \in X^\sharp$ and $y \in F$ (Banach space), we define the one rank operator $f \otimes y : X \longrightarrow F$ by $(f \otimes y)(x) = f(x)y$. The rank one operator is called an atom.

The next corollary is an immediate consequence of Theorem 4.2.

Corollary 4.5. *Let J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J preserves atoms.*

Now, we give in the next theorem the main result of this section. We adapt the proof in [13, Theorem 3.3] to Lipschitz situation.

Theorem 4.3. *Let $J : \mathcal{SN}_p^L(X, \ell_{p^*}) \longrightarrow \mathcal{SN}_p^L(X, \ell_{p^*})$. If J preserves rank and that preserves the basic atoms, then the following are equivalent:*

- (1) *J is an isometric onto operator.*
- (2) *There exist two isometric onto operators:*

$S : X^\sharp \rightarrow X^\sharp$ and $W : \ell_{p^} \rightarrow \ell_{p^*}$ and a sequence $(a_j)_j, |a_j| = 1$ for all j , with*

$$J(T) = \sum_j a_j S f_j \otimes W \delta_j.$$

Proof. $1 \Rightarrow 2$. Let J be an isometric onto operator.

Step 1: Let $M = \{f \otimes \delta_1 : f \in X^\sharp\}$. Then

$$J(M) = \{g \otimes \delta_j : \text{for fixed } j, \forall g \in X^\sharp\}.$$

Assume $J(f_1 \otimes \delta_1) = \hat{f}_1 \otimes \delta_{j_1}$ and $J(f_2 \otimes \delta_1) = \hat{f}_2 \otimes \delta_{j_2}$ where $\hat{f}_1, \hat{f}_2 \in X^\sharp$, and $\delta_{j_1} \neq \delta_{j_2}$. Then for some $g \in X^\sharp$ and $j \in \mathbb{N}$:

$$\begin{aligned}
\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2} &= J(f_1 \otimes \delta_1) + J(f_2 \otimes \delta_1) \\
&= J(f_1 \otimes \delta_1 + f_2 \otimes \delta_1) \\
&= J(f_1 \otimes (\delta_1 + \delta_1)) \\
&= g \otimes \delta_j
\end{aligned}$$

This implies $\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2} = g \otimes \delta_j$ with $\delta_{j_1} \neq \delta_{j_2}$. But $f_1 \otimes \delta_1 + f_2 \otimes \delta_1 = (f_1 + f_2) \otimes \delta_1$ is a basic atom, and J preserves basic atoms which is a contradiction because $\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2}$ is not a basic atom, since $\delta_{j_1} \neq \delta_{j_2}$. So $J(f \otimes \delta_1) = g \otimes \delta_j$ for fixed $j \in \mathbb{N}$. Similarly for $\delta_2, \dots$

Step 2: Let $J(f \otimes \delta_1) = g \otimes \delta_j$. Define $W : \ell_{p^*} \rightarrow \ell_{p^*}, W\delta_1 = \delta_j$, W it permutes the basis (δ_j) , so W is an onto and $W\delta_1 = \delta_{\varphi(j)}$, where φ is a permutation on the set of natural numbers $\mathbb{N}$. Since J is an isometric, then W is an isometric. Similarly for $\delta_2, \dots$

Step 3: Assume $f \otimes \delta_1 + f \otimes \delta_2 = f \otimes (\delta_1 + \delta_2)$ be a basic atom (1-rank operator). Let $J(f \otimes \delta_1) = g \otimes \delta_{\varphi(1)}$ and $J(f \otimes \delta_2) = h \otimes \delta_{\varphi(2)}$, such that $g \neq h$. Then $J(f \otimes (\delta_1 + \delta_2)) = g \otimes \delta_{\varphi(1)} + h \otimes \delta_{\varphi(2)}$. Since J is an isometry, then $Lip(f) = Lip(g) = Lip(h)$, but this implies that either g, h are independent or dependent i.e $g = \pm h$. If g, h are independent, then $J(f \otimes (\delta_1 + \delta_2)) = g \otimes \delta_{\varphi(1)} + h \otimes \delta_{\varphi(2)}$ is two rank operator which is a contradiction, because J preserves rank. Hence $J(f \otimes \delta_1) = a_1 g \otimes \delta_{\varphi(1)}$ and $J(f \otimes \delta_2) = a_2 g \otimes \delta_{\varphi(2)}$, with $|a_j| = 1$. Similarly we prove

$$\{J(f \otimes \delta_j) : j \in \mathbb{N}\} = \{a_j g \otimes \delta_{\varphi(j)} : |a_j| = 1, j \in \mathbb{N}\}.$$

Step 4: Define $S : X^\sharp \rightarrow X^\sharp, S(f) = g$, where $J(f \otimes \delta_j) = g \otimes a_j \delta_{\varphi(j)}$ with $|a_j| = 1$. It's easy to check that S is well-defined isometric linear operator. Let $g \in X^\sharp : J(f \otimes \delta_{\varphi^{-1}(j)}) = a_j g \otimes \delta_j$, where $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is one-to-one and onto map on the set of natural numbers $\mathbb{N}$. Since J is onto, then for some $g = S(f)$. This implies S is onto.

We show now that $J(T) = \sum_j a_j S f_j \otimes W \delta_j$. We have

$$J(\sum_j f_j \otimes \delta_j) = \sum_j J(f_j \otimes \delta_j) = \sum_j a_j g_j \otimes \delta_{\varphi(j)} = \sum_j a_j S f_j \otimes W \delta_j.$$

$2 \Rightarrow 1$. Let $T \in \mathcal{SN}_p^L(X, \ell_{p^*})$. Since S, W is an isometry we have

$$J(T) = \sum_j a_j S f_j \otimes W \delta_j = \sum_j a_j g_j \otimes \delta_{\varphi(j)} = \tilde{T},$$

where $Lip(f_j) = Lip(g_j)$, with

$$sv_p^L(J(T)) = sv_p^L(\tilde{T}) = (\sum_j |\lambda_j|^p)^{\frac{1}{p}} = sv_p^L(T).$$

Thus J is an isometry.

Now we show that J is onto, let $T = \sum_j f_j \otimes \delta_j$ be a strongly Lipschitz p -nuclear operator, where $f_j = a_j g_j$, with $|a_j| = 1$, let $\tilde{T} = \sum_j S^{-1} g_j \otimes \delta_{\varphi^{-1}(j)}$, so

$$J(\tilde{T}) = \sum_j a_j g_j \otimes \delta_j = \sum_j f_j \otimes \delta_j = T,$$

this implies that J is onto.

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