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Bu's theorem in the positive situation

AMAR BOUGOUTAIA¹ AND AMAR BELACEL²

ABSTRACT. In this paper, we valorize the relationship between positive p -summing operators and positive strongly q -summing operators using (Contemp. Math. 328, 145 – 149 (2003)).

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1. Introduction and preliminaries

Let E be a Banach lattice. We denote by E^+ the positive cone $\{x \in E, x \geq 0\}$. For $x \in E$, let $x^+ := \sup \{x, 0\}$, $x^- := \sup \{-x, 0\}$, be the positive part, the negative part of x , respectively. For any $x \in E$, we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Let F be a Banach lattice, and let X and Y be Banach spaces. X^* denote the topological dual of X and B_X denote its closed unit ball.

A linear operator $u : E \rightarrow F$ is called positive if $u(E^+) \subset F^+$.

Let $1 \leq p \leq \infty$, p^* is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$, we denote by $\ell_p^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

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¹ *Laboratory of Pure and Applied Mathematics (LPAM), University of Laghouat, Laghouat, Algeria.
e-mail: amarbou28@gmail.com (Corresponding Author)*

² *Laboratory of Pure and Applied Mathematics (LPAM), University of Laghouat, Laghouat, Algeria.
e-mail: amarbelacel@yahoo.fr.*

$$\begin{cases} \|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}; & 1 \leq p < \infty. \\ \|(x_i)_{i=1}^n\|_\infty = \sup_{1 \leq i \leq n} \|x_i\|; & p = \infty \end{cases}$$

and by $\ell_{p,weak}^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\begin{cases} w_p((x_i)_{i=1}^n) = \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n(X)} = \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, \varphi \rangle|^p \right)^{\frac{1}{p}}; & 1 \leq p < \infty. \\ \|(x_i)_{i=1}^n\|_{\ell_{\infty,weak}^n(X)} = \|(x_i)_{i=1}^n\|_\infty = \sup_{1 \leq i \leq n} \|x_i\|; & p = \infty. \end{cases}$$

Consider the case where X is replaced by a Banach lattice E , and define

$$\ell_{p,|weak|}^n(E) := \left\{ (x_i)_{i=1}^n : (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E) \right\}.$$

The compact set $B_{E^*}^+ = \{\xi \in B_{E^*} : \xi \geq 0\} = B_{E^*} \cap E^{*+}$.

If $x_1, \dots, x_n \geq 0$, we have that

$$\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x_i, \xi \rangle^p \right)^{\frac{1}{p}} = w_p((x_i)_{i=1}^n).$$

Definition 1.1 ([2]). Let $1 \leq p < \infty$. An operator $u : E \rightarrow X$ is said to be positive p -summing if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}, x_1, \dots, x_n \in E$, the inequality

$$\|u(x_i)_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)}. \quad (1.1)$$

For $p = \infty$

$$\sup_{1 \leq i \leq n} \|u(x_i)\| \leq C \|(x_i)_{i=1}^n\|_{\ell_{\infty,|weak|}^n(E)}.$$

We denote by $\Pi_p^+(E, X)$, the space of positive p -summing operators from E into X . $\Pi_p^+(E, X)$ becomes a Banach space with the norm $\pi_p^+(\cdot)$, such that $\pi_p^+(u)$ given by the infimum of the constant $C > 0$ that verify the inequality (1.1). We have $\Pi_\infty^+(E, X) = \mathcal{L}(E, X)$.

Definition 1.2 ([1]). Let $1 \leq p < \infty$. An operator $u : X \rightarrow F$ is positive strongly p -summing if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}, (x_i)_{i=1}^n \subset X, (y_i^*)_{i=1}^n \subset F^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq C \|(x_i)_{i=1}^n\|_p \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}. \quad (1.2)$$

The class of all positive strongly p -summing operators between X and F is denoted by $\mathcal{D}_p^+(X, F)$, the infimum of all the constant C in the inequality (1.2) defines a norm $d_p^+(\cdot)$ on $\mathcal{D}_p^+(X, F)$.

Formally, the Rademacher functions

$$r_n(t) : [0, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}$$

are defined by setting $r_n(t) = \text{sign}(\sin 2^n \pi t)$.

Khinchin's Inequality [6, p 10].

For any $0 < p < +\infty$, there are positive constants A_p, B_p , such that for any scalars $a_1, a_2, \dots, a_n$ we have

$$A_p \cdot \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \cdot \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}}.$$

Kahane's Inequality [6, p 211].

If $0 < p, q < +\infty$, then there is a constant $K_{p,q} > 0$ for which

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^p dt \right)^{1/p}.$$

2. The main result

In [3], Q. Bu generalized Theorem 4.2.2 of [5] for all p and q ($1 < p, q < +\infty$), i.e. $\Pi_p(H, Y) \subset \mathcal{D}_q(H, Y)$, where H is a Hilbert space. In this section, we compare the space of positive p -summing operators and that of positive strongly q -summing operators by generalizing the last result to the positive situation. We prove that:

Theorem 2.1 *Let $1 < p, q < +\infty$; and let H be a Hilbert space. Then*

$$\Pi_p^+(H, F) \subset \mathcal{D}_q^+(H, F).$$

Proof. First consider $H = \ell_2^n$ ($n \in \mathbb{N}$). Let $u \in \Pi_p^+(H, F)$. By Pietsch's domination theorem [1, Proposition 3.4] there exists a probability measure μ on the set $B_{\ell_2^n}^+$ provided with the weak* topology, and a constant $C > 0$, such that for any $x \in \ell_2^n$

$$\|u(x)\| \leq \pi_p^+(u) \left(\int_{B_{\ell_2^n}^+} \langle |x|, x^* \rangle^p d\mu(x^*) \right)^{\frac{1}{p}}. \quad (2.1)$$

Now for $x_1, \dots, x_m \in \ell_2^n$ and $y_1^*, \dots, y_m^* \in F^*$, we have

$$x_k = \sum_{i=1}^n x_{k,i} e_i, \quad k = 1, 2, \dots, m$$

Then

$$\begin{aligned} \sum_{k=1}^m |\langle u(x_k), y_k^* \rangle| &= \sum_{k=1}^m |\langle u(x_k), y_k^{*+} - y_k^{*-} \rangle| \\ &= \sum_{k=1}^m \left| \left\langle \sum_{i=1}^n x_{k,i} u(e_i), y_k^{*+} - y_k^{*-} \right\rangle \right| \\ &\leq \sum_{k=1}^m \left(\left| \left\langle \sum_{i=1}^n x_{k,i} u(e_i), y_k^{*+} \right\rangle \right| + \left| \left\langle \sum_{i=1}^n x_{k,i} u(e_i), y_k^{*-} \right\rangle \right| \right) \\ &\leq \sum_{k=1}^m \left(\sum_{i=1}^n (x_{k,i})^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\times \left[\left(\sum_{i=1}^n |\langle u(e_i), y_k^{*+} \rangle|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^n |\langle u(e_i), y_k^{*-} \rangle|^2 \right)^{\frac{1}{2}} \right],$$

by Khinchin's inequality,

$$\begin{aligned} &\leq \sum_{k=1}^m \|x_k\| \cdot \frac{1}{A_{q^*}} \left(\int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*+} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad + \sum_{k=1}^m \|x_k\| \frac{1}{A'_{q^*}} \left(\int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*-} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\leq \frac{1}{A_{q^*}} \left(\sum_{k=1}^m \|x_k\|^q \right)^{\frac{1}{q}} \cdot \left(\sum_{k=1}^m \int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*+} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad + \frac{1}{A'_{q^*}} \left(\sum_{k=1}^m \|x_k\|^q \right)^{\frac{1}{q}} \cdot \left(\sum_{k=1}^m \int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*-} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &= \|(x_k)_1^m\|_q \frac{1}{A_{q^*}} \left(\int_0^1 \sum_{k=1}^m \left| \left\langle \sum_{i=1}^n r_i(t) u(e_i), y_k^{*+} \right\rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad + \left(\sum_{k=1}^m \|x_k\|^q \right)^{\frac{1}{q}} \frac{1}{A'_{q^*}} \left(\sum_{k=1}^m \int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*-} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &= \left(\sum_{k=1}^m \|x_k\|^q \right)^{\frac{1}{q}} \frac{1}{A_{q^*}} \left(\sum_{k=1}^m \int_0^1 \left| \sum_{i=1}^n r_i(t) \langle u(e_i), y_k^{*+} \rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad + \|(x_k)_1^m\|_q \frac{1}{A'_{q^*}} \left(\int_0^1 \sum_{k=1}^m \left| \left\langle \sum_{i=1}^n r_i(t) u(e_i), y_k^{*-} \right\rangle \right|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\leq \|(x_k)_1^m\|_q \frac{1}{A_{q^*}} \left(\int_0^1 \left(\left\| \sum_{i=1}^n r_i(t) u(e_i) \right\| \cdot \|y_k^{*+}\|_{\ell_{q^*,|weak|}(F^*)} \right)^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad + \|(x_k)_1^m\|_q \frac{1}{A'_{q^*}} \left(\int_0^1 \left(\left\| \sum_{i=1}^n r_i(t) u(e_i) \right\| \cdot \|y_k^{*-}\|_{\ell_{q^*,|weak|}(F^*)} \right)^{q^*} dt \right)^{\frac{1}{q^*}} \\ &= \|(x_k)_1^m\|_q \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^{q^*} dt \right)^{\frac{1}{q^*}} \\ &\quad \times \left(\frac{1}{A_{q^*}} \|y_k^{*+}\|_{\ell_{q^*,|weak|}(F^*)} + \frac{1}{A'_{q^*}} \|y_k^{*-}\|_{\ell_{q^*,|weak|}(F^*)} \right). \end{aligned}$$

Consider $\frac{1}{A''_{q^*}} = \max \left\{ \frac{1}{A_{q^*}}, \frac{1}{A'_{q^*}} \right\}$, then

$$\sum_{k=1}^m |\langle u(x_k), y_k^* \rangle| \leq \frac{2}{A_{q^*}''} \|(x_k)_1^m\|_q \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^{q^*} dt \right)^{\frac{1}{q^*}} \|y_k^*\|_{\ell_{q^*, |weak|}(F^*)}. \quad (2.2)$$

Now by Kahane's inequality and (2.1), we have

$$\begin{aligned} \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^{q^*} dt \right)^{\frac{1}{q^*}} &\leq K_{p,q^*} \cdot \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^p dt \right)^{\frac{1}{p}} \\ &\leq K_{p,q^*} \cdot \pi_p^+(u) \cdot \left(\int_0^1 \left(\int_{B_{\ell_2^n}^+} \left\langle \left| \sum_{i=1}^n r_i(t) (e_i) \right|, x^* \right\rangle^p d\mu(x^*) \right) dt \right)^{\frac{1}{p}} \\ &\leq K_{p,q^*} \cdot \pi_p^+(u) \cdot \left(\int_0^1 \left(\int_{B_{\ell_2^n}^+} \left\langle \sum_{i=1}^n |r_i(t) (e_i)|, x^* \right\rangle^p d\mu(x^*) \right) dt \right)^{\frac{1}{p}} \\ &\leq K_{p,q^*} \cdot \pi_p^+(u) \cdot \left(\int_{B_{\ell_2^n}^+} \left(\int_0^1 \left\langle \sum_{i=1}^n |r_i(t) (e_i)|, x^* \right\rangle^p dt \right) d\mu(x^*) \right)^{\frac{1}{p}}, \end{aligned}$$

by Khinchin's inequality

$$\begin{aligned} \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^{q^*} dt \right)^{\frac{1}{q^*}} &\leq K_{p,q^*} \cdot \pi_p^+(u) \cdot \left(\int_{B_{\ell_2^n}^+} \left(B_p \left(\sum_{i=1}^n \langle e_i, x^* \rangle^2 \right)^{\frac{1}{2}} \right) d\mu(x^*) \right)^{\frac{1}{p}} \\ &\leq K_{p,q^*} \cdot \pi_p^+(u) \cdot B_p \left(\int_{B_{\ell_2^n}^+} \|x^*\|^p d\mu(x^*) \right)^{\frac{1}{p}}, \end{aligned}$$

then

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) u(e_i) \right\|^{q^*} dt \right)^{\frac{1}{q^*}} \leq K_{p,q^*} \pi_p^+(u) B_p. \quad (2.3)$$

Combining (2.2) and (2.3)

$$\sum_{k=1}^m |\langle u(x_k), y_k^* \rangle| \leq \frac{2}{A_{q^*}''} K_{p,q^*} \cdot \pi_p^+(u) \cdot B_p \|(x_k)_1^m\|_q \|y_k^*\|_{\ell_{q^*, |weak|}(F^*)}. \quad (2.4)$$

So by [1, Definition 4.1], we obtain $u \in \mathcal{D}_q^+(\ell_2^n, F)$ with

$$d_q^+(u) \leq \frac{2}{A_{q^*}''} K_{p,q^*} \cdot B_p \cdot \pi_p^+(u).$$

Now, consider a general Hilbert space H . Let $x_1, \dots, x_m \in H$ and $y_1^*, \dots, y_m^* \in F^*$. Then there is an $n \in \mathbb{N}$ such that $\text{span} \{x_k\}_{k=1}^m$ is isometrically isomorphic to ℓ_2^n . Let P be the orthogonal projection from H onto $\text{span} \{x_k\}_{k=1}^m$.

Then by (2.4)

$$\begin{aligned}
 \sum_{k=1}^m |\langle u(x_k), y_k^* \rangle| &= \sum_{k=1}^m |\langle u(P(x_k)), y_k^* \rangle| \\
 &\leq \frac{2}{A_{q^*}''} K_{p,q^*} \cdot \pi_p^+(u) \cdot B_p \|P(x_k)_1^m\|_q \|y_k^*\|_{\ell_{q^*,|weak|}(F^*)} \\
 &= \frac{2}{A_{q^*}''} K_{p,q^*} \cdot \pi_p^+(u) \cdot B_p \|(x_k)_1^m\|_q \|y_k^*\|_{\ell_{q^*,|weak|}(F^*)}.
 \end{aligned}$$

This shows us that $u \in \mathcal{D}_q^+(H, F)$ with

$$d_q^+(u) \leq \frac{2}{A_{q^*}''} K_{p,q^*} \cdot B_p \cdot \pi_p^+(u).$$

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