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Existence result for a fractional differential equation involving a special derivative

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ABSTRACT. In this article, we establish certain sufficient conditions to show the existence of solutions of an initial value problem of fractional-ordinary differential equation in Banach space. Our approach is based on a combination of Mönch fixed point theorem and a suitable measure of non-compactness. Also an example is given to illustrate our approach.

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1. Introduction

Differential equations of fractional order is an emerging field with ramifications and excellent applications in several fields of science and engineering. The monographs [12, 15, 16, 18, 20] contain basic concepts and theories in fractional differential equations and fractional calculus.

Great consideration has been accorded to the existence of solution of initial and boundary value problems for fractional differential equations and inclusions involving by Caputo and

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Riemann-Liouville derivative, see [3, 9, 10, 14].

In the last years, many interesting works have appeared in the study of fractional differential equations over Banach spaces, some of them examined the existence results of solutions, by using certain basic tools from functional analysis, we refer the reader to [4, 5, 6, 17].

We consider, in this work, the following initial value problem of fractional-ordinary differential equation:

$$(\mathbf{P}) \quad \begin{cases} {}^{rl}D_{0+}^{\alpha} y'(r) = f(r, y(r), y'(r)), & r \in (0, T], \\ I_{0+}^{1-\alpha} y'(0) = a, \\ y(0) = b, \end{cases}$$

where $0 < \alpha < 1$ and ${}^{rl}D_{0+}^{\alpha}$ denote the left-sided Riemann-Liouville fractional derivative. The operator $I_{0+}^{1-\alpha}$ denotes the left-sided Riemann-Liouville fractional integral, E is a Banach space with the norm $\|\cdot\|$, $a, b \in E$ and $f : (0, T] \times E \times E \rightarrow E$ a function satisfying some specified conditions (see, section 3).

We note that in our problem, the derivative that we have taken is a composition of fractional and integer derivatives. This is the motivation to study Problem (P) involving this special derivative.

The present work is organized in the following way: to make the readers understand our initial value problem, we give in Section 2 some definitions, lemmas and basic results. Next, in Section 3 we present our main results by using a new methode to showing the existence of solutions of Problem (P). Finally an illustrative example will be presented in Section 4.

2. Background and basic results

We introduce in this section some notation and technical results which are used throughout this paper. Let $C([0, T], E)$ be the space of E -valued continuous functions on the interval $[0, T]$ endowed with the following uniform norm topology:

$$\|u\|_{\infty} = \sup\{\|u(r)\|, r \in [0, T]\},$$

and let $L^1([0, T], E)$ be the space of E -valued Bochner integrable functions on the interval $[0, T]$ with the norm

$$\|v\|_{L^1} = \int_0^T \|v(r)\| dr.$$

For all $0 < \alpha < 1$, let $C_{1-\alpha}([0, T], E)$ be the Banach spaces of functions from the interval $[0, T]$ into E which is defined as:

$$C_{1-\alpha}([0, T], E) = \{w \in C((0, T], E) : \lim_{r \rightarrow 0^+} r^{1-\alpha} w(r) \text{ exists and finite}\}.$$

A norm in this space is given by

$$\|w\|_\alpha = \sup_{r \in [0, T]} r^{1-\alpha} \|w(r)\|,$$

and

$$C_\alpha^1([0, T], E) = \{w : [0, T] \rightarrow E : w \in C([0, T], E) \text{ and } w' \in C_{1-\alpha}([0, T], E)\}$$

with the norm

$$\|w\|_{C_\alpha^1} = \sup_{r \in [0, T]} \|w(r)\| + \sup_{r \in [0, T]} r^{1-\alpha} \|w'(r)\|$$

For any subset N of $C_{1-\alpha}([0, T], E)$, we put $N_\alpha =: \{w_\alpha, w \in N\}$, where

$$w_\alpha(r) = \begin{cases} r^{1-\alpha} w(r), & \text{if } r \in (0, T] \\ \lim_{r \rightarrow 0} r^{1-\alpha} w(r), & \text{if } r = 0. \end{cases}$$

Clearly the function w_α belongs to $C([0, T], E)$, and hence $N_\alpha \subseteq C([0, T], E)$.

Lemma 2.1 ([9]). *Let $\lambda, \nu, \omega > 0$. Then for all real number $r > 0$, we have*

$$\int_0^r (r-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \leq M r^{\nu-1},$$

where

$$M = \left(\frac{\Gamma(\lambda)(1 + \lambda(1 + \lambda)/\nu)}{\omega^\lambda} \right) \max\{2^{1-\nu}, 1\}.$$

Definition 2.1 ([12]). (i) Let Γ be the gamma function and α a non-negative real number. We recall that the fractional (arbitrary) integral of order α of a function $u \in L^1((0, b], E)$ is given by

$$I_{a+}^\alpha u(r) = \frac{1}{\Gamma(\alpha)} \int_a^r (r-s)^{\alpha-1} u(s) ds.$$

(ii) Let $0 < \alpha < 1$. The Riemann-Liouville fractional derivative of order α of the function u is given as:

$${}^r I_{a+}^\alpha u(r) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dr} \left(\int_0^r (r-s)^{-\alpha} u(s) ds \right) = \frac{d}{dr} I^{1-\alpha} u(r).$$

Lemma 2.2 ([8]). *We have the following properties of I^α .*

$$I_{a+}^\alpha (t-a)^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\alpha+\lambda)} (t-a)^{\alpha+\lambda-1}, \quad \alpha, \lambda \in \mathbb{R}_+,$$

and hence

$${}^r I_{a+}^\alpha (t-a)^{\alpha-1} = 0, \quad 0 < \alpha < 1.$$

Lemma 2.3 ([18, 9]). *Let $\alpha > 0$ and $0 \leq \gamma < 1$. If $\gamma \leq \alpha$. Then I_{0+}^α is bounded from $C_\gamma([0, T], E)$ into $C([0, T], E)$.*

Lemma 2.4 ([9, 13]). Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $y \in C_\gamma([0, T], E)$ and $I_{0+}^{1-\alpha} y \in C_\gamma^1([0, T], E)$, then

$$I_{0+}^\alpha D_{0+}^\alpha y(r) = y(r) - \frac{I_{0+}^{1-\alpha} y(0)}{\Gamma(\alpha)} r^{\alpha-1}, \text{ for all } r \in (0, T].$$

We recall here the definition and properties of measure of noncompactness, and for more details, we refer the reader [1, 2, 7, 11, 19].

Definition 2.2. The Kuratowski measure of non-compactness γ is defined on each bounded subset Ω of E by

$$\gamma(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}.$$

Lemma 2.5. Let A and B be two bounded subsets of Banach space E . Then following properties hold:

- (i₁) A is relatively compact if and only if $\gamma(A) = 0$,
- (i₂) $\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ denotes the closed of A ,
- (i₃) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$,
- (i₄) $A \subset B$ implies $\gamma(A) \leq \gamma(B)$,
- (i₅) $\gamma(a.A) = \|a\| \cdot \gamma(A)$ for all $a \in E$,
- (i₆) $\gamma(\{a\} \cup A) = \gamma(A)$ for all $a \in E$,
- (i₇) $\gamma(A) = \gamma(\text{Conv}(A))$, where $\text{Conv}(A)$ denotes the convex hull of A .

Lemma 2.6 ([11]). Let Ω be an equicontinuous subset of $C([a, b], E)$. If Ω is bounded, then $\gamma(\Omega(\cdot))$ is continuous on $[a, b]$ and

$$\gamma_C(\Omega) = \max_{r \in [a, b]} \{\gamma(\Omega(r))\} \quad \text{and} \quad \gamma\left(\int_a^b x(r) dr : x \in \Omega\right) \leq \int_a^b \gamma(\Omega(r)) dr,$$

where $\Omega(r) = \{x(r) : x \in \Omega\}$ and γ_C is the Kuratowski non-compactness measure on the space $C([a, b], E)$.

Theorem 2.1 ([1]). Let E be a Banach space and D a closed and convex subset of E such that D is bounded and contains 0, and let $N : D \rightarrow D$ be a continuous mapping. If for every subset V of D , we have the following implication:

$$V = N(V) \cup \{0\} \quad \text{or} \quad V = \overline{\text{conv}} N(V) \implies \gamma(V) = 0,$$

then N has a fixed point.

Lemma 2.7. Let h be a function of $C([0, T], E)$ and y a function in $C_a^1([0, T], E)$. Then y is a solution of the following integral equation:

$$y(r) = b + \frac{ar^\alpha}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} h(\tau) d\tau ds. \quad (2.1)$$

If and only if y is a solution of the following problem

$$(\mathbf{P}^*) \quad \begin{cases} {}^{rl}D_{0+}^\alpha y'(r) = h(r), & r \in (0, T], \\ I_{0+}^{1-\alpha} y'(0) = a, \\ y(0) = b, \end{cases}$$

Proof. Let $y \in C_\alpha^1([0, T], E)$ be a solution of (2.1), we have $y(0) = b$. Next, by applying $\frac{d}{dr}$ to both sides of (2.1), we get

$$y'(r) = \frac{ar^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} h(s) ds. \quad (2.2)$$

Applying $I_{0+}^{1-\alpha}$ to both sides of the equation (2.2) and utilizing Lemma 2.2, we get

$$I_{0+}^{1-\alpha} y'(r) = a + I_{0+}^1 h(r)$$

Taking $r \rightarrow 0$, we get

$$I_{0+}^{1-\alpha} y'(0^+) = a.$$

Conversely, let $y \in C_\alpha^1([0, T], E)$ be a solution of problem (P^*) . We want to prove that y is a solution of (2.1). By the definition of $C_\alpha^1([0, T], E)$, Lemma 2.3 and Definition of $I_{0+}^{1-\alpha}$, we have

$$I_{0+}^{1-\alpha} y' \in C([0, T], E) \text{ and } \frac{d}{dr} \left(I_{0+}^{1-\alpha} y'(r) \right) = {}^{lr}D_{0+}^\alpha y'(r) \in C_{1-\alpha}([0, T], E).$$

Thus, we have

$$I_{0+}^{1-\alpha} y' \in C_\alpha^1([0, T], E).$$

Applying Lemma 2.4, we get

$$I_{0+}^\alpha D_{0+}^\alpha y'(r) = y'(r) - \frac{I_{0+}^{1-\alpha} y'(0)}{\Gamma(\alpha)} r^{\alpha-1} = I_{0+}^\alpha h(r).$$

Which implies

$$y'(r) = \frac{ar^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} h(s) ds.$$

Then

$$\begin{aligned} y(r) &= y(0) + \frac{ar^\alpha}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s-\tau)^{\alpha-1} h(\tau) d\tau ds \\ &= b + \frac{ar^\alpha}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s-\tau)^{\alpha-1} h(\tau) d\tau ds. \end{aligned}$$

Thus y is a solution of (2.1).

3. Existence of the solution

We assume the following hypotheses:

(H₁) The function $f : (0, T] \times E \times E \rightarrow E$ is continuous and for all x, y, u, v and $r \in (0, T]$:

$$\|f(r, x, y) - f(r, u, v)\| \leq A\|x - u\| + r^{1-\alpha} B\|y - v\|,$$

where $A, B \in \mathbb{R}^+$.

(H₂) For $r \in (0, T]$ and $u, v \in E$,

$$\|f(r, u, v)\| \leq a(r)\|u\| + r^\lambda e^{-\sigma r} b(r)\|v\|,$$

where $\sigma > 0, \lambda \geq 1 - \alpha$ and $a(\cdot), b(\cdot) : [0, \infty) \rightarrow \mathbb{R}^+$ are continuous functions,

(H₃) There exists two functions $\ell, j \in C([0, T], \mathbb{R}^+)$ such that for each bounded nonempty subset $\Omega \subset C_\alpha^1([0, T], E)$,

$$\gamma(f(r, \Omega(r), \Omega'(r))) \leq \ell(r)\gamma(\Omega(r)) + j(r)\gamma(r^{1-\alpha}\Omega'(r)), \quad \text{for all } r \in (0, T],$$

and

$$TF'(T) + F(T) < 1,$$

where $\Omega(r) = \{y(r) : y \in \Omega\}$, $\Omega'(r) = \{y'(r) : y \in \Omega\}$ and

$$F(r) = \frac{2}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} (\ell(\tau) + j(\tau)) d\tau ds, \quad \text{for all } r \in (0, T],$$

(H₄) $\alpha(1 + \alpha)b^*C + (1 + \alpha)[a^*T + b^*CT^\alpha] + a^*T^{1+\alpha} < \Gamma(2 + \alpha)$,
where $a^* = \sup_{[0, T]} a(r)$ and $b^* = \sup_{[0, T]} b(r)$.

Define the operator $N : C_\alpha^1([0, T], E) \rightarrow C_\alpha^1([0, T], E)$ by

$$Ny(r) = b + \frac{ar^\alpha}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} f(\tau, y(\tau), y'(\tau)) d\tau ds,$$

and the operator $N' : C_{1-\alpha}([0, T], E) \rightarrow C_{1-\alpha}([0, T], E)$ by

$$N'y(r) = (Ny)'(r) = \frac{ar^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r (r - s)^{\alpha-1} f(s, y(s), y'(s)) ds.$$

We will give some useful lemmas to prove our main result.

Lemma 3.1. *Suppose that (H₁) and (H₂) hold. Then*

- (1) *N is continuous and bounded.*
- (2) *NB is equicontinuous for all bounded subset of $C_\alpha^1([0, T], E)$.*

Proof. Let's prove (1), we start to prove that N is bounded. Let $y \in C_\alpha^1([0, T], E)$, from (H₂) it is easy to deduce that $Ny \in C_\alpha^1([0, T], E)$. Using (H₂) and Lemma 2.1, for all $y \in B_\kappa = \{y \in C_\alpha^1([0, T], E) : \|y\|_{C_\alpha^1} < \kappa\}$ and $r \in (0, T]$ we get

$$\begin{aligned} \|Ny(r)\| &\leq \|b\| + \frac{\|a\|r^\alpha}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} \|f(\tau, y(\tau), y'(\tau))\| d\tau ds \\ &\leq \|b\| + \frac{\|a\|T^\alpha}{\Gamma(\alpha + 1)} + \frac{a^*\|y\|_\infty}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} d\tau ds \\ &\quad + \frac{b^*\|y'\|_\alpha}{\Gamma(\alpha)} \int_0^r \int_0^s (s - \tau)^{\alpha-1} \tau^{\lambda+\alpha-1} e^{-\sigma\tau} d\tau ds. \end{aligned}$$

So,

$$\|Ny\|_\infty \leq \|b\| + \frac{\|a\|T^\alpha}{\Gamma(1 + \alpha)} + \frac{a^*\kappa T^{1+\alpha}}{\Gamma(2 + \alpha)} + \frac{b^*\kappa CT^\alpha}{\Gamma(1 + \alpha)} = M_1. \quad (3.1)$$

And we have alos

$$\|r^{1-\alpha}(Ny)'(r)\| \leq \frac{\|a\|}{\Gamma(\alpha)} + \frac{r^{1-\alpha}}{\Gamma(\alpha)} \int_0^r (r - s)^{\alpha-1} \|f(s, y(s), y'(s))\| ds$$

$$\begin{aligned} &\leq \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\|y\|_\infty r^{1-\alpha}}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} ds \\ &\quad + \frac{b^*\|y'\|_{C_\alpha} r^{1-\alpha}}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} s^{\lambda+\alpha-1} e^{-\sigma s} ds. \end{aligned}$$

So,

$$\|(Ny)'\|_\alpha \leq \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\kappa T}{\Gamma(\alpha+1)} + \frac{b^*\kappa C}{\Gamma(\alpha)} = M_2, \quad (3.2)$$

where

$$C = \max\{1, 2^{1-\alpha}\} \Gamma(\lambda + \alpha) [1 + (\lambda + \alpha)(\lambda + \alpha + 1)/\alpha] \sigma^{-(\lambda+\alpha)}.$$

From (3.1) and (3.2), we get

$$\|(Ny)\|_{C_\alpha^1} \leq M = M_1 + M_2.$$

Now we will prove that N is continuous. Let $\{y_n\}_{n \in \mathbb{N}} \rightarrow y$ in $C_\alpha^1([0, T], E)$ and $\epsilon > 0$. The hypothesis (H_1) confirm the existence of an integer $m \in \mathbb{N}$ such that, for all $n \geq m$ and $r \in (0, T]$,

$$\|f(r, y_n(r), y_n'(r)) - f(r, y(r), y'(r))\| < \frac{\Gamma(1+\alpha)}{T} \left[\frac{1+\alpha}{1+\alpha+T^\alpha} \right] \epsilon. \quad (3.3)$$

Thus, $\forall r \in (0, T]$,

$$\begin{aligned} &\|Ny_n(r) - Ny(r)\| + r^{1-\alpha} \|(Ny_n)'(r) - (Ny)'(r)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^s (s-\tau)^{\alpha-1} \|f(\tau, y_n(\tau), y_n'(\tau)) - f(\tau, y(\tau), y'(\tau))\| d\tau ds \\ &\quad + \frac{r^{1-\alpha}}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} \|f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))\| ds. \end{aligned}$$

From (3.3), we conclude that for all $n \geq m$:

$$\|Ny_n - Ny\|_{C_\alpha^1} < \epsilon.$$

To prove (2), it suffices to show that NB_κ and $(NB_\kappa)'$ are equicontinuous respectively in $C([0, T], E)$ and in $C_{1-\alpha}([0, T], E)$. Let $y \in B_\kappa$ and $r_1, r_2 \in (0, T]$ with $r_1 < r_2$. First of all, we have

$$\begin{aligned} \|N(y)(r_2) - N(y)(r_1)\| &\leq \frac{a(r_2^\alpha - r_1^\alpha)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_{r_1}^{r_2} \int_0^s (s-\tau)^{\alpha-1} \|f(\tau, y(\tau), y'(\tau))\| d\tau ds \\ &\leq \frac{a(r_2^\alpha - r_1^\alpha)}{\Gamma(\alpha+1)} + \frac{a^*\kappa}{\Gamma(\alpha)} \int_{r_1}^{r_2} \int_0^s (s-\tau)^{\alpha-1} d\tau ds + \frac{b^*\kappa}{\Gamma(\alpha)} \int_{r_1}^{r_2} \int_0^s (s-\tau)^{\alpha-1} \tau^{\lambda+\alpha-1} e^{-\sigma\tau} d\tau ds \\ &\leq \frac{a+b^*\kappa C}{\Gamma(\alpha+1)} (r_2^\alpha - r_1^\alpha) + \frac{a^*\kappa}{\Gamma(\alpha+2)} (r_2^{\alpha+1} - r_1^{\alpha+1}). \end{aligned}$$

By taking r_2 tends to r_1 , the right-hand side of the last inequality approaches to 0. Therefore NB_κ is equicontinuous in $C([0, T], E)$.

And, we also have

$$\|r_2^{1-\alpha} N'(y)(r_2) - r_1^{1-\alpha} N'(y)(r_1)\|$$

$$\begin{aligned}
&\leq \left\| \frac{r_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_2} (r_2 - s)^{\alpha-1} f(s, y(s), y'(s)) ds - \frac{r_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_1} (r_1 - s)^{\alpha-1} f(s, y(s), y'(s)) ds \right\| \\
&\leq \frac{r_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_1} [(r_1 - s)^{\alpha-1} - (r_2 - s)^{\alpha-1}] \|f(s, y(s), y'(s))\| ds \\
&\quad + \frac{r_2^{1-\alpha} - r_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_1} (r_2 - s)^{\alpha-1} \|f(s, y(s), y'(s))\| ds \\
&\quad + \frac{r_2^{1-\alpha}}{\Gamma(\alpha)} \int_{r_1}^{r_2} (r_2 - s)^{\alpha-1} \|f(s, y(s), y'(s))\| ds \\
&\leq \frac{a^* \kappa r_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_1} [(r_1 - s)^{\alpha-1} - (r_2 - s)^{\alpha-1}] ds \\
&\quad + \frac{b^* \kappa r_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{r_1} [(r_1 - s)^{\alpha-1} - (r_2 - s)^{\alpha-1}] s^{\lambda+\alpha-1} ds \\
&\quad + \frac{a^* \kappa (r_2^{1-\alpha} - r_1^{1-\alpha})}{\Gamma(\alpha)} \int_0^{r_1} (r_2 - s)^{\alpha-1} ds + \frac{b^* \kappa (r_2^{1-\alpha} - r_1^{1-\alpha})}{\Gamma(\alpha)} \int_0^{r_1} (r_2 - s)^{\alpha-1} s^{\lambda+\alpha-1} ds \\
&\quad + \frac{a^* \kappa r_2^{1-\alpha}}{\Gamma(\alpha)} \int_{r_1}^{r_2} (r_2 - s)^{\alpha-1} ds + \frac{a^* \kappa r_2^{1-\alpha}}{\Gamma(\alpha)} \int_{r_1}^{r_2} (r_2 - s)^{\alpha-1} s^{\lambda+\alpha-1} ds \\
&\leq \frac{a^* \kappa T^{1-\alpha} + b^* \kappa T^\lambda}{\Gamma(\alpha+1)} [r_1^\alpha - r_2^\alpha + (r_2 - r_1)^\alpha] + \frac{a^* \kappa T^\alpha + b^* \kappa T^{\lambda+2\alpha-1}}{\Gamma(\alpha+1)} (r_2^{1-\alpha} - r_1^{1-\alpha}) \\
&\quad + \frac{a^* \kappa T^{1-\alpha} + b^* \kappa T^\lambda}{\Gamma(\alpha+1)} (r_2 - r_1)^\alpha.
\end{aligned}$$

By taking r_2 tends to r_1 , the right-hand side of the last inequality approaches to 0, and hence $(NB_\kappa)'$ is equicontinuous in $C_{1-\alpha}([0, T], E)$.

To present our main result, we recall here some definitions and notations. Let

$$B = \{y \in C_\alpha^1([0, T], E) : \|y\|_{C_\alpha^1} \leq R\}.$$

We denote by γ_α the Kuratowski measure of non-compactness defined on any bounded subset of $C_\alpha^1([0, T], E)$. We can easily show the following inequality

$$\gamma_\alpha(D) \leq \sup_{r \in [0, T]} \gamma(D(r)) + \sup_{r \in [0, T]} \gamma(D'_\alpha(r)) \leq 2\gamma_\alpha(D), \quad (3.4)$$

where D is a bounded and equicontinuous subset of $C_\alpha^1([0, T], E)$, $D(r) = \{y(r) : y \in D\}$ and $D'_\alpha(r) = \{y'_\alpha(r) : y \in D\}$.

Theorem 3.1. *Suppose that conditions $(H_1) - (H_4)$ are valid. If*

$$R > \frac{\|b\| \Gamma(2+\alpha) + (1+\alpha) \|a\| T^\alpha + \alpha(1+\alpha) \|a\|}{\Gamma(2+\alpha) - [\alpha(1+\alpha) b^* C + (1+\alpha) [a^* T + b^* C T^\alpha] + a^* T^{1+\alpha}}. \quad (3.5)$$

Then Problem (P) has at least one solution.

Proof. Using the lemma 2.7, it is clear that the fixed points of the operator N , defined previously, are solutions of Problem (P). We will verify that N satisfies the assumptions of Mönch fixed point theorem (Theorem 2.1).

First, we show that N is well-defined from B to B , indeed: let $y \in B$. By using the condition (H₂) and after some calculations, we get

$$\begin{aligned} \|Ny(r)\| + \|r^{1-\alpha}(Ny)'(r)\| &\leq \|b\| + \frac{\|a\|T^\alpha}{\Gamma(\alpha+1)} + \frac{\|a\|}{\Gamma(\alpha)} \\ &+ \left[\frac{\alpha(\alpha+1)b^*C + (\alpha+1)[a^*T + b^*CT^\alpha] + a^*T^{\alpha+1}}{\Gamma(\alpha+2)} \right] R. \end{aligned}$$

From (H₄) and the inequality (3.5), we obtain

$$\forall y \in B : \|Ny\|_{C_\alpha^1} < R.$$

Note that B is bounded, convex and closed subset of $C_\alpha^1([0, T], E)$ and N is continuous on B . Next, we need to prove the following implication

$$V \subset \overline{\text{conv}}\{N(V) \cup \{0\}\} \implies \gamma_\alpha(V) = 0, \text{ for any } V \subset B.$$

Let $V \subset B$ such that $V \subset \overline{\text{conv}}\{N(V) \cup \{0\}\}$. From Lemmas 2.6, 3.1, (H₃), the inequality (3.4) and the previous steps, we arrive at

$$\begin{aligned} \gamma(NV(r)) + \gamma(r^{1-\alpha}(NV)'(r)) &\leq \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^s (s-\tau)^{\alpha-1} \ell(\tau) \gamma(NV(\tau)) d\tau ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^s (s-\tau)^{\alpha-1} J(\tau) \gamma((NV)'_\alpha(\tau)) d\tau ds \\ &+ \frac{T}{\Gamma(\alpha)} \int_0^T (r-s)^{\alpha-1} \ell(s) \gamma(NV(s)) ds + \frac{T}{\Gamma(\alpha)} \int_0^T (r-s)^{\alpha-1} J(s) \gamma((NV)'_\alpha(s)) ds \end{aligned}$$

Thus

$$\gamma_\alpha(NV) \leq [F(T) + TF'(T)] \gamma_\alpha(NV).$$

By condition (H₃). We get $\gamma_\alpha(N(V)) = 0$, that is $\gamma_\alpha(V) = 0$. From the theorem 2.1, thus N has a fixed point $y \in B$ which is a solution of Problem (P).

4. Example

As an example of the use of the main result, we shall now consider the following fractional-ordinary differential equation

$${}^rI^{\frac{1}{2}}y'(r) = \left(\frac{\sin(r)}{20(1+r^2)}y_n(r) + \frac{\sqrt{r}}{20(1+r^2)e^{5\sqrt{2}r}}y'_n(r) \right)_{n=1}^\infty, \quad r \in (0, 1], \quad (4.1)$$

$$I_{0+}^{\frac{1}{2}}y'(0^+) = (1, 0, \dots, 0, \dots), \quad (4.2)$$

$$y(0) = (1, 0, \dots, 0, \dots). \quad (4.3)$$

Let

$$E = \{(y_1, y_2, \dots, y_n, \dots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|\cdot\|)$ consists a Banach space, by comparing with the equations of Problem (P), we notice that

$$\alpha = \lambda = \frac{1}{2} \text{ and } f(r, y(r), y'(r)) = (f(r, y_1(r), y'_1(r)), \dots, f(r, y_n(r), y'_n(r)), \dots),$$

where

$$f(r, y_n(r), y'_n(r)) = \frac{\sin(r)}{20(1+r^2)} y_n(r) + \frac{\sqrt{r}}{20(1+r^2)e^{5\sqrt{2}r}} y'_n(r), n \in \mathbb{N}^*.$$

Clear that $f : (0, 1] \times E \times E \rightarrow E$ is continuous and

$$\|f(r, y(r), y'(r))\| \leq \frac{1}{20(1+r^2)} \|y(r)\| + \frac{\sqrt{r}}{20(1+r^2)e^{5\sqrt{2}r}} \|y'(r)\|.$$

Hence (H_1) and (H_2) are satisfied. Next, For any bounded set $B \subset C_\alpha^1([0, 1], E)$, we have

$$f(r, B(r), B'(r)) = \frac{\sin(r)}{20(1+r^2)} B(r) + \frac{\sqrt{r}}{20(1+r^2)} B'(r).$$

Then

$$\gamma(f(r, B(r), B'(r))) \leq \frac{\sin(r)}{20(1+r^2)} \gamma(B(r)) + \frac{1}{20(1+r^2)} \gamma(\sqrt{r} B'(r)),$$

since $F(1) + F'(1) \leq \frac{1}{5\Gamma(2.5)} + \frac{1}{5\Gamma(1.5)} \simeq 0.3760 < 1$. So, (H_3) holds. We have $C = 1$, thus

$$\alpha(1+\alpha)b^*C + (1+\alpha)[a^*T + b^*CT^\alpha] + a^*T^{\alpha+1} = 0.475 < \Gamma(2+\alpha) = 1.3293.$$

So, (H_4) holds. Therefore, Theorem 3.1 ensures that problem (4.1)-(4.3) has a solution.

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