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# Certain Classes of Analytic Functions Associated With $q$-Analogue of $p$-Valent Cătaş Operator

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Abstract. In this paper, we investigate several interesting properties for certain class of analytic functions defined by q-analogue of p-valent Cătaş operator. All our results are sharp.

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## 1. Introduction

Let $\mathcal{A}_j(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=j+p}^{\infty} a_k z^k \ \ (j, p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. We note that $\mathcal{A}_1(p) = \mathcal{A}(p)$ and $\mathcal{A}_1(1) = \mathcal{A}$. Following the investigations of Aouf [10] and Altintas [19], Chen et. al [21], Keerthi et. al [28], Orhan [30] and Srivastava et. al [38], we denote $T_j(p)$ the subclass of $A_j(p)$ consisting of analytic and p-valent functions with negative coefficients which can expressed in the form:

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$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \ \ (a_k \geq 0, \ j, p \in \mathbb{N}). \tag{1.2}$$

A function $f(z) \in T_j(p)$ is said to be p-valently starlike of order $\sigma$ if it satisfies the following inequality:

$$Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \sigma \quad (0 \leq \sigma < p; z \in \mathbb{U}). \tag{1.3}$$

We denote by $T_j^*(p,\sigma)$ the class of all p-valently starlike functions of order $\sigma$. Also a function $f(z) \in T_j(p)$ is said to be p-valently convex of order $\sigma$ if it satisfies the following inequality:

$$Re\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > \sigma \quad (0 \leq \sigma < p; z \in \mathbb{U}). \tag{1.4}$$

We denote by $C_j(p,\sigma)$ the class of all p-valently convex functions of order $\alpha$. It follows from (1.3) and (1.4) that

$$f(z) \in C_j(p,\sigma) \Leftrightarrow \frac{zf^{'}(z)}{p} \in T_j^*(p,\sigma) \ \ (0 \leq \sigma < p). \tag{1.5}$$

The classes $T_j^*(p,\sigma)$ and $C_j(p,\sigma)$ are studied by Owa [31,32], Aouf [8,9,11] and Yamaka [43].

The fractional $q-$calculus and the fractional of $q-$derivative operators in Geometric Function Theory are investigated by sturdy of researchers (see, for example, [1,3,4,7,18,24,26,29,36,37, 39–42,44]). As well as this are generalized in one or more operators. In the general run, the $q-$calculus is used in various fields of mathematics and physics.
For a function $f(z) \in \mathcal{A}(p)$ given by (1.1) (with $j = 1$) and $0 < q < 1$. Jackson's q-derivative (or $q-$difference) $D_{q,p}$ of a function defined on a subset of the complex space $C$ is defined as follows:

$$D_{q,p}f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & for\ z \neq 0, \\ \\ f'(0) & for\ z = 0, \end{cases} \tag{1.6}$$

provided that $f'(0)$ exists. From (1.1) (with $j = 1$) and (1.6), we deduce that

$$D_{q,p}f(z) = [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q a_k z^{k-1}, \tag{1.7}$$

where $[k]_q$ is the q-integer number $k$ defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + \sum_{k=1}^{n-1} q^k, \ [0]_q = 0, \ 0 < q < 1. \tag{1.8}$$

We note that

$$\lim_{q \to 1^-} D_{q,p} f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a function $f$ which is differentiable in a given subset of $\mathbb{C}$. For a function $f(z) \in \mathcal{A}_j(p)$, Aouf and Madian [15] defined the q-analogue p-valent Cătaş operator $I^n_{q,p}(\lambda, l) f(z) : \mathcal{A}_j(p) \to \mathcal{A}_j(p)$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, j, p \in \mathbb{N}, \ l, \lambda \geq 0, \ 0 < q < 1)$ as follows:

$$I^0_{q,p}(\lambda, l) f(z) = f(z),$$

$$\begin{aligned} I^1_{q,p}(\lambda, l) f(z) &= (1-\lambda) f(z) + \frac{\lambda}{[p+l]_q z^{l-1}} D_{q,p}(z^l f(z)) \\ &= z^p + \sum_{k=j+p}^{\infty} \left[ \frac{[p+l]_q + \lambda \left( [k+l]_q - [p+l]_q \right)}{[p+l]_q} \right] a_k z^k, \end{aligned}$$

$$\begin{aligned} I^2_{q,p}(\lambda, l) f(z) &= (1-\lambda) I^1_{q,p}(\lambda, l) f(z) + \frac{\lambda}{[p+l]_q z^{l-1}} D_{q,p}(z^l I^1_{q,p}(\lambda, l) f(z)) \\ &= z^p + \sum_{k=j+p}^{\infty} \left[ \frac{[p+l]_q + \lambda \left( [k+l]_q - [p+l]_q \right)}{[p+l]_q} \right]^2 a_k z^k, \end{aligned}$$

$$\vdots$$

$$I^n_{q,p}(\lambda, l) f(z) = (1-\lambda) I^{n-1}_{q,p}(\lambda, l) f(z) + \frac{\lambda}{[p+l]_q z^{l-1}} D_{q,p}(z^l I^{n-1}_{q,p}(\lambda, l) f(z)) \ (n \in \mathbb{N}). \tag{1.9}$$

From (1.1) and (1.9), we can obtain

$$I^n_{q,p}(\lambda, l) f(z) = z^p + \sum_{k=j+p}^{\infty} \Psi^n_{q,p}(k, \lambda, l) a_k z^k, \tag{1.10}$$

where

$$\Psi^n_{q,p}(k, \lambda, l) = \left[ \frac{[p+l]_q + \lambda \left( [k+l]_q - [p+l]_q \right)}{[p+l]_q} \right]^n \ (n \in \mathbb{N}_0, \ j, p \in \mathbb{N}, l, \ \lambda \geq 0). \tag{1.11}$$

We note that

(i) $\lim_{q\to 1^-} I^n_{q,p}(\lambda,l)f(z) = I^n_p(\lambda,l)f(z)$, $I^n_{q,1}(\lambda,l)f(z) = I^n_q(\lambda,l)f(z)$, $\lim_{q\to 1^-} I^n_q(\lambda,l)f(z) = I^n(\lambda,l)f(z)$ (see Cătaş [20]);

(ii) $I^n_{q,p}(1,l)f(z) = I^n_{q,p}(l)f(z)$

$$= \left\{ \begin{array}{c} f \in \mathcal{A}_j(p) : I^n_{q,p}(l)f(z) = z^p + \sum_{k=j+p}^{\infty} \left(\frac{[k+l]_q}{[p+l]_q}\right)^n a_k z^k, \\ n \in \mathbb{N}_0,\ j,p \in \mathbb{N}, l \geq 0,\ 0<q<1 \end{array} \right\};$$

(iii) $I^n_{q,p}(\lambda,0)f(z) = D^n_{q,p,\lambda}f(z)$

$$= \left\{ \begin{array}{c} f \in \mathcal{A}_j(p) : D^n_{q,p,\lambda}f(z) = z^p + \sum_{k=j+p}^{\infty} \left[\frac{[p]_q+\lambda\left([k]_q-[p]_q\right)}{[p]_q}\right]^n a_k z^k, \\ n \in \mathbb{N}_0,\ j,p \in \mathbb{N},\ \lambda \geq 0,\ 0<q<1 \end{array} \right\};$$

(iv) $I^n_{q,p}(1,0)f(z) = D^n_{q,p}f(z)$

$$= \left\{ \begin{array}{c} f \in \mathcal{A}_j(p) : D^n_{q,p}f(z) = z^p + \sum_{k=j+p}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^k, \\ n \in \mathbb{N}_0,\ j,p \in \mathbb{N},\ 0<q<1 \end{array} \right\},$$

where $\lim_{q\to 1^-} D^n_{q,p}f(z) = D^n_p f(z)$ (see [13,16,27]);

(v) $I^n_{q,1}(1,l)f(z) = I^n_q(l)f(z)$, $\lim_{q\to 1^-} I^n_q(l)f(z) = I^n(l)f(z)$ (see [22,23]);

(vi) $I^n_{q,1}(\lambda,0)f(z) = D^n_{q,\lambda}f(z)$, $\lim_{q\to 1^-} D^n_{q,\lambda}f(z) = D^n_\lambda f(z)$ (see [2,17]);

(vii) $I^n_{q,1}(1,0)f(z) = D^n_q f(z)$ (see [25]), $\lim_{q\to 1^-} D^n_q f(z) = D^n f(z)$ (see Sălăgean [33]) and see also [12,14].

Now by using q-analogue p-valent Cătaş operator $I^n_{q,p}(\lambda,l)f(z)$, we say that $f(z) \in T_j(p)$ is in the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$ *if and only if*

$$Re\left\{\frac{(1-\alpha)zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))+\alpha zD_{q,p}(zD_{q,p}(I^n_{q,p}(\lambda,l)f(z)))}{(1-\alpha)I^n_{q,p}(\lambda,l)f(z)+\alpha zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))}\right\} > \beta$$

$$(n \in \mathbb{N}_0, p,j \in \mathbb{N}, 0<q<1, l,\lambda \geq 0, 0 \leq \alpha \leq 1, 0 \leq \beta < [p]_q). \tag{1.12}$$

We note that:

(i) $\lim_{q\to 1^-} SC^n_{p,q}(j,\lambda,l,\alpha,\beta) = SC^n_p(j,\lambda,l,\alpha,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{z(I^n_p(\lambda,l)f(z))'+\alpha z^2(I^n_p(\lambda,l)f(z))''}{(1-\alpha)I^n_p(\lambda,l)f(z)+\alpha z(I^n_p(\lambda,l)f(z))'}\right\} > \beta\,, 0 \leq \beta < p\right\};$$

(ii) $SC^0_{p,q}(j,\lambda,l,\alpha,\beta) = SC_{p,q}(j,\alpha,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{(1-\alpha)zD_{q,p}f(z)+\alpha zD_{q,p}(zD_{q,p}f(z))}{(1-\alpha)f(z)+\alpha zD_{q,p}f(z)}\right\} > \beta\,, 0 \leq \beta < [p]_q\right\};$$

(iii) $SC^n_{p,q}(j,\lambda,l,0,\beta) = S^n_{p,q}(j,\lambda,l,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))}{I^n_{q,p}(\lambda,l)f(z)}\right\} > \beta \,, 0 \le \beta < [p]_q\right\}$$

$S^0_{p,q}(j,\lambda,l,\beta) = S_{p,q}(j,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{zD_{q,p}f(z)}{f(z)}\right\} > \beta \,, 0 \le \beta < [p]_q\right\};$$

(iv) $SC^n_{p,q}(j,\lambda,l,1,\beta) = C^n_{p,q}(j,\lambda,l,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{D_{q,p}(zD_{q,p}(I^n_{q,p}(\lambda,l)f(z)))}{D_{q,p}(I^n_{q,p}(\lambda,l)f(z))}\right\} > \beta \,, 0 \le \beta < [p]_q\right\}$$

$C^0_{p,q}(j,\lambda,l,\beta) = C_{p,q}(j,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{D_{q,p}(zD_{q,p}f(z))}{D_{q,p}f(z)}\right\} > \beta \,, 0 \le \beta < [p]_q\right\};$$

(vi) $\lim\limits_{q\to 1^-} S_{p,q}(j,\sigma) = T^*_j(p,\sigma) \;\; (0 \le \sigma < p)\,,\; \lim\limits_{q\to 1^-} C_{p,q}(j,\sigma) = C_j(p,\sigma) \;\; (0 \le \sigma < p)\,;$

(vii) $\lim\limits_{q\to 1^-} S^0_{p,q}(j,\lambda,l,\beta) = T(j,p,\beta)$ (see Altintas et al. [6]) and

$\lim\limits_{q\to 1^-} S^0_{1,q}(j,\lambda,l,\beta) = P(j,\beta)$ (see Altintas [5]);

(viii) $\lim\limits_{q\to 1^-} SC^n_{p,q}(j,\lambda,l,\alpha,\beta) = SC^n_p(j,\lambda,l,\alpha,\beta)$

$$\left\{f(z) \in T_j(p) : Re\left\{\frac{z(I^n_p(\lambda,l)f(z))' + \alpha z^2(I^n_p(\lambda,l)f(z))''}{(1-\alpha)I^n_p(\lambda,l)f(z) + \alpha z(I^n_p(\lambda,l)f(z))'}\right\} > \beta \,, 0 \le \beta < p\right\}.$$

The class $SC^n_p(j,\lambda,l,\alpha,\beta)$ explain the important relation between the new class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$ and Catas operator.

## 2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $n \in \mathbb{N}_0$, $p,j \in \mathbb{N}$, $l,\lambda \ge 0$, $0 < q < 1, 0 \le \alpha \le 1$, $0 \le \beta < [p]_q$ and $\Psi^n_{q,p}(k,\lambda,l)$ is given by (1.11).

**Theorem 1.** *Let the function $f(z)$ defined by (1.2). then $f(z)$ belongs to the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$ if and only if*

$$\sum_{k=j+p}^{\infty} ([k]_q - \beta)[1 + \alpha([k]_q - 1)]\Psi^n_{q,p}(k,\lambda,l)a_k \le ([p]_q - \beta)[1 + \alpha([p]_q - 1)]. \tag{2.1}$$

**Proof.** Assume that the inequality (2.1) holds true. Then we have

$$([j+p]_q - \beta)\sum_{k=j+p}^{\infty} [1 + \alpha([k]_q - 1)]\Psi^n_{q,p}(k,\lambda,l)a_k \le$$

$$\sum_{k=j+p}^{\infty}([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k \leq ([p]_q-\beta)[1+\alpha([p]_q-1)],$$

that is, that

$$\sum_{k=j+p}^{\infty}[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k \leq \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)}. \tag{2.2}$$

Since

$$\begin{aligned}
&\left|[1+\alpha([p]_q-1)]-\sum_{k=j+p}^{\infty}([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k z^{k-p}\right| \\
&\geq [1+\alpha([p]_q-1)]-\sum_{k=j+p}^{\infty}([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k \\
&\geq \frac{[1+\alpha([p]_q-1)]([j+p]_q-[p]_q)}{[j+p]_q-\beta}>0.
\end{aligned}$$

Then we find that

$$\begin{aligned}
&\left|\frac{(1-\alpha)zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))+\alpha zD_{q,p}(zD_{q,p}(I^n_{q,p}(\lambda,l)f(z)))}{(1-\alpha)I^n_{q,p}(\lambda,l)f(z)+\alpha zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))}-[p]_q\right| \\
&\leq \frac{\sum\limits_{k=j+p}^{\infty}([k]_q-[p]_q)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\,|z|^{k-p}}{[1+\alpha([p]_q-1)]-\sum\limits_{k=j+p}^{\infty}[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\,|z|^{k-p}} \\
&\leq \frac{\sum\limits_{k=j+p}^{\infty}([k]_q-[p]_q)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k}{[1+\alpha([p]_q-1)]-\sum\limits_{k=j+p}^{\infty}[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k} \leq [p]_q-\beta.
\end{aligned}$$

This shows that the values of the function

$$\Phi(z)=\frac{(1-\alpha)zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))+\alpha zD_{q,p}(zD_{q,p}(I^n_{q,p}(\lambda,l)f(z)))}{(1-\alpha)I^n_{q,p}(\lambda,l)f(z)+\alpha zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))}, \tag{2.3}$$

lie in a circle which is centered at $w=[p]_q$ and whose radius is $([p]_q-\beta)$. Hence $f(z)$ satisfies the condition (1.12).

Conversely, assume that the function $f(z)$ is in the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then we have

$$Re\left\{\frac{(1-\alpha)zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))+\alpha zD_{q,p}(zD_{q,p}(I^n_{q,p}(\lambda,l)f(z)))}{(1-\alpha)I^n_{q,p}(\lambda,l)f(z)+\alpha zD_{q,p}(I^n_{q,p}(\lambda,l)f(z))}\right\}$$

$$= Re\left\{\frac{[p]_q[1+\alpha([p]_q-1)]-\sum\limits_{k=j+p}^{\infty}[k]_q[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\ z^{k-p}}{[1+\alpha([p]_q-1)]-\sum\limits_{k=j+p}^{\infty}[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\ z^{k-p}}\right\} > \beta. \tag{2.4}$$

For some $n \in \mathbb{N}_0,\ p,j \in \mathbb{N},\ l,\lambda \geq 0,\ 0 < q < 1, 0 \leq \alpha \leq 1,\ 0 \leq \beta < [p]_q$ and $z \in \mathbb{U}$. Choose values of $z$ on real axis so that $\Phi(z)$ given by (2.3) is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we can see that

$$\begin{aligned}&[p]_q\,[1+\alpha([p]_q-1)] - \sum_{k=j+p}^{\infty}[k]_q\,[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\\ &\geq \beta\left\{[1+\alpha([p]_q-1)] - \sum_{k=j+p}^{\infty}[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)a_k\right\}.\end{aligned} \tag{2.5}$$

Thus we have the inequality (2.1). This completes the proof.

**Corollary 1.** *Let the function $f(z)$ defined by (1.2) be in the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then*

$$a_k \leq \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}(k \geq j+p, j,p \in \mathbb{N}). \tag{2.6}$$

*The result is sharp for the function $f(z)$ given by*

$$f(z) = z^p - \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}z^k$$

$$(k \geq j+p,\ p,j \in \mathbb{N},\ l,\lambda \geq 0,\ 0 < q < 1, 0 \leq \alpha \leq 1,\ 0 \leq \beta < [p]_q). \tag{2.7}$$

Putting $\alpha = 0$ in Theorem 1, we obtain the following corollary.
**Corollary 2.** *Let the function $f(z)$ defined by (1.1). Then $f(z) \in S^n_{p,q}(j,\lambda,l,\beta)$ if and only if*

$$\sum_{k=j+p}^{\infty}([k]_q-\beta)\Psi^n_{q,p}(k,\lambda,l)a_k \leq ([p]_q-\beta).$$

Putting $\alpha = 1$ in Theorem 1, we obtain the following corollary.
**Corollary 3.** *Let the function $f(z)$ defined by (1.1). Then $f(z) \in C^n_{p,q}(j,\lambda,l,\beta)$ if and only if*

$$\sum_{k=j+p}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)([k]_q-\beta)\Psi^n_{q,p}(k,\lambda,l)a_k \leq ([p]_q-\beta).$$

**Remark**. Putting $n = 0,\ j = 1$ and letting $q \to 1^-$ in Corollaries 2 and 3, we obtain the results obtained by Owa [32] and Aouf [9] with $A = 1$ and $B = -1$.

## 3. Distortion theorems

**Theorem 2.** *Let a function $f(z)$ of the form (1.2) belongs to the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then*

$$|z|^p - \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^{j+p} \le |f(z)|$$
$$\le |z|^p + \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^{j+p} \quad (z\in\mathbb{U}). \tag{3.1}$$

*The result is sharp.*

**Proof.** Since $f(z)\in SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$, in view of Theorem 1, we have

$$\frac{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}\sum_{k=j+p}^{\infty}a_k$$
$$\le \sum_{k=j+p}^{\infty}\frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}a_k \le 1, \tag{3.2}$$

which evidently yields

$$\sum_{k=j+p}^{\infty}a_k \le \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}. \tag{3.3}$$

Consequently, we obtain

$$|f(z)| \ge |z|^p - |z|^{j+p}\sum_{k=j+p}^{\infty}a_k$$
$$\ge |z|^p - \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^{j+p},$$

and

$$|f(z)| \le |z|^p + |z|^{j+p}\sum_{k=j+p}^{\infty}a_k$$
$$\le |z|^p + \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^{j+p},$$

which prove the assertion (3.1) of Theorem 2. The bounds in (3.1) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}z^{j+p}. \tag{3.4}$$

**Theorem 3.** *Let a function $f(z)$ in the form (1.2) belongs to the class $SC_{p,q}^{n}(j,\lambda,l,\alpha,\beta)$. Then*

$$\left|f'(z)\right| \geq p\,|z|^{p-1} - \frac{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}\,|z|^{j+p-1} \tag{3.5}$$

*and*

$$\left|f'(z)\right| \leq p\,|z|^{p-1} + \frac{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}\,|z|^{j+p-1} \quad (z\in\mathbb{U})\,. \tag{3.6}$$

*The bounds in (3.5) and (3.6) are attained for the function $f(z)$ given by (3.4).*

**Proof.** From Theorem 1, we have

$$\frac{([j+p]_q-\beta)[1+\alpha([p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}\sum_{k=j+p}^{\infty}ka_k \leq 1,$$

which evidently yields

$$\sum_{k=j+p}^{\infty}ka_k \leq \frac{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}. \tag{3.7}$$

Consequently, we obtain

$$\begin{aligned}\left|f'(z)\right| &\geq p\,|z|^{p-1} - |z|^{j+p-1}\sum_{k=j+p}^{\infty}ka_k \\ &\geq p\,|z|^{p-1} - \frac{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}\,|z|^{j+p-1}\,,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\left|f'(z)\right| &\leq p\,|z|^{p-1} + |z|^{j+p-1}\sum_{k=j+p}^{\infty}ka_k \\ &\leq p\,|z|^{p-1} + \frac{(j+p)\,([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)}\,|z|^{j+p-1}\,,\end{aligned} \tag{3.9}$$

which prove the assertions (3.5) and (3.6) of Theorem 2. The bounds in (3.5) and (3.6) are attained for the function $f(z)$ given by (3.4).

**Corollary 4.** *Under the hypothesis of Theorem 2, $f(z)$ is included in the disc with center at the origin and radius $r$ given by*

$$r = 1 + \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi_{q,p}^{n}(j+p,\lambda,l)},$$

*and $f'(z)$ is included in the disc with center at the origin and radius $R$ given by*

$$R = p + \frac{(j+p)\left([p]_q - \beta\right)[1+\alpha([p]_q - 1)]}{([j+p]_q - \beta)[1+\alpha([j+p]_q - 1)]\Psi^n_{q,p}(j+p,\lambda,l)}.$$

**Theorem 4.** *Let a function $f(z)$ in the form (1.2) belongs to the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then*

$$\left\{\frac{p!}{(p-m)!} - \frac{(j+p)!([p]_q-\beta)[1+\alpha([p]_q-1)]}{(j+p-m)!([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^j\right\}|z|^{p-m}$$

$$\leq \left|f^{(m)}(z)\right| \leq \left\{\frac{p!}{(p-m)!} + \frac{(j+p)!([p]_q-\beta)[1+\alpha([p]_q-1)]}{(j+p-m)!([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}\,|z|^j\right\}|z|^{p-m}$$

$$(n \in \mathbb{N}_0,\ p,j \in \mathbb{N},\ l,\lambda \geq 0,\ 0 < q < 1, 0 \leq \alpha \leq 1,\ 0 \leq \beta < [p]_q, p > m). \tag{3.10}$$

**Proof.** From Theorem 1, we have

$$\frac{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}{(j+p)!([p]_q-\beta)[1+\alpha([p]_q-1)]}\sum_{k=j+p}^{\infty} k!a_k$$

$$\leq \sum_{k=j+p}^{\infty} \frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}a_k \leq 1,$$

which evidently yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(j+p)!([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)}. \tag{3.11}$$

Now, by differentiating both sides of (1.2) $m$ times, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!}z^{p-m} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-m)!}z^{k-m},\ (k \geq j+p, p, j \in \mathbb{N}, p > m), \tag{3.12}$$

and Theorem 4 follows from (3.11) and (3.12), respectively.

**Remark.** Putting $m = 0$ and $m = 1$, respectively, in Theorem 4, we obtain Theorems 2 and 3, respectively.

## 4. Radii of p- valently close-to-convexity, starlikeness and convexity

Our results in this section (Theorems 5 and 6 and Corollary 5 below) provide the radii of p-valently close-to-convexity, starlikeness and convexity for the class $SC_{p,q}^{n}(j,\lambda,l,\alpha,\beta)$ .

**Theorem 5**. *Let a function $f(z)$ in the form (1.2) belongs to the class $SC_{p,q}^{n}(j,\lambda,l,\alpha,\beta)$. Then $f(z)$ is p-valently close-to-convex of order $\delta$ $(0\leq\delta<p)$ in the disc $|z|<r_1$, where*

$$r_1=\inf_k\left\{\frac{(p-\delta)([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi_{q,p}^{n}(k,\lambda,l)}{k([p]_q-\beta)[1+\alpha([p]_q-1)]}\right\}^{\frac{1}{k-p}}\ (k\geq j+p,p,j\in\mathbb{N},0<q<1). \tag{4.1}$$

*The result is sharp with the extremal function $f(z)$ given by (2.7).*

**Proof.** From (1.2), we easily get

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq\sum_{k=j+p}^{\infty}ka_k\left|z\right|^{k-p}. \tag{4.2}$$

Thus we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\delta,$$

if

$$\sum_{k=j+p}^{\infty}\left(\frac{k}{p-\delta}\right)a_k\left|z\right|^{k-p}\leq 1,$$

that is, if

$$\left(\frac{k}{p-\delta}\right)\left|z\right|^{k-p}\leq\frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi_{q,p}^{n}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}, \tag{4.3}$$

where we have made used the assertion (2.1) of Theorem 1. The last inequality (4.3) leads us immediately to the disc $|z|<r_1$, where $r_1$ given by (4.1), and the proof of Theorem 5 is completed.

**Theorem 6**. *Let a function $f(z)$ in the form (1.2) belongs to the class $SC_{p,q}^{n}(j,\lambda,l,\alpha,\beta)$. Then $f(z)$ is p-valently starlike of order $\delta$ $(0\leq\delta<p)$ in the disc $|z|<r_2$, where*

$$r_2=\inf_k\left\{\frac{(p-\delta)([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi_{q,p}^{n}(k,\lambda,l)}{(k-\delta)([p]_q-\beta)[1+\alpha([p]_q-1)]}\right\}^{\frac{1}{k-p}}\ (k\geq j+p,p,j\in\mathbb{N},0<q<1). \tag{4.4}$$

*The result is sharp with the extremal function $f(z)$ given by (2.7).*

**Proof.** Making use of the definition (1.2), we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum\limits_{k=j+p}^{\infty}(k-p)a_k\,|z|^{k-p}}{1-\sum\limits_{k=j+p}^{\infty}a_k\,|z|^{k-p}} \leq p-\delta,$$

if

$$\sum_{k=j+p}^{\infty}\left(\frac{k-\delta}{p-\delta}\right)a_k\,|z|^{k-p} \leq 1,$$

that is, if

$$\left(\tfrac{k-p}{p-\delta}\right)|z|^{k-p} \leq \frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}, (k \geq j+p, p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0<q<1), \tag{4.5}$$

where we have also used the assertion (2.1) of Theorem 1. The last inequality (4.5) leads us to the disc $|z| < r_2$, where $r_2$ given by (4.4) and the proof of Theorem 6 is evidently completed.

**Corollary 5**. *Let a function $f(z)$ in the form (1.2) belongs to the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then $f(z)$ is p-valently convex of order $\delta$ $(0 \leq \delta < p)$ in the disc $|z| < r_3$, where*

$$r_3 = \inf_k \left\{\frac{(p-\delta)([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{k(k-\delta)([p]_q-\beta)[1+\alpha([p]_q-1)]}\right\}^{\frac{1}{k-p}} (k \geq j+p, p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0<q<1). \tag{4.6}$$

*The result is sharp with the extremal function $f(z)$ given by (2.7).*

## 5. Modified Hadamard products

For the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by

$$f_\nu(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,\nu}z^k \;(a_{k,\nu} \geq 0,\ \nu = 1,2), \tag{5.1}$$

we denote by $(f_1 * f_2)(z)$ modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$ defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1}a_{k,2}z^k\ . \tag{5.2}$$

**Theorem 7**. *Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by (5.1) be in the class $SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$. Then $(f_1 * f_2)(z) \in SC^n_{p,q}(j,\lambda,l,\alpha,\beta,\gamma)$, where*

$$\gamma = [p]_q - \frac{([j+p]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)^2[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)-([p]_q-\beta)^2[1+\alpha([p]_q-1)]}. \tag{5.3}$$

*The result is sharp for the functions*

$$f_\nu(z) = z^p - \frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)} z^{j+p} \ (\nu = 1,2, p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0 < q < 1). \quad (5.4)$$

**Proof.** Employing the technique used earlier by Schild and Silverman [35], we need to find the largest $\gamma$ such that

$$\sum_{k=j+p}^{\infty} \frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]} a_{k,1}a_{k,2} \leq 1 \ (f_\nu(z) \in SC^n_{p,q}(j,\lambda,l,\alpha,\beta), \ \nu = 1,2).$$

Since $f_\nu(z) \in SC^n_{p,q}(j,\lambda,l,\alpha,\beta)$ $(\nu = 1,2)$, we readily see that

$$\sum_{k=j+p}^{\infty} \frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]} a_{k,\nu} \leq 1 \ (\nu = 1,2). \quad (5.5)$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=j+p}^{\infty} \frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]} \sqrt{a_{k,1}a_{k,2}} \leq 1\ . \quad (5.6)$$

Thus we only need to show that

$$\frac{([k]_q-\gamma)}{([p]_q-\gamma)} a_{k,1}a_{k,2} \leq \frac{([k]_q-\beta)}{([p]_q-\beta)} \sqrt{a_{k,1}a_{k,2}} \ (k \geq j+p, p, j \in \mathbb{N}), \quad (5.7)$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{([k]_q-\gamma)([p]_q-\beta)}{([p]_q-\beta)([k]_q-\gamma)} \ \ (k \geq j+p, p, j \in \mathbb{N}). \quad (5.8)$$

Hence, in light of the inequality (5.6), it is sufficient to prove that

$$\frac{([p]_q-\beta)[1+\alpha([p]_q-1)]}{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)} \leq \frac{([p]_q-\gamma)}{([p]_q-\beta)} \frac{([k]_q-\beta)}{([k]_q-\gamma)} \ (k \geq j+p, p, j \in \mathbb{N}). \quad (5.9)$$

It follows from (5.9) that

$$\gamma \leq [p]_q - \frac{([k]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([k]_q-\beta)^2[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)-([p]_q-\beta)^2[1+\alpha([p]_q-1)]} (k \geq j+p, p, j \in \mathbb{N}). \quad (5.10)$$

Now, defining the function $G_q(k)$ by

$$G_q(k) = [p]_q - \frac{([k]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([k]_q-\beta)^2[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)-([p]_q-\beta)^2[1+\alpha([p]_q-1)]}, \quad (5.11)$$

we see that $G_q(k)$ is an increasing function of $k$. Therefore we conclude that

$$\gamma \leq G_q(j+p) = [p]_q - \frac{([j+p]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)^2[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)-([p]_q-\beta)^2[1+\alpha([p]_q-1)]}, \quad (5.12)$$

which evidently completes the proof of Theorem 7.

Putting $\alpha = 0$ and $\alpha = 1$, respectively, in Theorem 7, we obtain

**Corollary 6.** *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by (5.1) be in the class* $S^n_{p,q}(j, \lambda, l, \beta)$. *Then* $(f_1 * f_2)(z) \in S^n_{p,q}(j, \lambda, l, \gamma)$, *where*

$$\gamma = [p]_q - \frac{([j+p]_q - [p]_q)([p]_q - \beta)^2}{([j+p]_q - \beta)^2 \Psi^n_{q,p}(j+p, \lambda, l) - ([p]_q - \beta)^2} (p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0 < q < 1).$$

*The result is sharp.*

**Corollary 7.** *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by (5.1) be in the class* $C^n_{p,q}(j, \lambda, l, \beta)$. *Then* $(f_1 * f_2)(z) \in C^n_{p,q}(j, \lambda, l, \gamma)$, *where*

$$\gamma = [p]_q \left\{ 1 - \frac{([j+p]_q - [p]_q)([p]_q - \beta)^2}{[j+p]_q([j+p]_q - \beta)^2 \Psi^n_{q,p}(j+p, \lambda, l) - [p]_q([p]_q - \beta)^2} \right\} (p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0 < q < 1).$$

*The result is sharp.*

Using the arguments similar to those in the proof of Theorem 7, we obtain the following result.

**Theorem 8**. *Let the function* $f_1(z)$ *defined by (5.1) be in the class* $SC^n_{p,q}(j, \lambda, l, \alpha, \beta)$.

*Suppose also that the function* $f_2(z)$ *defined by (5.1) be in the class* $SC^n_{p,q}(j, \lambda, l, \alpha, \eta)$. *Then* $(f_1 * f_2)(z) \in SC^n_{p,q}(j, \lambda, l, \alpha, \beta, \zeta)$, *where*

$$\zeta = [p]_q - \frac{([j+p]_q - [p]_q)([p]_q - \beta)([p]_q - \eta)[1 + \alpha([p]_q - 1)]}{([j+p]_q - \beta)([j+p]_q - \eta)[1 + \alpha([j+p]_q - 1)]\Psi^n_{q,p}(j+p, \lambda, l) - \Omega}, \tag{5.13}$$

*where*

$$\Omega = ([p]_q - \beta)([p]_q - \eta)[1 + \alpha([p]_q - 1)]. \tag{5.14}$$

*The result is the best possible for the functions*

$$f_1(z) = z^p - \frac{([p]_q - \beta)[1 + \alpha([p]_q - 1)]}{([j+p]_q - \beta)[1 + \alpha([j+p]_q - 1)]\Psi^n_{q,p}(j+p, \lambda, l)} z^{j+p} \ (p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0 < q < 1), \tag{5.15}$$

*and*

$$f_2(z) = z^p - \frac{([p]_q - \eta)[1 + \alpha([p]_q - 1)]}{([j+p]_q - \eta)[1 + \alpha([j+p]_q - 1)]\Psi^n_{q,p}(j+p, \lambda, l)} z^{j+p} \ (p, j \in \mathbb{N}, n \in \mathbb{N}_0, 0 < q < 1). \tag{5.16}$$

**Theorem 9**. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2, ..., m)$ *defined by (5.1) be in the class* $SC^n_{p,q}(j, \lambda, l, \alpha, \beta)$. *Then the function*

$$f(z) = z^p - \sum_{k=j+p}^{\infty} \left( \sum_{\nu=1}^{m} a^2_{k,\nu} z^k \right) , \tag{5.17}$$

*belongs to the class* $SC^{n}_{p,q}(j,\lambda,l,\alpha,\xi)$, *where*

$$\xi=[p]_q-\frac{m([j+p]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)^2[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)-m([p]_q-\beta)^2[1+\alpha([p]_q-1)]}. \tag{5.18}$$

*The result is the best possible for the functions* $f_\nu(z)$ $(\nu=1,2,...,m)$ *given by (5.4).*

**Proof.** Noting that

$$\sum_{k=j+p}^{\infty}\left\{\frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}\right\}^2 a^2_{k,\nu}$$
$$\leq \sum_{k=j+p}^{\infty}\left\{\frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}a_{k,\nu}\right\}^2,$$
$$(f_\nu(z)\in SC^n_{p,q}(j,\lambda,l,\alpha,\beta),\ (\nu=1,2,...,m)), \tag{5.19}$$

we have

$$\sum_{k=j+p}^{\infty}\frac{1}{m}\left\{\frac{([k]_q-\beta)[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{([p]_q-\beta)[1+\alpha([p]_q-1)]}\right\}^2\left(\sum_{k=j+p}^{\infty}a^2_{k,\nu}\right)\leq 1. \tag{5.20}$$

Therefore, we have find the largest $\xi$ such that

$$\frac{([k]_q-\xi)}{([p]_q-\xi)}\leq\frac{([k]_q-\beta)^2[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)}{m([p]_q-\beta)^2[1+\alpha([p]_q-1)]}\quad (k\geq j+p,p,j\in\mathbb{N}), \tag{5.21}$$

that is, that

$$\xi\leq[p]_q-\frac{m([k]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([k]_q-\beta)^2[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)-m([p]_q-\beta)^2[1+\alpha([p]_q-1)]}\ (k\geq j+p,p,j\in\mathbb{N}). \tag{5.22}$$

Now, defining the function $\Psi_q(k)$ by

$$\Psi_q(k)=[p]_q-\frac{m([k]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([k]_q-\beta)^2[1+\alpha([k]_q-1)]\Psi^n_{q,p}(k,\lambda,l)-m([p]_q-\beta)^2[1+\alpha([p]_q-1)]}$$
$$(k\geq j+p,p,j\in\mathbb{N}). \tag{5.23}$$

We observe that $\Psi_q(k)$ an increasing function of $k$. Therefore we conclude that

$$\xi\leq\Psi_q(j+p)=[p]_q-\frac{m([j+p]_q-[p]_q)([p]_q-\beta)^2[1+\alpha([p]_q-1)]}{([j+p]_q-\beta)^2[1+\alpha([j+p]_q-1)]\Psi^n_{q,p}(j+p,\lambda,l)-m([p]_q-\beta)^2[1+\alpha([p]_q-1)]}, \tag{5.24}$$

which completes the proof of Theorem 9.

**Remarks.** *(i) Letting* $q\to 1^-$ *in our results, we obtain new results for the class* $SC^n_p(j,\lambda,l,\alpha,\beta)$;
(ii) *Putting* $n=0$ *in our results, we obtain new results for the class* $SC^n_{p,q}(j,\alpha,\beta)$;

(iii) *Putting* $\alpha = 0$ *and* $\alpha = 1$*, respectively, in our results, we obtain new results for the classes* $S^n_{p,q}(j,\lambda,l,\beta)$ *and* $C^n_{p,q}(j,\lambda,l,\beta)$*, respectively;*
(iv) *(a) Letting* $q \to 1^-$ *and putting* $n = \alpha = 0$ *(b) Letting* $q \to 1^-$ *and putting* $n = 0$ *and* $\alpha = 1$ *in our results, we obtain the results obtained by Owa [31] and Sălăgean et al. [34];*
(v) *Letting* $q \to 1^-$ *and putting* $n = 0$ *and* $p = 1$ *in our results, we obtain new results for the class* $P(j,\beta)$.

## Compliance with ethical standards

*Ethical approval.* This article does not contain any studies with human participants or animals performed by any of the authors.
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