FROM UNEQUAL CHANCE TO A COIN GAME DANCE: VARIANTS OF PENNEY’S GAME

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Abstract

We start by exploring and analyzing the various aspects of Penney’s game, examining its possible outcomes as well as its fairness (or lack thereof). In search of a fairer game, we create many variations of the original Penney’s game by altering its rules. Specifically, we introduce the Head-Start Penney’s game, the Post-a-Bobalypsic Penney’s game, the Second-Occurrence Penney’s game, the Two-Coin game, the No-Flippancy game, and the Blended game. We then analyze each of these games and the odds of winning for both players.

1 Penney’s game

Alice and Bob, two friends with spare change and spare time, have a fair coin to flip. We show them a game that is played as follows: Alice selects a string of flip outcomes (heads or tails) of length n , after which Bob chooses his own string of flip outcomes, also of length n . They begin by tossing a coin to form a sequence of heads and tails until one of the players’ strings appears; this player is then declared the victor.

This game is known as “Penney’s game” or “paradoxical pennies,” as the game was named in [WW04]. We denote flip outcomes with H for heads and T for tails.

Alice and Bob play the game for $n = 3$, where Alice selects the string HHH and Bob selects the string THH. They then toss the coin several times and yield the following sequence of flips:

H H T T H T H H

We see that Bob's string, emphasized in the sequence of flip outcomes above, appears before Alice's string HHH does; therefore, he is the winner of this round.

The duo decide to play several rounds with these same strings. They quickly find that Bob wins significantly more than Alice.

Suspecting something, they examine Penney's game a little more closely. Alice correctly commences with the claim that each string of length 3 comprised of Hs and Ts has the same probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ to appear in three consecutive tosses.

"This must mean that the game is fair, right?" Bob asks.

"It can't be — you won most of the time!" Alice replies.

"Hmm, maybe you're right," Bob nods. "Let's consider a simpler case where $n = 2$. How about HH and HT?"

"For both strings," Alice starts, "we need to wait for the first H. After that, an unfavorable flip for HH means restarting. However, while an unfavorable flip for HT means it fails to progress, it doesn't need to restart entirely!"

"So HT is more likely to appear first!" Bob chimes. "Our initial assumption that the game is fair was incorrect. Perhaps the key lies with the expected number of flips it takes for each string to appear."

"I think that's called the 'expected wait time' of the string."

Bob nods. "Is there a way to find the expected wait time for each string in terms of flips?"

In fact, there is. We cover this later in Section 5. For the time being, we provide Alice and Bob with the following data for reference: the expected wait time for HT is 4 flips while the expected wait time for HH is 6 flips. For strings of length 3, the expected wait time for HHH and TTT is 16 flips, the expected wait time for HTH and THT is 14 flips, and the expected wait time for HHT, HTT, THH, and TTH is 8 flips.

"Aha! If a string has a shorter expected wait time," Bob states, "it must have a greater probability of winning since it is more likely to appear first. If two strings have the same wait time, then their probabilities of winning are equal."

However, he is still wrong. We tell Alice and Bob to consider the case where Alice selects the string HHH and Bob selects the string HHT. Alice's string has an expected wait time of 16 flips while Bob's has a shorter expected wait time of 8 flips.

Now we look at how these strings would play out in the actual game. Both players wait for two Hs in a row since that segment of their strings is the same. After two consecutive Hs appear, each of them wins with probability $\frac{1}{2}$.

"So," Alice concludes, "despite our strings having different wait times, our probabilities of winning are equal."

Another example where the string with a longer expected wait time has an advantage occurs when Alice selects THTH and Bob selects HTHH. The string THTH has an expected wait time of 20 flips, and the string HTHH has an expected wait time of 18 flips. Despite having a higher expected wait time, THTH has a higher winning probability of $\frac{9}{14}$.

The formula for winning probabilities in Penney's game is well-known. We present it in Section 5. Table 1 displays the probability of Bob winning in the game with strings of length 3. Alice's possible strings fill the top row and Bob's possible strings are in the left column. We only show Bob's winning probabilities for Alice's potential strings that begin with H as we can calculate the rest by symmetry.

	HHH	HHT	HTH	HTT
HHH	*	1/2	2/5	2/5
HHT	1/2	*	2/3	2/3
HTH	3/5	1/3	*	1/2
HTT	3/5	1/3	1/2	*
THH	7/8	3/4	1/2	1/2
THT	7/12	3/8	1/2	1/2
TTH	7/10	1/2	5/8	1/4
TTT	1/2	3/10	5/12	1/8

Table 1: Bob’s probability of winning.

Table 2 below displays the best choice for Bob given Alice’s string and the odds of Bob winning for that choice. Once again, we only need to consider Alice’s strings which start with H.

Alice’s string	Bob’s best choice	Bob’s odds
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1

Table 2: Bob’s best choice and odds.

Figure 1 shows Bob’s best choice. The arrow points from Alice’s string to Bob’s best choice.

Figure 1: Bob’s best choice given Alice’s string.

Interestingly, there is an algorithm that Bob can employ to choose his string once provided Alice’s string such that he has the optimal probability of winning in Penney’s game. Namely, given Alice’s string of Hs and Ts, Bob takes the second letter from Alice’s string, reverses it (turns it to H if it’s T and vice versa), and adds the first $n - 1$ elements of Alice’s string to it (see [GO81]). From paper [Fel06], we know that this optimal string is unique.

To demonstrate this with an example, suppose Alice chooses the string HTT. According to the algorithm described above, Bob chooses HHT, which gives a win ratio of 2:1 against Alice. If Alice chooses HHT instead, then Bob selects THH, yielding a win ratio of 3:1. Hence, one may conclude that THH is a better choice than both HTT and HHT. However, this is incorrect; THH, for all

practical purposes, is the same as HTT. Thus, we attain a non-transitive cycle: HHT beats HTT, which beats TTH, which beats THH, which beats HHT.

Since there is no best string that Alice can select, this game gives Alice a major disadvantage. Given Alice's string, Bob can construct his own string to ensure that he is more likely to win. Now, alongside Alice and Bob, we seek to construct a coin game that, for once and for all, gives the two equitable odds.

The games we invent and discuss in this paper are listed and summarized below:

1. *The Head-Start Penney's game.* We give Alice an edge by allowing her a one-flip head-start. We discuss this coin game in detail in Section 2. Bob's expected wait time increases by 1 and Alice's chances of winning improve. However, given Alice's string, Bob can select his own such that his probability of winning is better than a half. Interestingly, given Alice's string, Bob's optimal string in this game is the same as Bob's optimal string in Penney's game.
2. *The Post-a-Bobalyptic Penney's game.* We discuss this game in Section 3. In this variant of Penney's game, Alice is allowed an additional toss at the end if Bob's string appears before hers does. If this extra toss completes her string, the game is a tie. Otherwise, Bob wins. For every choice that Alice can make, her worst-case odds are better than in Penney's game. Additionally, if Alice picks a favorable string, this game is fair for $n = 3$.
3. *The Second-Occurrence Penney's game.* We explore and discuss this coin game in Section 4. Here, the first player whose string occurs twice wins. This game does improve the odds in favor of Alice, with a few exceptions.
4. *The Two-Coin game.* In this game, which we analyze in Section 6, Alice and Bob flip separate coins. Whoever sees their string appear first in their own series of flips wins. The Two-Coin game differs from other games in that the winning probabilities only depend on expected wait times as the two strings cannot interact with one another. This game is fair.
5. *The No-Flippancy game.* We discuss this game in detail in Section 7. This game, being deterministic, is rather distinct from the others here as it does not involve a coin. Each move is decided according to some simple rules. Alice has the first turn, and the two players alternate turns afterwards. The nature of this variant is such that the outcome of each game is always either a win, a loss, or an infinite game. Unfortunately, these new rules are not enough to give Alice a fair chance at winning. Interestingly, for $n = 3$, Bob can follow the algorithm that obtains an optimal string in Penney's game to pick a winning string in the No-Flippancy game. As this game has numerous highly intriguing properties, we have written an in-depth paper about it [Fli20].
6. *The Blended game.* This game is a friendly mixture of the No-Flippancy and Penney's game. We cover it in Section 8. When Alice's and Bob's wanted outcomes coincide, that is the outcome they receive, similar to the No-Flippancy Game. If not, they flip a coin. Once again, whoever's string appears first is triumphant. One interesting feature of this game is that it also has a non-transitive cycle of length 6.

It is worth noting that, in Penney's game, the odds of a string S_1 winning against a string S_2 are independent of which player selects which string. This is due to the symmetry present in the game with respect to Alice and Bob.

However, this property is not true in the Head-Start Penney’s game, the Post-a-Bobalyptic Penney’s game, and the No-Flippancy game. Nevertheless, the remaining three games — the Second-Occurrence game, Two-Coins game, and Blended game — retain this characteristic.

Interestingly, the Second-Occurrence game has a non-transitive cycle of strings congruent to that of Penney’s game in Figure 1. Furthermore, the Blended game surprisingly has a non-transitive cycle of length six.

We cover some general theory in Section 5. Specifically, we discuss a way to measure how self-overlapping a string is and we use this knowledge to explain the formula for the odds in Penney’s game. Additionally, we mention the probability that a string appears for the first time in a given number of flips.

Below are conventions we utilize throughout the paper.

- We assume that Alice’s and Bob’s string are different as most games would end in a tie otherwise.
- Unless otherwise noted, we assume that Alice’s string starts with H as the results for strings starting with T can be extended by symmetry.
- Alice and Bob always choose strings of a common length n .
- In all of the tables with winning probabilities, unless otherwise noted, the top row shows Alice’s choices, the left column shows Bob’s choices, and the entries show Bob’s probabilities of winning.

Alice and Bob begin their journey with the simplest variation of Penney’s game: the Head-Start Penney’s game.

2 The Head-Start Penney’s game

Alice and Bob, unsatisfied by the unfairness of Penney’s game, agree to even the odds by playing the Head-Start Penney’s game instead. In this variation of Penney’s game, the first toss counts only towards Alice’s string. Just as with Penney’s game, the two players cannot tie.

Before they play, the two companions wish to investigate how much the game leverages the odds in Alice’s favor. After a few rounds, they realize that the first toss is only beneficial to Alice if the first n tosses are her string, which occurs with probability $\frac{1}{2^n}$. Otherwise, with probability $1 - \frac{1}{2^n}$ the game continues as Penney’s game.

They denote the probability of Alice winning in Penney’s game as $p(A)$. With this notation, Alice’s chance of winning in this new game is $\frac{1}{2^n} + (1 - \frac{1}{2^n})p(A)$, which can be rewritten as $p(A) + (1 - p(A))\frac{1}{2^n}$. Therefore, Bob’s probability of winning is $1 - \frac{1}{2^n} - (1 - \frac{1}{2^n})p(A) = (1 - p(A))(1 - \frac{1}{2^n})$. His chance of winning in Penney’s game is $1 - p(A)$, so his probability of victory in the Head-Start Penney’s game is directly proportional to his probability of victory in Penney’s game.

Note that the larger $p(A)$ is and the smaller n is, the greater the increase in Alice’s chance of winning the Head-Start game compared to Penney’s game. Furthermore, given Alice’s string, Bob’s optimal string in the Head-Start game is the same as the one for Penney’s game, since the probabilities are directly proportional.

For example, suppose $n = 3$. Then, Alice's chance of winning is $\frac{1+7p(A)}{8}$ and Bob's chance of winning is $\frac{7(1-p(A))}{8}$. Thus, using the winning probabilities for Penney's game when $n = 3$ from Table 1, we obtain Table 3 below, which shows Bob's probabilities of winning in the Head-Start Penney's game.

	HHH	HHT	HTH	HTT
HHH	*	7/16	7/20	7/20
HHT	7/16	*	7/12	7/12
HTH	21/40	7/24	*	7/16
HTT	21/40	7/24	7/16	*
THH	49/64	21/32	7/16	7/16
THT	49/96	21/64	7/16	7/16
TTH	49/80	7/16	35/64	7/32
TTT	7/16	21/80	35/96	7/64

Table 3: Bob's probabilities of winning.

Table 4 shows Bob's optimal string choice and his odds of winning given Alice's string. As noted above, the optimal string for Bob is the same here as it is in Penney's game. This signifies that Figure 1 represents this game as well. Nevertheless, the odds in the Head-Start Penney's game are certainly more favorable for Alice in comparison to Penney's game. Despite this, Bob still has an advantage because he can always pick a string with favorable odds.

Alice's string	Bob's best choice	Bob's odds
HHH	THH	49 to 15
HHT	THH	21 to 11
HTH	HHT	7 to 5
HTT	HHT	7 to 5

Table 4: Bob's best choice and odds.

As Bob is one toss behind, the expected wait time for his string increases by 1 flip. In this game, as in Penney's game, the string bearing a longer wait time may still have a higher success rate. In Penney's game, this only occurs for strings where $n \geq 4$. In the Head-Start Penney's game, this can occur for $n = 3$ as well. All such cases for strings of length 3 are displayed in Table 5, where "Winner" denotes the player with the advantage.

Winner	Alice	Bob	Wait Time
Alice	HTH	THH	(10, 9)
Bob	HHT	THH	(8, 9)
Alice	HTH	HTT	(10, 9)
Bob	HTT	HHT	(8, 9)
Alice	HHH	HHT	(14, 9)

Table 5: Winning versus wait time.

The last line in Table 5 is particularly interesting since the winner has a string with a much higher wait time. This phenomenon continues occurring for greater string lengths. For instance, if Alice selects a string comprised completely of Hs and Bob chooses a string congruent to Alice's except for the last term, Alice has a higher winning probability despite her string's expected wait time being almost twice as large as Bob's.

Alice and Bob fail to find a fair game with the Head-Start Penney's game, yet the knowledge that allowing Alice an extra flip at the beginning helped her odds encourages them. But what would happen if Alice were given an extra flip at the end instead?

3 The Post-a-Bobalyptic Penney's game

This game is distinguished from Penney's game by the fact that, if Alice loses, she receives an extra toss at the very end. If this toss allows her string to appear, then the outcome is a tie. However, the additional flip is only useful if the last $n - 1$ terms of Bob's string coincide with the first $n - 1$ terms of Alice's string. Note that Bob's algorithm for obtaining an optimal string in Penney's game satisfies this condition.

For example, suppose Alice selects the string HHH. The best-odds algorithm for Bob yields THH. However, the Post-a-Bobalyptic game gives Alice a chance to tie things up with the extra toss at the end.

Therefore, when Bob uses his optimal string, there is a $\frac{1}{2}$ probability that Alice will receive her string's last character, in which case they tie. However, if Bob selects a string where the last $n - 1$ terms do not match Alice's first $n - 1$ terms, his probability of victory is the same as that in the Penney's game.

Suppose, for some set of strings chosen by Alice and Bob, Alice comes out the winner with probability $p(A)$ in Penney's game and with the probability $p'(A)$ in the Post-a-Bobalyptic Penney's game. Correspondingly, suppose Bob wins against Alice in Penney's game with probability $p(B)$ and in the Post-a-Bobalyptic Penney's game with probability $p'(B)$. We furthermore denote by $p'(T)$ the probability that Alice and Bob obtain a tie in the Post-a-Bobalyptic Penney's game. So, $p(A) + p(B) = 1$ and $p'(A) + p'(B) + p'(T) = 1$.

Applying this notation to the reasoning above, if the first $n - 1$ elements of Alice's string fail to match with the last $n - 1$ elements of Bob's string, then

$$p'(T) = 0 \quad p'(A) = p(A) \quad p'(B) = p(B)$$

Otherwise,

$$p'(T) = \frac{p(B)}{2} \quad p'(A) = p(A) \quad p'(B) = \frac{p(B)}{2}$$

Therefore, by performing the best-odds algorithm, Bob's odds of winning the Post-a-Bobalyptic Penney's game are half his odds of winning Penney's game. Therefore, another string might be a better choice for Bob in the Post-a-Bobalyptic Penney's game. In any case, his odds are not reduced by more than a factor of 2.

Table 6 displays probabilities of different game outcomes for strings of length 3. The first term represents the probability of Bob winning, the second one is the probability of Alice winning, and the last one is the probability of a tie.

	HHH	HHT	HTH	HTT
HHH	*	1/4, 1/2, 1/4	2/5, 3/5, 0	2/5, 3/5, 0
HHT	1/2, 1/2, 0	*	1/3, 1/3, 1/3	1/3, 1/3, 1/3
HTH	3/5, 2/5, 0	1/3, 2/3, 0	*	1/2, 1/2, 0
HTT	3/5, 2/5, 0	1/3, 2/3, 0	1/2, 1/2, 0	*
THH	7/16, 1/8, 7/16	3/8, 1/4, 3/8	1/2, 1/2, 0	1/2, 1/2, 0
THT	7/12, 5/12, 0	3/8, 5/8, 0	1/4, 1/2, 1/4	1/4, 1/2, 1/4
TTH	7/10, 3/10, 0	1/2, 1/2, 0	5/8, 3/8, 0	1/4, 3/4, 0
TTT	1/2, 1/2, 0	3/10, 7/10, 0	5/12, 7/12, 0	1/8, 7/8, 0

Table 6: Probabilities of different outcomes for the Post-a-Bobalyptic Penney's game.

Table 7 shows the odds of Bob winning.

	HHH	HHT	HTH	HTT
HHH	*	1 to 2	2 to 3	2 to 3
HHT	1 to 1	*	1 to 1	1 to 1
HTH	3 to 2	1 to 2	*	1 to 1
HTT	3 to 2	1 to 2	1 to 1	*
THH	7 to 2	3 to 2	1 to 1	1 to 1
THT	7 to 5	3 to 5	1 to 2	1 to 2
TTH	7 to 3	1 to 1	5 to 3	1 to 3
TTT	1 to 1	3 to 7	5 to 7	1 to 7

Table 7: Bob's odds of victory in the Post-a-Bobalyptic Penney's game.

As seen in the table, this game is now fair for $n = 3$. If Alice picks HTT for her string, the best choice for Bob results in equal odds. However, for any other string Alice chooses, Bob would still have an advantage.

Table 8 shows the best choice for Bob and the odds of Bob winning.

Alice	Bob	Odds
HHH	THH	7 to 2
HHT	THH	3 to 2
HTH	TTH	5 to 3
HTT	HHT, HTH, THH	1 to 1

Table 8: Bob’s best choice and odds.

Figure 2 shows Bob’s best choice. The arrow points from Alice’s string to Bob’s best choice.

Figure 2: Bob’s best choice.

Studying the preceding data tables, we see that the optimal string choice for Bob in Penney’s game remains the optimal choice in the Post-a-Bobalyptic Penney’s game when $n = 3$. However, in the Post-a-Bobalyptic game, there may be multiple optimal strings.

We observed that, in Penney’s game, a string with a greater expected wait time may have an advantage. This is true in the Post-a-Bobalyptic Penney’s game as well. However, we must have $n \geq 4$ for this to occur. Table 9 shows all such possibilities for $n = 4$. The second and third columns show Alice’s and Bob’s strings, correspondingly. The last column represents two numbers: Alice’s string’s expected wait time and Bob’s string’s expected wait time. The second to last line is particularly interesting as it shows that Bob can also win with a higher expected wait time.

Although the Post-a-Bobalyptic game is fairer, Alice and Bob continue on their quest to find a better game.

Winner	Alice	Bob	Expected Wait Time
Alice	HTTH	TTHH	(18,16)
Alice	HTHT	THTT	(20,18)
Alice	HTHH	THHH	(18,16)
Bob	HHTT	HTHH	(16,18)
Alice	HTHH	HHTT	(18,16)

Table 9: Pairs of strings for $n = 4$ where the string with the longer wait time has higher winning probabilities.

4 The Second-Occurrence Penney's game

Alice and Bob have seen that adding a flip in Alice's favor has not quite worked out as intended. While there was a slight improvement in her odds, the games have failed to be fair. Instead, Alice suggests a new game — the *Second-Occurrence Penney's game*. Here, a player wins once their string appears a second time in the sequence of coin flips.

As an example, suppose Alice's string is HHH and Bob's string is HHT. The game output HHTHHHH is a win for Alice, as HHH appears twice whereas HHT only appears once.

As is true with Penney's game, ties are not possible in the Second-Occurrence Penney's game. Additionally, if Alice and Bob choose complementary strings, then, by symmetry, they both win with an equal probability.

We say that a string of length n is *self-overlapping* if one can fit two occurrences of the string into less than $2n$ tosses. These strings may have better odds in this game than in the original Penney's game. For example, HHH is an unfavorable string selection for Alice in Penney's game. However, with this string in the new game, Alice has a $1/2$ chance of attaining another H immediately after the first H and thus receiving the second occurrence of her string.

We used a computer to calculate the probabilities of different outcomes for $n = 3$. The results are presented in Table 10. As before, the numbers below are Bob's winning probabilities.

	HHH	HHT	HTH	HTT
HHH	*	$1/2$	$56/125$	$56/125$
HHT	$1/2$	*	$16/27$	$16/27$
HTH	$69/125$	$11/27$	*	$1/2$
HTT	$69/125$	$11/27$	$1/2$	*
THH	$11/16$	$3/4$	$1/2$	$1/2$
THT	$59/108$	$7/16$	$1/2$	$1/2$
TTH	$151/250$	$1/2$	$9/16$	$1/4$
TTT	$1/2$	$99/250$	$49/108$	$5/16$

Table 10: Bob's winning probabilities in the Second-Occurrence Penney's game

Table 11 displays Bob's optimal string choice for different string selections made by Alice and the corresponding odds. Bob's best choice when $n = 3$ is the same as in Penney's game and is represented in Figure 1.

Alice's string	Bob's best choice	Bob's odds
HHH	THH	11 to 5
HHT	THH	3 to 1
HTH	HHT	16 to 11
HTT	HHT	16 to 11

Table 11: Bob's best choice and odds.

We now discuss observations regarding the data presented in the tables.

1. If Alice and Bob select complementary strings, the probability of either winning is $1/2$, as expected.
2. We assumed that the Second-Occurrence Penney's game is fairer than the original Penney's game. The probabilities are not altered between Penney's game and this game in cases where both players have a winning probability of $1/2$. This result is not surprising; the probability $1/2$ is achieved for pairs of complementary strings and pairs of strings that differ only in the last character.
3. For the string pair HHT and THH (and, similarly, the complementary pair TTH and HTT), the probabilities did not change between this game and Penney's game. We can show not only for the Second-Occurrence Penney's game but for any k th-Occurrence Penney's game that the probability for this pair of strings fails to change. Indeed, if we have a long string of tosses, HHT and THH must alternate; THH always appears between two occurrences of HHT and, likewise, one HHT always occurs between two occurrences of THH. Therefore, whoever obtains the first occurrence of their string wins any k th-Occurrence Penney's game. The probability of THH winning will always be $3/4$.
4. For the remaining pairs of strings, the probabilities are closer to $1/2$ than in the original Penney's game.

5 Some theory

Penney's game has been extensively studied. The theory presented in this section is available in many places, including [WW04, Col82, Fel06, MGS74, MGT88, GO81, HN10, Pen69, Val17]. Although the theory is developed for alphabets of any size, we only discuss the case with the alphabet of size 2 corresponding to the two outcomes of coin flips.

5.1 Autocorrelation polynomial and Conway's Leading Number

The autocorrelation vector of a word $w = w_1 \dots w_n$, where w_i is the i -th character in the word, is $c = (c_0, \dots, c_{n-1})$, with c_i being 1 if the prefix of length $n - i$ equals the suffix of length $n - i$, and 0 otherwise. That is, value c_i indicates whether $w_{i+1} \dots w_n = w_1 \dots w_{n-i}$.

For example, the autocorrelation vector of HHH is $(1, 1, 1)$ since all the prefixes are equal to all the suffixes. The autocorrelation vector of THH is $(1, 0, 0)$ since no proper prefix is equal to a proper suffix. Finally, the autocorrelation vector of HHTTHH is $(1, 0, 0, 0, 1, 1)$.

Note that c_0 is always equal to 1 since the prefix and the suffix of length n are both equal to the word w . Similarly, c_{n-1} is 1 if and only if the first and the last letters are the same. If $c_1 = 1$, then all letters in the word are the same. It follows that if $c_1 = 1$, then for any i we have $c_i = 1$. If c_0 is the only value that is 1, we call the word w a *non-self-overlapping word*.

The autocorrelation polynomial of w is defined as $c_w(z) = c_0z^0 + \cdots + c_{n-1}z^{n-1}$. It is a polynomial of degree at most $n - 1$. For example, HHH's autocorrelation polynomial is $1 + z + z^2$, THH's is 1, and HHTTHH's is $1 + z^4 + z^5$.

The Conway Leading Number is the integer value of the autocorrelation vector viewed as a string of zeros and ones and evaluated in binary. Thus, HHH's Conway Leading Number is 7, THH's is 4, and HHTTHH's is 35. We denote the Conway Leading Number of a word w as C_w .

We can express the Conway Leading Number through the autocorrelation polynomial:

$$C_w = 2^{n-1} c_w \left(\frac{1}{2} \right)$$

5.2 Probabilities

Suppose we have a string A of length n . We call a string S that ends with A an *A-first-timer* if the last n characters of S is the only occurrence of A in S . We denote the number of *A*-first-timers of length i as $a_i(A)$ for $i \geq 0$. We call the sequence $a_1(A), a_2(A), \dots$ and so on an *A-first timer sequence*.

Let $p_i(A)$ denote the probability that we first see the string A after the i -th toss of a coin. The probabilities $p_i(A)$ and integers $a_i(A)$ are connected by the following equation:

$$p_i(A) = \frac{a_i(A)}{2^i}$$

For example, we know that $a_i(HT) = i - 1$. Thus, $p_i(HT) = \frac{i-1}{2^i}$.

The generating function for the w -first-timer sequence, where word w has length k , is known [GO81]:

$$G(z) = \frac{z^k}{z^k + (1 - 2z)c_w(z)}$$

If we plug in $z = \frac{1}{2}$, we get 1, which is the probability that the string ever occurs.

This generating function allows us to derive the formula for the expected wait time.

5.3 The expected wait time

The expected wait time of string A can be calculated as

$$\sum_0^\infty i p_i = \sum_0^\infty i a_i(A) \frac{1}{2^i}$$

Given that $G(z)$ above is the generating function for the sequence $a_i(A)$, the generating function for the sequence $ia_i(A)$ is $zG'(z)$, which equals

$$\frac{z(kz^{k-1}(z^k + (1 - 2z)c(z)) - z^k(kz^{k-1} - 2c(z) + (1 - 2z)c'(z)))}{(z^k + (1 - 2z)c(z))^2}$$

Minor manipulation produces

$$\frac{kz^k((1-2z)c(z)) + 2z^{k+1}c(z) - z^{k+1}(1-2z)c'(z))}{(z^k + (1-2z)c(z))^2}$$

When we plug in $z = \frac{1}{2}$, we get

$$\frac{2(\frac{1}{2})^{k+1}c(z)}{(\frac{1}{2})^{2k}} = 2^k c(z)$$

We can see that the expected wait time for a string is its Conway leading number times 2, which is a well-known fact.

Here are examples of the wait time for strings of length 3: HHH — 14; HTH — 10; HHT and HTT — 8.

We can see some patterns: If all letters are the same, then the wait time for a string of length n is $2^{n+1} - 2$. This is the longest possible wait time for strings of length n . Strings that are reverses of each other have the same wait time as they have the same autocorrelation polynomial. The shortest wait time is 2^n , which occurs when the strings are non-self-overlapping. We see that the wait time for some strings is almost twice the wait time for others.

5.4 Examples

Table 12 shows the examples of the first-timer sequences for strings of length less than or equal to 3 and some especially interesting examples for strings of length 4.

We group strings with the same autocorrelation polynomial, that is, strings with the same Conway Leading Number (CLN), into one line.

String	CLN	Sequence
H,T	1	1,1,1,1,1,1,1,1
HH, TT	11	0,1,1,2,3,5,8,13,21
HT, TH	10	0,1,2,3,4,5,6,7,8
HHH, TTT	111	0,0,1,1,2,4,7,13,24
HHT, HTT, THH, TTH	100	0,0,1,2,4,7,12,20,33
HTH, THT	101	0,0,1,2,3,5,9,16,28
HHHH, TTTT	1111	0,0,0,1,1,2,4,8,15
HHHT,HHT,HTTT,TTTH,TTHH,THHH	1000	0,0,0,1,2,4,8,15,28

Table 12: First-timer sequences for some strings of length up to 4.

Table 13 shows, given a Conway Leading Number, the sequence number in the OEIS and its linear recurrence. The sequence in the OEIS is often shifted. We also provide the name of the sequence if it is short.

CLN	OEIS	Name	Recurrence
1	A000012	Only 1s	$a_n = a_{n-1}$
11	A000045	Fibonacci Numbers	$a_n = a_{n-1} + a_{n-2}$
10	A001477	Whole numbers	$a_n = 2a_{n-1} - a_{n-2}$
111	A000073	Tribonacci Numbers	$a_n = a_{n-1} + a_{n-2} + a_{n-3}$
100	A000071	Fibonacci Numbers -1	$a_n = 2a_{n-1}a_{n-3}$
101	A005314	(No short description)	$a_n = a_{n-1} + a_{n-2} + a_{n-4}$
1111	A000078	Tetranacci numbers	$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$
1000	A008937	Partial sums of Tribonacci numbers	$a_n = 2a_{n-1} - a_{n-4}$

Table 13: First-timer sequences for small strings in the OEIS.

For example, consider the string HHT of length 3. The corresponding first-timer sequence is:

$$0, 0, 1, 2, 4, 7, 12, \dots$$

with the recurrence $a_i = 2a_{i-1} - a_{i-3}$ for $i > 2$.

This is sequence A000071 in the OEIS. We see that $a_i(\text{HHT}) = F_{i+1} - 1$, where F_i is the i -th Fibonacci number. This is also sequence of partial sums of the Fibonacci sequence shifted one place to the right. One of the descriptions of this sequence in the OEIS says that this sequence is the number of 001-avoiding binary words of length $i - 3$, which is how we defined our sequence. Indeed, removing HHT from the end of HHT-first-timers creates a bijection between our strings and strings of length $i - 3$ avoiding HHT. The generating function for the HHT-first-timer sequence is

$$\frac{x^2}{(1 - x - x^2)(1 - x)}$$

Next, let's look at strings that consist of the same letter. In our table, these correspond to the following sequences: Only 1s, Fibonacci numbers, Tribonacci numbers, and Tetranacci numbers. They form a clear pattern: for strings of length n , each next term of the sequence is the sum of n previous terms. The sequence of only 1s belongs to this pattern. One might call it the "mononacci" numbers.

Now, consider non-self-overlapping strings — strings with Conway Leading Numbers that are powers of two. In our table, they correspond to the following sequences: Only 1s, Whole numbers, Fibonacci numbers -1 , Partial sums of Tribonacci numbers.

At first glance, there is no clear pattern. However, we can relate these sequences using the following famous identity for the Fibonacci numbers:

$$\sum_{i=0}^n F_i = F_{n+2} - 1$$

That is, the Fibonacci sequence is the sequence of partial sums of the Fibonacci numbers shifted by two terms. Now we can see a pattern. The sequence of whole numbers is also the sequence of partial sums of the sequence of all ones (our mononacci sequence). Thus, for a non-self-overlapping string of length n , the sequence is the partial sums of the $n - 1$ -nacci numbers.

5.5 Game results

To obtain an explicit formula for winning probabilities, we need to extend the notion of Conway Leading Numbers. Conway Leading Numbers quantify the overlaps of a word with itself. However, we can also quantify the overlap of a word with another word. We only define the correlation of two words of the same length n , although it also can be defined for words of different lengths.

The correlation vector of two words w_1 and w_2 is $c_{w_1, w_2} = (c_0, \dots, c_{n-1})$, with c_i being 1 if the prefix of length $n - i$ of w_2 equals the suffix of length $n - i$ of w_1 and 0 otherwise.

The correlation vector defines the correlation polynomial, the same way as it is defined by one word. It also defines the Conway Leading Number for two words, which we denote as C_{w_1, w_2} .

If A is Alice's string and B is Bob's string, then the formula for Bob's odds in Penney's game is

$$\frac{C_{A,A} - C_{A,B}}{C_{B,B} - C_{B,A}}$$

For example, let's look at HHT and THT. For $i = 0$, we see that HHT is different from THT, so $c_0 = 0$. For $i = 1$, we see that HT is different from TH, meaning that $c_1 = 0$. For $i = 2$, since the last character of HHT is the same as the first character of THT, we get $c_2 = 1$. Therefore, $c_{HHT, THT} = (0, 0, 1)$ and $C_{HHT, THT} = 001_2 = 1$. Similarly, we get that $C_{HHT, HHT} = 4$, $C_{THT, THT} = 5$, and $C_{THT, HHT} = 0$. This means that Bob's comparative odds are $\frac{4-1}{5-0} = \frac{3}{5}$, and his probability of winning is $\frac{3}{8}$.

Based on this formula, the strategy for Bob to pick the best string when Alice's string is given in [Fel06]: Bob takes the second character in Alice's string and flips it. Then, he appends Alice's prefix of size $n - 1$ to it. For example, if Alice's string is HHT, Bob flips the second character H and uses T as his first character. Then he appends HH. Thus Bob's best choice is THH.

We are now ready to describe a new game that might be considered a fair game.

6 The Two-Coin game

Alice and Bob, dismayed by their repeated failure to attain a fair game, decide to invent a game involving two coins. In this new game, Alice and Bob toss their coins side-by-side, thus simultaneously developing separate sequences of flip outcomes. The first person to attain their string through their sequence of flips is the winner. A tie is possible if both players get their strings at the same time.

This game finally evens the odds. We show that the winning probabilities for Alice and Bob are solely dependent on the expected wait times.

Proposition 6.1. *The probabilities of a win/loss/tie depend only on the expected wait times of Alice's and Bob's strings.*

Proof. Let Alice's string be A and Bob's string be B . Suppose that p_i and q_i are the probabilities that Alice and Bob, respectively, see their string for the first time at the i th toss. These probabilities allow us to calculate the winning, losing, and tying probabilities for Alice.

Alice wins with probability

$$\sum_{i=1}^{\infty} p_i \left(1 - \sum_{k=1}^i q_k \right)$$

Alice loses with probability

$$\sum_{i=1}^{\infty} q_i \left(1 - \sum_{k=1}^i p_k \right)$$

There is a tie with probability

$$\sum_{i=1}^{\infty} p_i q_i$$

Since the sequences p_i and q_i depend only on the autocorrelation of A and B , respectively, the winning probabilities depend only on the autocorrelation of each sequence. In other words, they only depend on the Conway Leading Numbers, which are half the expected wait time. $\square$

Note that these probabilities sum up to 1. That is,

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} q_i = 1$$

It follows that the probabilities of a win, loss, or a tie sum up to 1. Equivalently, the probability of an infinite game is 0.

Example. Suppose Alice and Bob both choose HT. As we showed before, $p_i(HT) = \frac{i-1}{2^i}$. Thus, the probability of a tie is

$$\sum_{i=1}^{\infty} \frac{(i-1)^2}{4^i} = \sum_{i=0}^{\infty} \frac{1}{4} \cdot \frac{i^2}{4^i}$$

The generating function of the squares sequence is $\frac{x(1+x)}{(1-x)^3}$. Plugging in $x = \frac{1}{4}$ and dividing by 4, we get $\frac{5}{27}$. That is, if both Alice and Bob choose HT or TH, then a tie occurs with probability $\frac{5}{27}$, and each of them wins with probability $\frac{11}{27}$.

We wrote a program to calculate probabilities for strings of length 3. They are displayed in Table 14. The top row represents Alice's string's Conway Leading Number. The first column shows the Conway Leading Number for Bob's string. The numbers in each cell, from left to right, represent the probabilities of Bob winning, Alice winning, and a tie.

	111	101	100
111	$\frac{435}{913}, \frac{435}{913}, \frac{43}{913}$	$\frac{3289}{8691}, \frac{1643}{2897}, \frac{473}{8691}$	$\frac{23327}{73057}, \frac{45409}{73057}, \frac{4321}{73057}$
101	$\frac{1643}{2897}, \frac{3289}{8691}, \frac{473}{8691}$	$\frac{487}{1045}, \frac{487}{1045}, \frac{71}{1045}$	$\frac{22431}{55265}, \frac{28673}{55265}, \frac{4161}{55265}$
100	$\frac{45409}{73057}, \frac{23327}{73057}, \frac{4321}{73057}$	$\frac{28673}{55265}, \frac{22431}{55265}, \frac{4161}{55265}$	$\frac{377}{825}, \frac{377}{825}, \frac{71}{825}$

Table 14: Probabilities of different outcomes.

In Table 15, we approximated the values from Table 14 to make it easier to compare.

	111	101	100
111	.48, .48, .047	.38, .57, .055	.32, .62, .059
101	.57, .38, .054	.47, .47, .068	.41, .52, .075
100	.62, .32, .059	.52, .41, .075	.46, .46, .086

Table 15: Approximate probabilities of different outcomes.

In Figure 3, the arrows point from Alice’s Conway Leading Numbers to the Conway Leading Number of Bob’s best choice.

Figure 3: Bob’s best choice.

The results are less surprising for this game when compared to Penney’s game. The strings with shorter expected wait times have better winning odds. Thus, both Alice and Bob should choose a string with a Conway Leading Number of 100 — one of HHT, HTT, THH, or TTH.

Alice and Bob finally found a fair game. They also got tired of probabilities. They now look to create a coin-flipping game involving no coin flips.

7 The No-Flippancy Game

Alice and Bob create a game based only on skill and strategy. Instead of relying on luck, they must use logic to win.

In this game, Alice chooses her string of length n . Bob then chooses his string of the same length. The outcome of the game is decided deterministically. Alice and Bob take turns looking

at the current sequence of tosses and determining whether they need heads or tails next in order to progress. They do this by calculating the largest suffix of length i which matches the prefix of their string. The next move is the character in position $i + 1$ of their string.

We study the games where $n = 3$.

Table 16 shows how the game proceeds. The top row shows Alice's choice and the left column shows Bob's choice.

	HHH	HHT	HTH	HTT
HHH	HHH	HHT	HHTH	HHTHHT...
HHT	HHH	HHT	HHT	HHT
HTH	HTH	HTH	HTH	HTT
HTT	HTHTHT...	HTHTHT...	HTH	HTT
THH	HTHH	HTHH	HTH	HTT
THT	HTHT	HTHT	HTH	HTT
TTH	HTHTHT...	HTHTHT...	HTH	HTT
TTT	HTHTHT...	HTHTHT...	HTH	HTT

Table 16: The output of the game.

Table 17 displays who wins, followed by the number of turns in the game.

	HHH	HHT	HTH	HTT
HHH	Tie, 3	Alice, 3	Alice, 4	Tie, infinite
HHT	Alice, 3	Tie, 3	Bob, 3	Bob, 3
HTH	Bob, 3	Bob, 3	Tie, 3	Alice, 3
HTT	Tie, infinite	Tie, infinite	Alice, 3	Tie, 3
THH	Bob, 4	Bob, 4	Alice, 3	Alice, 3
THT	Bob, 4	Bob, 4	Alice, 3	Alice, 3
TTH	Tie, infinite	Tie, infinite	Alice, 3	Alice, 3
TTT	Tie, infinite	Tie, infinite	Alice, 3	Alice, 3

Table 17: The output of the game.

We see that if Alice chooses her string first, Bob can always choose his string so that he wins. In fact, Bob can use his optimal strategy in Penney's game to pick his winning string.

Also, Alice loses even though she goes first. This game has many other intriguing properties, which we discuss in a separate paper [Fli20].

Alice realizes that removing probabilities was a mistake, so she decides to add them into the No-Flippancy game and create a new game.

8 The Blended game

In this new game, Alice and Bob choose strings of the same length. As in the No-Flippancy game, they each want to increase their prefix in the output by 1. If they want the same outcome from the next coin toss, they get it. If they want different outcomes, they flip a coin.

If Alice's first $n - 1$ characters are the same as Bob's, then the probability of each of them winning is $1/2$. This is because they both want the same toss for the first $n - 1$ tosses, so the game progresses to that state. Then, the winner depends on the outcome of the last toss, so they both win with $1/2$ chance.

Additionally, complementary strings will have equal probabilities of winning by symmetry.

We used a computer to calculate the results for other pairs of strings where $n = 3$. Table 18 shows Bob's winning probabilities.

3	HHH	HHT	HTH	HTT
HHH	*	1/2	1/4	2/5
HHT	1/2	*	1/2	2/3
HTH	3/4	1/2	*	1/2
HTT	3/5	1/3	1/2	*
THH	7/8	3/4	1/4	1/2
THT	7/12	3/8	1/2	3/4
TTH	7/10	1/2	5/8	1/4
TTT	1/2	3/10	5/12	1/8

Table 18: Bob's winning probabilities.

We see that the majority of entries in this table are the same as in Penney's game, although there are some differences.

Listed below in Table 19 are Bob's best choice and odds.

Alice's string	Bob's best choice	odds
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	TTH	5 to 3
HTT	THT	3 to 1

Table 19: Bob's best choice and odds.

Figure 4 shows Bob's best choice. The arrow points from Alice's string to Bob's best choice. The below graphic demonstrates something rather beautiful about this game: it has a non-transitive cycle of length 6. Due to this, Bob can always pick a winning string as in Penney's game.

Figure 4: Bob's best choice.

After playing these numerous coin games, Alice and Bob have had enough. They decide to donate their lustrous coin to charity and their discoveries to the world of mathematical research.

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