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Integral inequalities via harmonically h -convexity

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ABSTRACT. In this paper, we establish some estimates of the left side of the generalized Gauss-Jacobi quadrature formula for harmonic h -preinvex functions involving Euler's beta and hypergeometric functions. The obtained results are mainly based on the identity given by M. A. Noor, K. I. Noor and S. Iftikhar in [17].

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1. Introduction

It is well known that convexity plays an important role in many areas, such as economics, finances, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

We first recall some definitions, and assume that I is a real interval, such that $I = (0, \infty)$.

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Definition 1.1 ([14]). A set I is said to be an harmonically invex set with respect to the continuous function $\eta : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{R}$, if

$$\frac{\vartheta(\vartheta+\eta(y,\vartheta))}{\vartheta+(1-\epsilon)\eta(y,\vartheta)} \in I$$

for all $\vartheta, y \in I$ and $\epsilon \in [0, 1]$.

Definition 1.2 ([15]). Let $f : I \rightarrow \mathbb{R}$, f is said to be harmonically preinvex with respect to the continuous function $\eta : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{R}$, if

$$f\left(\frac{\vartheta(\vartheta+\eta(y,\vartheta))}{\vartheta+(1-\epsilon)\eta(y,\vartheta)}\right) \leq h(1-\epsilon)f(\vartheta) + h(\epsilon)f(y)$$

holds for all $\vartheta, y \in I$ and $\epsilon \in [0, 1]$.

Remark 1.1. Definition 1.2, will be reduced to the following classes according to the values of $h(t)$: harmonically P -preinvex function [15], harmonically preinvex function [14], harmonically s -preinvex function in the second sense [15], harmonically Godunova-Levin-preinvex function [15], harmonically s -Godunova-Levin-preinvex function [15], harmonically tgs-preinvex function [17], harmonically MT-preinvex function [16], and harmonically beta-preinvex function [17], for $h(t) = 1, t, t^s, t^{-1}, t^{-s}, t(1-t), \frac{\sqrt{1-t}}{2\sqrt{t}}$ and $t^\alpha(1-t)^\beta$ respectively.

The generalized quadrature formula of Gauss-Jacobi type has the following form

$$\int_{\alpha}^{\beta} (\vartheta - \alpha)^p (\beta - \vartheta)^q f(\vartheta) d\vartheta = \sum_{k=0}^m B_{m,k} f(\gamma_k) + \mathfrak{R}_m[f], \quad (1.1)$$

where $B_{m,k}$ are the Christoffel coefficients, γ_k are the roots of the Jacobi polynomial of degree m , and $\mathfrak{R}_m[f]$ is the remainder term (see [20]).

Özdemir et al. [19], gave the estimate of the left side of equality (1.1) when the function f is quasi-convex, using the following identity

$$\int_a^b (\vartheta - a)^p (b - \vartheta)^q f(\vartheta) d\vartheta = (b - a)^{p+q+1} \int_a^b (1-t)^p t^q f(ta + (1-t)b) dt. \quad (1.2)$$

About some papers related to identity (1.2), we mention [2, 6, 7, 8, 13].

In [1], Ahmad used the following identity

$$\begin{aligned} & \int_{\alpha}^{\alpha+\tilde{\eta}(b,\alpha)} (\vartheta - \alpha)^p (\alpha + \tilde{\eta}(b,\alpha) - \vartheta)^q f(\vartheta) d\vartheta \\ &= (\tilde{\eta}(b,\alpha))^{p+q+1} \int_0^1 (1-\tau)^p \tau^q f(\tau\alpha + (1-\tau)b) d\tau, \end{aligned} \quad (1.3)$$

and gave some estimates of the left-hand side of equality (1.3) when $|f|, |f|^{\frac{k}{k-1}}$ and $|f|^l$ are P -preinvex and prequasiinvex functions, see also [4, 5, 9, 10, 11, 12], which have a relationship with the equality (1.3).

İşcan et al. [3], discussed the cases where $|f|$, $|f|^{\frac{k}{k-1}}$ and $|f|^l$ are harmonically convex functions based on the following identity, see also [18].

$$\begin{aligned} & \int_a^b (\vartheta - a)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq a^{p+1} b^{q+1} (b - a)^{p+q+1} \int_a^b \frac{(1-\tau)^p \tau^q}{(\tau a + (1-\tau)b)^{p+q+2}} f\left(\frac{ab}{\tau a + (1-\tau)b}\right) d\tau. \end{aligned} \quad (1.4)$$

Noor et al. [17], gave the following identity and establish the estimate of the left hand side of the generalized quadrature formula of Gauss-Jacobi for harmonically β -preinvex functions

Lemma 1.1 ([17]). *Let f be a positive and integrable real function on $[a, a + \eta(b, a)]$, then for some fixed $p, q > 0$, we have*

$$\begin{aligned} & \int_a^{a+\eta(b,a)} (\vartheta - a)^p (a + \eta(b, a) - \vartheta)^q f(\vartheta) d\vartheta \\ & = a^{p+1} (a + \eta(b, a))^{q+1} (\eta(b, a))^{p+q+1} \int_0^1 \frac{t^p (1-t)^q}{(a + (1-t)\eta(b, a))^{p+q+2}} f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) dt. \end{aligned}$$

In the present study, by using the identity given by Noor et al. [17], we establish some estimates of the left-hand side of the generalized Gauss-Jacobi quadrature formula in cases where the function f and $|f|$ at a certain power are harmonically h -preinvex functions. We also discuss some special cases.

2. Main results

We first recall that the hypergeometric function is defined as follows

$${}_2F_1(\zeta, \varepsilon; \mu; b) = \frac{1}{B(\varepsilon, \mu - \varepsilon)} \int_0^1 \tau^{\varepsilon-1} (1-\tau)^{\mu-\varepsilon-1} (1-b\tau)^{-\zeta} d\tau,$$

where $\mu > \varepsilon > 0$, $|b| < 1$, and B is the Euler beta function given by

$$B(\vartheta, \chi) = \int_0^1 \tau^{\vartheta-1} (1-\tau)^{\chi-1} d\tau \text{ for } \vartheta, \chi > 0.$$

Theorem 2.1. *Let $f : [v, v + \eta(b, v)] \rightarrow \mathbb{R}$ be a positive and integrable function on $(v, v + \eta(b, v))$ with $b, v \in I$ and $\eta(b, v) > 0$. If f is harmonically h -preinvex on $[v, v + \eta(b, v)]$, then for all $p, q > 0$, we have*

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta$$

$$\leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \\ \times \left(f(v) \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(1-\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon + f(b) \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon \right).$$

Proof. From Lemma 1.1, and from the harmonically h -preinvexity of f , we have

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ \leq v^{p+1} (v + \eta(b,v))^{q+1} (\eta(b,v))^{p+q+1} \\ \times \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q}{(v + (1-\varepsilon)\eta(b,v))^{p+q+2}} (h(1-\varepsilon)f(v) + h(\varepsilon)f(b)) d\varepsilon \\ = \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} (\eta(b,v))^{p+q+1} \\ \times \left(f(v) \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(1-\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon + f(b) \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon \right),$$

which is the desired result.

Remark 2.1. Theorem 2.1 will be reduced to Theorem 1 from [3], if we take $\eta(b,v) = b - v$ and $h(\varepsilon) = \varepsilon$.

Corollary 2.1. In Theorem 2.1, if we take $h(\varepsilon) = \varepsilon^\alpha (1-\varepsilon)^\beta$, we obtain

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} (B(p+\beta+1, q+\alpha+1) \\ \times {}_2F_1\left(p+q+2, p+\beta+1, p+q+\alpha+\beta+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right) f(v) \\ + B(p+\alpha+1, q+\beta+1) \\ \times {}_2F_1\left(p+q+2, p+\alpha+1, p+q+\alpha+\beta+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right) f(b)).$$

Moreover, if we choose $\eta(b,v) = b - v$, we get

$$\int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{b} \right)^{p+1} (b - v)^{p+q+1} (B(p+\beta+1, q+\alpha+1)$$

$$\begin{aligned} & \times {}_2F_1(p+q+2, p+\beta+1, p+q+\alpha+\beta+2; 1-\frac{v}{b}) f(v) \\ & + B(p+\alpha+1, q+\beta+1) \\ & \times {}_2F_1(p+q+2, p+\alpha+1, p+q+\alpha+\beta+2; 1-\frac{v}{b}) f(b) \end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.2. Let $f : [v, v + \eta(b, v)] \rightarrow \mathbb{R}$ be a positive and integrable function on $(v, v + \eta(b, v))$ with $b, v \in I$ and $\eta(b, v) > 0$, and let $\lambda > 1$. If $|f|^\lambda$ is harmonically h -preinvex on $[v, v + \eta(b, v)]$, then for all $p, q > 0$, we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} \\ & \times \left(B(p+1, q+1) {}_2F_1\left(p+q+2, p+1, p+q+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right) \right)^{1-\frac{1}{\lambda}} \\ & \times \left(|f(v)|^\lambda \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(1-\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon + |f(b)|^\lambda \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon \right)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ and β are the hypergeometric and the beta functions respectively.

Proof. From Lemma 1.1, properties of modulus, power mean inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq v^{p+1} (v + \eta(b, v))^{q+1} (\eta(b, v))^{p+q+1} \left(\int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q}{(v+(1-\varepsilon)\eta(b,v))^{p+q+2}} d\varepsilon \right)^{1-\frac{1}{\lambda}} \\ & \times \left(\int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q}{(v+(1-\varepsilon)\eta(b,v))^{p+q+2}} \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda \right) d\varepsilon \right)^{\frac{1}{\lambda}} \\ & = \frac{v^{p+1} (\eta(b,v))^{p+q+1}}{(v+\eta(b,v))^{q+1}} \left(B(p+1, q+1) {}_2F_1\left(p+q+2, p+1, p+q+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right) \right)^{1-\frac{1}{\lambda}} \\ & \times \left(|f(v)|^\lambda \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(1-\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon + |f(b)|^\lambda \int_0^1 \frac{\varepsilon^p (1-\varepsilon)^q h(\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{p+q+2}} d\varepsilon \right)^{\frac{1}{\lambda}}, \end{aligned}$$

which is the desired result.

Remark 2.2. Theorem 2.2 will be reduced to Theorem 2 from [3], if we take $\eta(b, v) = b - v$ and $h(\varepsilon) = \varepsilon$.

Corollary 2.2. In Theorem 2.2, if we take $h(\varepsilon) = \varepsilon^s$, we obtain

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} \\ & \quad \times \left(B(p+1, q+1) {}_2F_1 \left(p+q+2, p+1, p+q+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{1-\frac{1}{\lambda}} \\ & \quad \times (B(p+1, q+s+1) \\ & \quad {}_2F_1 \left(p+q+2, p+1, p+q+s+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) |f(v)|^\lambda \\ & \quad + B(p+s+1, q+1) \\ & \quad \times {}_2F_1 \left(p+q+2, p+s+1, p+q+s+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) |f(b)|^\lambda)^{\frac{1}{\lambda}}. \end{aligned}$$

Moreover, if we choose $\eta(b, v) = b - v$, we get

$$\begin{aligned} & \int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{b} \right)^{p+1} (b - v)^{p+q+1} \\ & \quad \times \left(B(p+1, q+1) {}_2F_1 \left(p+q+2, p+1, p+q+2; 1 - \frac{v}{b} \right) \right)^{1-\frac{1}{\lambda}} \\ & \quad \times (B(p+1, q+s+1) \\ & \quad {}_2F_1 \left(p+q+2, p+1, p+q+s+2; 1 - \frac{v}{b} \right) |f(v)|^\lambda \\ & \quad + B(p+s+1, q+1) \\ & \quad \times {}_2F_1 \left(p+q+2, p+s+1, p+q+s+2; 1 - \frac{v}{b} \right) |f(b)|^\lambda)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.3. Let $f : [v, v + \eta(b, v)] \rightarrow \mathbb{R}$ be a positive and integrable function on $(v, v + \eta(b, v))$ with $b, v \in I$ and $\eta(b, v) > 0$, and let $\lambda, k > 1$ with $\frac{1}{\lambda} + \frac{1}{k} = 1$. If $|f|^\lambda$ is harmonically h -preinvex on $[v, v + \eta(b, v)]$, then for all $p, q > 0$, we have

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta$$

$$\leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(|f(v)|^\lambda + |f(b)|^\lambda \right)^{\frac{1}{\lambda}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{\lambda}} \\ \times \left(B(kp+1, kq+1) {}_2F_1 \left((p+q+2)k, kp+1, kp+kq+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{\frac{1}{k}},$$

where ${}_2F_1$ and β are the hypergeometric and the beta functions respectively.

Proof. From Lemma 1.1, properties of modulus, Hölder's inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ \leq v^{p+1} (v + \eta(b,v))^{q+1} (\eta(b,v))^{p+q+1} \left(\int_0^1 \frac{\varepsilon^{pk}(1-\varepsilon)^{qk}}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)k}} d\varepsilon \right)^{\frac{1}{k}} \\ \times \left(\int_0^1 \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda \right) d\varepsilon \right)^{\frac{1}{\lambda}} \\ = \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\int_0^1 \frac{\varepsilon^{pk}(1-\varepsilon)^{qk}}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)k}} d\varepsilon \right)^{\frac{1}{k}} \\ \times \left(|f(v)|^\lambda + |f(b)|^\lambda \right)^{\frac{1}{\lambda}} \left(\int_0^1 h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}} \\ = \frac{v^{p+1}(\eta(b,v))^{p+q+1}}{(v+\eta(b,v))^{p+1}} \left(|f(v)|^\lambda + |f(b)|^\lambda \right)^{\frac{1}{\lambda}} \left(\int_0^1 h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}} \\ \times \left(B(kp+1, kq+1) {}_2F_1 \left((p+q+2)k, kp+1, kp+kq+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{\frac{1}{k}},$$

which is the desired result.

Corollary 2.3. In Theorem 2.3, if we take $h(\varepsilon) = \frac{\sqrt{1-\varepsilon}}{2\sqrt{\varepsilon}}$, we obtain

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{\pi |f(v)|^\lambda + \pi |f(b)|^\lambda}{4} \right)^{\frac{1}{\lambda}} (B(kp+1, kq+1))$$

$$\times {}_2F_1 \left((p+q+2)k, kp+1, kp+kq+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \Big)^{\frac{1}{k}}.$$

Moreover, if we choose $\eta(b, v) = b - v$, we get

$$\begin{aligned} & \int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{b} \right)^{p+1} (b - v)^{p+q+1} \left(\frac{\pi |f(v)|^\lambda + \pi |f(b)|^\lambda}{4} \right)^{\frac{1}{\lambda}} (B(kp+1, kq+1) \\ & \quad \times {}_2F_1 \left((p+q+2)k, kp+1, kp+kq+2; 1 - \frac{v}{b} \right) \Big)^{\frac{1}{k}}, \end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.4. Assume that all the assumptions of Theorem 2.3 hold, then we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} (B(pk+1, qk+1))^{\frac{1}{k}} \\ & \quad \times \left(|f(v)|^\lambda \int_0^1 \frac{h(1-t)}{\left(1-t \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)\lambda}} dt + |f(b)|^\lambda \int_0^1 \frac{h(t)}{\left(1-t \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)\lambda}} dt \right)^{\frac{1}{\lambda}}, \end{aligned}$$

where $B(.,.)$ is the beta function.

Proof. From Lemma 1.1, Hölder's inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq v^{p+1} (v + \eta(b, v))^{q+1} (\eta(b, v))^{p+q+1} \left(\int_0^1 \varepsilon^{pk} (1 - \varepsilon)^{qk} d\varepsilon \right)^{\frac{1}{k}} \\ & \quad \times \left(\int_0^1 \frac{1}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)\lambda}} \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda \right) d\varepsilon \right)^{\frac{1}{\lambda}} \\ & = v^{p+1} (v + \eta(b, v))^{q+1} (\eta(b, v))^{p+q+1} (B(pk+1, qk+1))^{\frac{1}{k}} \\ & \quad \times \left(|f(v)|^\lambda \int_0^1 \frac{h(1-\varepsilon)}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)\lambda}} d\varepsilon + |f(b)|^\lambda \int_0^1 \frac{h(\varepsilon)}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)\lambda}} d\varepsilon \right)^{\frac{1}{\lambda}} \end{aligned}$$

$$= \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} (B(pk+1, qk+1))^{\frac{1}{k}} \\ \times \left(|f(v)|^\lambda \int_0^1 \frac{h(1-\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)\lambda}} d\varepsilon + |f(b)|^\lambda \int_0^1 \frac{h(\varepsilon)}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)\lambda}} d\varepsilon \right)^{\frac{1}{\lambda}},$$

which is the desired result.

Corollary 2.4. *In Theorem 2.4, if we take $h(\varepsilon) = \varepsilon^\alpha (1-\varepsilon)^\beta$, we obtain*

$$\int_v^{v+\eta(b,v)} (\vartheta-v)^p (v+\eta(b,v)-\vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} (B(pk+1, qk+1))^{\frac{1}{k}} \\ \times \left(B(\beta+1, \alpha+1)_2 F_1 \left((p+q+2)\lambda, \beta+1, \alpha+\beta+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) |f(v)|^\lambda \right. \\ \left. + B(\alpha+1, \beta+1)_2 F_1 \left((p+q+2)\lambda, \alpha+1, \alpha+\beta+2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) |f(b)|^\lambda \right)^{\frac{1}{\lambda}}.$$

Moreover, if we choose $\eta(b,v) = b-v$, we get

$$\int_v^b (\vartheta-v)^p (b-\vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{b} \right)^{p+1} (b-v)^{p+q+1} (B(pk+1, qk+1))^{\frac{1}{k}} \\ \times \left(B(\beta+1, \alpha+1)_2 F_1 \left((p+q+2)\lambda, \beta+1, \alpha+\beta+2; 1-\frac{v}{b} \right) |f(v)|^\lambda \right. \\ \left. + B(\alpha+1, \beta+1)_2 F_1 \left((p+q+2)\lambda, \alpha+1, \alpha+\beta+2; 1-\frac{v}{b} \right) |f(b)|^\lambda \right)^{\frac{1}{\lambda}},$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.5. *Assume that all the assumptions of Theorem 2.3 hold, then we have*

$$\int_v^{v+\eta(b,v)} (\vartheta-v)^p (v+\eta(b,v)-\vartheta)^q f(\vartheta) d\vartheta \\ \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left({}_2F_1 \left((p+q+2)k, 1, 2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{\frac{1}{k}} \\ \times \left(|f(v)|^\lambda \int_0^1 t^{q\lambda} (1-t)^{p\lambda} h(t) dt + |f(b)|^\lambda \int_0^1 t^{p\lambda} (1-t)^{q\lambda} h(t) dt \right)^{\frac{1}{\lambda}},$$

where ${}_2F_1$ is the hypergeometric function.

Proof. From Lemma 1.1, Hölder's inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\begin{aligned}
& \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\
& \leq v^{p+1} (v + \eta(b, v))^{q+1} (\eta(b, v))^{p+q+1} \left(\int_0^1 \frac{1}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)k}} d\varepsilon \right)^{\frac{1}{k}} \\
& \quad \times \left(\int_0^1 \varepsilon^{p\lambda} (1-\varepsilon)^{q\lambda} \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda \right) d\varepsilon \right)^{\frac{1}{\lambda}} \\
& = \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} \left(\int_0^1 \frac{1}{\left(1-\varepsilon \frac{\eta(b,v)}{v+\eta(b,v)}\right)^{(p+q+2)k}} d\varepsilon \right)^{\frac{1}{k}} \\
& \quad \times \left(|f(v)|^\lambda \int_0^1 \varepsilon^{p\lambda} (1-\varepsilon)^{q\lambda} h(1-\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 \varepsilon^{p\lambda} (1-\varepsilon)^{q\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}} \\
& = \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} \left({}_2F_1 \left((p+q+2)k, 1, 2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{\frac{1}{k}} \\
& \quad \times \left(|f(v)|^\lambda \int_0^1 \varepsilon^{q\lambda} (1-\varepsilon)^{p\lambda} h(\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 \varepsilon^{p\lambda} (1-\varepsilon)^{q\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result.

Remark 2.3. Theorem 2.5 will be reduced to Theorem 5 from [3], if we take $\eta(b, v) = b - v$ and $h(\varepsilon) = \varepsilon$.

Corollary 2.5. In Theorem 2.5, if we take $h(\varepsilon) = 1$, we obtain

$$\begin{aligned}
& \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\
& \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b, v))^{p+q+1} \left({}_2F_1 \left((p+q+2)k, 1, 2; \frac{\eta(b,v)}{v+\eta(b,v)} \right) \right)^{\frac{1}{k}} \\
& \quad \times B(q\lambda + 1, p\lambda + 1)^{\frac{1}{\lambda}} \left(|f(v)|^\lambda + |f(b)|^\lambda \right)^{\frac{1}{\lambda}}.
\end{aligned}$$

Moreover, if we choose $\eta(b, v) = b - v$, we get

$$\begin{aligned} & \int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{b}\right)^{p+1} (b - v)^{p+q+1} \left({}_2F_1\left((p+q+2)k, 1, 2; 1 - \frac{v}{b}\right)\right)^{\frac{1}{k}} \\ & \quad \times B(q\lambda + 1, p\lambda + 1)^{\frac{1}{\lambda}} \left(|f(v)|^\lambda + |f(b)|^\lambda\right)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.6. Assume that all the assumptions of Theorem 2.3 hold, then we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{v+\eta(b,v)}\right)^{p+1} (\eta(b, v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, 1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1}\right)^{\frac{1}{k}} \\ & \quad \times \left(|f(v)|^\lambda \int_0^1 (1-\varepsilon)^{p\lambda} h(\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(\varepsilon) d\varepsilon\right)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function.

Proof. From Lemma 1.1, Hölder's inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\begin{aligned} & \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b, v) - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq v^{p+1} (v + \eta(b, v))^{q+1} (\eta(b, v))^{p+q+1} \left(\int_0^1 \frac{(1-\varepsilon)^{qk}}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)k}} d\varepsilon\right)^{\frac{1}{k}} \\ & \quad \times \left(\int_0^1 t^{p\lambda} \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda\right) d\varepsilon\right)^{\frac{1}{\lambda}} \\ & = \left(\frac{v}{v+\eta(b,v)}\right)^{p+1} (\eta(b, v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, 1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1}\right)^{\frac{1}{k}} \end{aligned}$$

$$\begin{aligned}
& \times \left(|f(v)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(1-\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}} \\
& = \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, 1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1} \right)^{\frac{1}{k}} \\
& \times \left(|f(v)|^\lambda \int_0^1 (1-\varepsilon)^{p\lambda} h(\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result.

Corollary 2.6. *In Theorem 2.6, if we take $h(\varepsilon) = \frac{\sqrt{1-\varepsilon}}{2\sqrt{\varepsilon}}$, we obtain*

$$\begin{aligned}
& \int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\
& \leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, 1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1} \right)^{\frac{1}{k}} \\
& \times \left(B\left(\frac{1}{2}, p\lambda + \frac{1}{2}\right) \right)^{\frac{1}{\lambda}} \left(\frac{(2p\lambda+1)|f(v)|^\lambda + |f(b)|^\lambda}{2(p\lambda+1)} \right)^{\frac{1}{\lambda}}.
\end{aligned}$$

Moreover, if we choose $\eta(b,v) = b - v$, we get

$$\begin{aligned}
& \int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\
& \leq \left(\frac{v}{b} \right)^{p+1} (b - v)^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, 1, qk+2; 1 - \frac{v}{b}\right)}{qk+1} \right)^{\frac{1}{k}} \\
& \times \left(B\left(\frac{1}{2}, p\lambda + \frac{1}{2}\right) \right)^{\frac{1}{\lambda}} \left(\frac{(2p\lambda+1)|f(v)|^\lambda + |f(b)|^\lambda}{2(p\lambda+1)} \right)^{\frac{1}{\lambda}},
\end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

Theorem 2.7. *Assume that all the assumptions of Theorem 2.3 hold, then we have*

$$\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta$$

$$\begin{aligned} &\leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, qk+1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1} \right)^{\frac{1}{k}} \\ &\quad \times \left(|f(v)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 (1-\varepsilon)^{p\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function.

Proof. From Lemma 1.1, Hölder's inequality, and the harmonically h -preinvexity of $|f|^\lambda$, we have

$$\begin{aligned} &\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ &\leq v^{p+1} (v + \eta(b,v))^{q+1} (\eta(b,v))^{p+q+1} \left(\int_0^1 \frac{\varepsilon^{qk}}{(v+(1-\varepsilon)\eta(b,v))^{(p+q+2)k}} d\varepsilon \right)^{\frac{1}{k}} \\ &\quad \times \left(\int_0^1 (1-\varepsilon)^{p\lambda} \left(h(1-\varepsilon) |f(v)|^\lambda + h(\varepsilon) |f(b)|^\lambda \right) d\varepsilon \right)^{\frac{1}{\lambda}} \\ &= \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, qk+1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1} \right)^{\frac{1}{k}} \\ &\quad \times \left(|f(v)|^\lambda \int_0^1 \varepsilon^{p\lambda} h(\varepsilon) d\varepsilon + |f(b)|^\lambda \int_0^1 (1-\varepsilon)^{p\lambda} h(\varepsilon) d\varepsilon \right)^{\frac{1}{\lambda}}, \end{aligned}$$

which is the desired result.

Corollary 2.7. In Theorem 2.7, if we take $h(\varepsilon) = \varepsilon^\alpha (1-\varepsilon)^\beta$, we obtain

$$\begin{aligned} &\int_v^{v+\eta(b,v)} (\vartheta - v)^p (v + \eta(b,v) - \vartheta)^q f(\vartheta) d\vartheta \\ &\leq \left(\frac{v}{v+\eta(b,v)} \right)^{p+1} (\eta(b,v))^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, qk+1, qk+2; \frac{\eta(b,v)}{v+\eta(b,v)}\right)}{qk+1} \right)^{\frac{1}{k}} \\ &\quad \times \left(B(p\lambda + 2, \alpha + 1) |f(v)|^\lambda + B(2, p\lambda + \beta + 1) |f(b)|^\lambda \right)^{\frac{1}{\lambda}}. \end{aligned}$$

Moreover, if we choose $\eta(b, v) = b - v$, we get

$$\begin{aligned} & \int_v^b (\vartheta - v)^p (b - \vartheta)^q f(\vartheta) d\vartheta \\ & \leq \left(\frac{v}{b}\right)^{p+1} (b - v)^{p+q+1} \left(\frac{{}_2F_1\left((p+q+2)k, qk+1, qk+2; 1-\frac{v}{b}\right)}{qk+1} \right)^{\frac{1}{k}} \\ & \quad \times \left(B(p\lambda + \alpha + 1, \beta + 1) |f(v)|^\lambda + B(\alpha + 1, p\lambda + \beta + 1) |f(b)|^\lambda \right)^{\frac{1}{\lambda}}, \end{aligned}$$

where ${}_2F_1$ and B are the hypergeometric and the beta functions respectively.

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