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MARGINAL UTILITY AND COURNOT STABILITY UNDER MARKET ENTRY

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Abstract:

This paper introduces into the discussion of the stability of quantity-oligopoly equilibrium, the role that the slope of the marginal utility curve and of market entry in shaping the equilibrium and its stability. It does so by considering inverse multivariate linear demand functions and the notion of stability related to multivariate mean value theorem. The equilibrium cluster of Cournot sellers is determined by the stability dictated by this theorem and the rate of decline of the marginal utility of the product under consideration. Strategic complementarity is found to be the case under product heterogeneity, while the strategic substitutability associated with product homogeneity, induces a modification of Cournot limit theorem.

Key words: *Cournot equilibrium stability, General quantity oligopoly, Quasilinear quadratic utility, Mean value theorem*

1. Introduction

This paper investigates the stability of Cournot equilibrium from the viewpoint of mean value theorem under inverse multivariate linear demand functions coming out of Levitan and Shubik's (Shubik 1980, Choné and Linnemer 2019) non-symmetric quasilinear quadratic utility model (QQUM); it is a utility model employed often for differentiated oligopoly analyses (see e.g. Ardabili and Liu 2017). Stability is thus related to the steepness of the marginal utility curve, which in turn shapes market entry. As Corchón and Torregrosa remarked in 2018 " 'Cournot equilibrium' has more than 50.000 Google citations." It should be added that most of them sprung out of the development of game theory and particularly, of its aggregative-games branch given that Cournot (and Bertrand) competition are susceptible to game-theoretic treatment anyway (Corchón and Marini 2018). Also, the dynamic version of this competition renders the subject attractive as an application area of mathematics and particularly, of nonlinear differential equations and chaos theory (Al-khedhairi 2019, Pecora and Sodini 2018, Zhou et al. 2018). In addition, there are combinations of the two approaches (Yu and Yu 2018, Askar et al. 2018, Elsadany 2017) and extensions of the basics of quantity competition (Barthel and Hoffmann 2019, Delbono and Lambertini 2018, Kollias et al. 2017), particularly in

connection with the information requirements of the competition (Andaluz et al. 2020, Ueda 2019, Sarafopoulos and Papadopoulos 2018).

Methodologically, such papers put too much emphasis on the technical aspect of the argument made each time, leading to conclusions that do not quietly fall in the realm of what we know to have been Cournot's research agenda on the subject (Puu 2018). For example, Cournot oligopoly is a topic which is only subsumed by aggregative games and their generalizations (Garg and Jaakkola 2017); the economics of the topic is of secondary importance as the technical argument would be equally applicable to other topics like public goods, or cost and surplus schemes. Of course, the economics of Cournot oligopoly is not a doctrine, which is true for all economics and any science. But, there are certain stylized facts like that Cournot focused on quantity competition because he wanted to examine the effect that increasing market entry (towards perfect competition) would have on a monopolized market. "His" oligopoly is a market structure between the two extremes of monopoly and perfect competition, and should be studied as such.

The present paper is exactly in this spirit to the extent that the stability of Cournot equilibrium is examined by means of the mean value theorem. Applying this theorem to quantity competition, it states (in its classical, Lagrange's form) that: $dp(q)/dq = \{[p(q^M) - p(q^C)]/(q^M - q^C)\}$, where $p(q)$ is the indirect demand function, and q^M and q^C are the equilibrium quantities under monopoly and Cournot competition, respectively. It is not the difference between monopoly and perfectly competitive quantities, because competition is not the impersonal one of perfect competition that makes "partial productions *inappreciable*" (Cournot 1897[1838], Ch VIII, p. 90). He makes the behavioral assumption that competition takes place in steps and that in every step each firm takes for granted the last values that were taken by the competitors (without thinking about their future reaction). The number of firms that satisfy the mean value theorem is the optimal number from the viewpoint of stability as well. Since $p(q)$ is a continuous function on the closed interval $[q^M, q^C]$ and differentiable on the open interval (q^M, q^C) , there is a quantity $q = q^*$ in the latter interval, which quantity is the abscissa at the planar qp -space of the point at which the tangent to the planar demand curve is parallel to the secant between the two endpoints q^M and q^C . By the definition then of q^* and assuming convexity of $p(q)$ in the positive quadrant, the number n of firms corresponding to this quantity will be stable.

The next section makes a first attempt to see if there does exist a stability issue for a general Cournot oligopoly in the context of the multivariate mean value theorem; a first only attempt, because the discussion centers upon demand curves based on the QQUM. It does relate this issue to increasing n , viz. to market entry, guided by the slope of the marginal utility curve related to the product under consideration, in contrast to the only similar work on the subject by Gao and Du (2018), who leave unmotivated the linear indirect-demand function used. The analysis below considers the two extreme cases of differentiated oligopoly and different consumer preferences for each product variety, and of homogeneous symmetric oligopoly with preference indifference about the origin of the product. The stable market size is found to be unique, depending on the rate of decline of marginal utility, and on the difference between marginal utility and marginal cost, i.e. on marginal surplus, the sum of consumer and producer surplus. It also depends on product substitutability or complementarity in the case of differentiated oligopoly.

Theocharis' (1960) stability result and its reformulations (see for recent ones, Gao and Du 2018, and Hommes et al. 2018) is disputed. This paper, confirms in general the global stability results surrounding Cournot competition; results such as those by Okuguchi and Yamazaki (2008) though from the perspective of the mean value theorem and by emphasizing the role of the slope of the marginal utility curve. Put in terms of game theory, strategic substitutability or complementarity is found to be the case when this slope is less or greater than one, respectively. And, the multivariate version of the mean value theorem allows equilibrium n to be endogenously determined uniquely by stability considerations. Stability shapes the extent of entry into a monopolistic market towards the equilibrium cluster of firms given the slope of the marginal utility curve, which slope signals the attainment of equilibrium once this cluster has been formed. Section 3 concludes this paper with a further explanation of the character of the approach followed here in the light of previous similar work and in terms of aggregative game theory.

2. Formal Considerations

As noted above, there are n products and the consumer is assumed to behave in line with the QQU model, maximizing,

$$U(q_1, q_2, q_s) - \sum_1^n p_i q_i = \sum_1^n (a_i - p_i) q_i - \frac{1}{2\beta} [2\sigma \sum_i \sum_{j>i} q_i q_j + [n - (n-1)\sigma] \sum_1^n q_i^2] \quad (1)$$

where U is utility, and $i, j = 1, 2, \dots, n, i \neq j$. Parameter $\sigma \in (-\infty, 1)$ captures the substitutability or complementarity among the heterogeneous goods. Coefficient $\alpha > 0$ is a (marginal) quality (or utility) index, one for each variety i : Other things equal, an increase in α increases the marginal utility of consuming the good to which it refers. $\partial^2 U / \partial q_i^2 = 1/\beta > 0$ shows how quickly the marginal utility of each good declines; it is a concavity parameter. It is a very important parameter, because the decline of the marginal utility derived from additional consumption becomes smaller as β becomes larger, encouraging market entry. If people need much to consume of a good to get satisfied, it pays to enter such a market even with a slightly only differentiated product when goods are substitute. In the case of complementarity, market entry with a complementary product will have a good chance to succeed only when complementarity is strong. But, if the marginal utility curve is steep, it pays to enter the market only by selling what the competitor does in order to appropriate market share. This intuition is verified by our results below.

Now, the indirect and direct demand functions obtained from the maximization of (3) are:

$$p_i = a_i - \frac{1}{\beta} \{ \sigma Q_{-i} + [n(1 - \sigma) + \sigma] q_i \} \quad (2)$$

$$q_i = \frac{\beta}{n^2(1 - \sigma)} [(n - \sigma)(a_i - p_i) - \sigma \sum_j (a_j - p_j)] \quad (3)$$

respectively, where the summation $\sum_j (\cdot)$ applies to all firms except the i th one, and $Q_{-i} \equiv \sum_j q_j$. Each product is provided by one only firm at average and marginal cost c_i , obtaining the following Cournot equilibrium quantities and prices (Choné and Linnemer 2019):

$$q_i^c = \frac{\beta}{[2n - (2n - 1)\sigma]^2} [2n(1 - \sigma)(a_i - c_i) - \sigma \sum_j (a_j - c_j)] \quad (4)$$

$$p_i^C - c_i = \frac{n - (n-1)\sigma}{[2n - (2n-1)\sigma]^2} [2n(1-\sigma)(a_i - c_i) - \sigma \sum_j (a_j - c_j)] \quad (5)$$

where the superscript "C" refers to Cournot competition. The difference $(a_i - c_i)$ is the marginal surplus whose positiveness manifests that there does exist a market for the i th product. The case with monopoly by the i th firm is:

$$q_i^M = \frac{\beta(a_i - c_i)}{2} \quad (6)$$

$$p_i^M = \frac{(a_i + c_i)}{2} \quad (7)$$

where the superscript "M" connotes monopoly variables.

The multivariate mean value theorem has in our case as follows (*planetmath.org*):

$$p_i^M - p_i^C = \frac{dp_i}{dq_i} (q_i^M - q_i^C) + \frac{dp_i}{dQ_{-i}} (Q_{-i}^M - Q_{-i}^C) \quad (8)$$

in absolute values. Given n, σ , and β , it determines a single stable point in the three-dimensional space of $p_i - q_i - Q_{-i}$. If n is allowed to vary, (8) gives the various stable n 's depending on individual and aggregate marginal surpluses, given again σ and β . Inserting (4) in (2), yields p_i at q_i^C , while inserting (6) in (2), gives p_i at q_i^M , so that:

$$p_i^M - p_i^C = \frac{(a_i + c_i) - 2\beta a_i - 2\{\sigma Q_{-i}^C + [n(1-\sigma) + \sigma]q_i^C\}}{2\beta}$$

Subtracting (4) from (6):

$$q_i^M - q_i^C = \frac{\beta(a_i - c_i)\{[2n - (2n-1)\sigma]^2 - 4n(1-\sigma)\} + 2\beta\sigma \sum_j (a_j - c_j)}{2\beta}$$

while adding "(4)s": $Q_{-i}^C = 2\beta[n(1-\sigma) - \sigma] \sum_j (a_j - c_j) / [2n - (2n-1)\sigma]^2$, and adding "(6)s": $Q_{-i}^M = \beta \sum_j (a_j - c_j) / 2$, one obtains the difference:

$$Q_{-i}^M - Q_{-i}^C = \frac{\beta \sum_j (a_j - c_j) [2n - (2n-1)\sigma]^2 - 4\beta \sum_j (a_j - c_j) [n(1-\sigma) - \sigma]}{2[2n - (2n-1)\sigma]^2}$$

Of course, Q_{-i}^M would be the case if all firms but the i th were all together a monopoly, acting not as such, not as a "fusion or association" of oligopolists as Cournot might put it (p. 103), but by "separation of interests", as a sum of individual-seller outcomes. Given now that: $\partial p_i / \partial q_i = -[n(1-\sigma) + \sigma] / \beta$ and $\partial p_i / \partial Q_{-i} = -\sigma / \beta$, (8) becomes after some operations that:

$$z = \frac{y\{\beta[2n - (2n-1)\sigma]^2 + 2\beta n(1-\sigma) - [n(1-\sigma) + \sigma][2n - (2n-1)\sigma]^2\{[2n - (2n-1)\sigma]^2 - 4n(1-\sigma)\}\}}{\sigma\{\beta[2n - (2n-1)\sigma]^2 - 8\beta[n(1-\sigma) - \sigma] - 2[n(1-\sigma) + \sigma][2n - (2n-1)\sigma]^2 + 2\}}$$

where $z \equiv \sum_j (a_j - c_j)$ and $y \equiv (a_i - c_i)$.

This expression in z, y, n, β , and σ , call it "z function", is impossible to handle algebraically and consequently, insight is sought below through its calibrations. But, first note that when q_i and Q_{-i} are independent and $\sigma = 0$, the expression for z becomes:

$$2(a_i - c_i)(\beta + 2\beta n + 8n^3 - 8n^4) = 0$$

The solution to this quartic equation with respect to n – given that $a_i > c_i$ or the same, that there does exist a market for q_i by assumption – results in a fixed value of it (depending only on β), as one might plausibly expect given the independence of the operations in the market for q_i from the developments surrounding Q_{-i} . For example, when $\beta = 0.2$, the only real positive solution is: $n = 1.065$, implying that only one firm markets q_i , as it was

assumed in the first place. It takes a high $\beta = 12$ to get $n = 1.97 \approx 2$, that is, a homogeneous duopoly of the good marked by q_i .

We proceed with the calibrations based on the use of *wolframalpha.com*, with n being set equal to x , obtaining the following diagrams for $\sigma = 0.7, \sigma = 0.3, \sigma = -0.5$, and $\sigma = -1$, and $\beta = 1$ and $\beta = 5$.

Fig. 1(a): $\sigma = 0.7, \beta = 1, y = 20, n = 50$

Fig. 1(b): $\sigma = 0.7, \beta = 5, y = 20, n = 50$

Fig. 2(a): $\sigma = 0.3, \beta = 1, y = 20, n = 50$

Fig. 2(b): $\sigma = 0.3, \beta = 14, y = 20, n = 50$

Fig. 3: $\sigma = 0.7, \beta = 1, y = 20, n = 1000$

Fig. 4: $\sigma = -0.5, \beta = 5, y = 15, n = 1$

Fig. 5(a): $\sigma = -1, \beta = 1, y = 20, n = 50$

Fig. 5(b): $\sigma = -1, \beta = 5, y = 20, n = 5$

The contours on the surfaces of Figs. 1-5, illustrate stable combinations of n, y , and z . A given n can be consistent with various combinations of y and z . Since y is monopolist's marginal surplus before entry and hence, a datum, there is a one-to-one correspondence between n and z . What Figs. 1-3 illustrate is that as long as there is a substitute that can be produced and there is a market for it, rivalry cannot do much to prevent it from being provided. Fig. 1(b) suggests that parameter β has only a small upward effect on marginal surplus when substitutability is high but, according to Fig. 2(b), it has to increase considerably to have a similar effect when substitutability is small. These findings verify intuition about the positive role of β in encouraging market entry under substitutability, but up to a "point", which point takes on a concise meaning under complementarity. According to Fig. 4, β has to be large to have one at most firm in the market when complementarity is moderate. It helps the provision of a product that can complement a series of other complementary products and that would remain unproduced otherwise.

This is the most that the high β can do from the point of view that it does not encourage the supply of the goods that would complement the one which was helped enter the market. Calibrations with higher β show negative surplus. When complementarity is strong, Fig. 5(a) shows that the market result can be similar to that under substitutability, harmed according to Fig. 5(b) seriously when β is high. Now, judging this result in the light

of the findings about β in connection with Fig. 4 from the viewpoint of maximum z , one might safely conclude that there is a one-to-one correspondence among β , the optimal number of firms in the market, and the intensity of complementarity. To prove this formally, the quintic(?) equations $\partial z / \partial \beta = 0$ and $\partial z / \partial \sigma = 0$ have to be solved for β and $\sigma \in (-\infty, 0)$, respectively, given the proper codomain, (since we consider z being a bijective bivariate function in β and σ), which is a formidable task. Anyway, in a few words, under substitutability and strong complementarity with a "reasonable value" of β : "there is room for everyone". All $n - y - z$ configurations are stable, and market entry does not prompt instability because the total and thereby per-firm-surplus grow exponentially. Nothing can stop the marginal surplus from increasing until the market runs out of product "invention", of making n larger, all else, of course, being the same.

The role of parameter β is also the only factor shaping stability and market entry when a homogeneous product is provided under identical production costs as follows. In this case, indirect demand becomes:

$$p = a - \frac{1}{\beta}(q_i + Q_{-i})$$

while under Cournot equilibrium:

$$q_i^C = \frac{\beta(a - c)}{n + 1}$$

$$Q^C = \frac{n\beta(a - c)}{n + 1}$$

and:

$$p^C = \frac{a + nc}{n + 1}$$

where $q_i = q_j = q$. Also, since: $Q^M = nq^M = n\beta(a - c)/2n$:

$$(n - 1)q^M = Q_{-i}^M = \frac{(n - 1)\beta(a - c)}{2n}$$

Applying these values to multivariate mean value theorem formula, and carrying out operations, yields that:

$$n = \frac{1 + \sqrt{1 - \beta^2}}{\beta}$$

whose calibration appears in Fig. 6:

There is a bijection between β and n , with β being less than one, as expected to be the case under product homogeneity. The rapid decline of marginal utility, MU , is a motive to enter the market of this good given that the value of β allows such entry. If for example, β is between 0.7 and 0.8, the equilibrium number of firms is only one. Or, when β is around 0.35, the equilibrium number of firms will be five. These are all as stable equilibria as those under product heterogeneity, with the modification now that as β becomes smaller and more firms can become consistent with the equilibrium, profit-per-firm declines.

Fig. 6: $\sigma = 1, (a_i - c_i) = (a - c), x \equiv \beta, f(x) \equiv n$

As Fig. 6 implies, $n \rightarrow \infty$ as $dMU/dq \rightarrow 0$, which, in turn, implies that profit-per-firm can eventually fall to zero. The difference with Cournot and others like Amir and Lambson (2000) in obtaining this outcome is that it now depends on the value of β . Under homogeneous product, there can be no successful market entry if β does not allow it. Once the equilibrium cluster of firms has been formed, β signals the saturation of the market with sellers who behave in line with the best-reply rule. The Cournot limit theorem becomes sensible only from this point of view, which obtains by letting the slope coefficient of the indirect demand function be interpreted as the slope coefficient of the marginal curve, too. Let $\bar{n} = \sup n$ and $\underline{\beta} = \inf \beta$ so that $\lim_{\beta \rightarrow \underline{\beta}} n(\beta) = \bar{n}$. The theorem becomes now:

$$\lim_{\beta \rightarrow \underline{\beta}} Q^c(\beta) = \frac{\bar{n}\beta(a - c)}{\bar{n} + 1} \approx \underline{\beta}(a - c)$$

for large $\bar{n}$. There does have to exist a non-zero minimum value of β , because otherwise:

$$\lim_{\beta \rightarrow 0} n(\beta) = \infty \text{ and thereby, } \lim_{\beta \rightarrow 0} Q^c(\beta) = \infty, \text{ which is not plausible, given that } dn/d\beta < 0.$$

3. Concluding Remarks

It is self-evident that the studies on Cournot stability either hold up to the "tradition" of 2-5-firm stability (Theocharis 1960, Hahn 1962, Al-Nowaihi and Levine 1985) or make a case for global stability (Okuguchi and Yamazaki 2008). This paper should be taken to fall in the realm of the latter case from a perspective that tried to be as close to Cournot's thinking as possible. The assumptions and purpose of the approach adopted by each study play a critical role in assessing stability. The example of Gao and Du (2018), who apply the mean value theorem too, assuming product homogeneity, and end up supporting Theocharis' (1960) version of 2-firm stability, is quite instructive, simply because they (i)

use Lagrange's version of this theorem, and (i) do not motivate the standard linear demand curve employed by their analysis. The use of Lagrange's mean value theorem subsumes that $(Q_{-i}^M - Q_{-i}^C) = 0$ and subsequently, two in total rivals, a duopoly, in which market entrants are in effect excluded from playing the Cournot-Nash equilibrium strategy. Stability is based on this presumption rather than on a utility-motivated demand curve. A homogeneous oligopoly can be stable because its size is determined by the steepness of the marginal utility curve for its product, which is common knowledge and discourages entry once the equilibrium number of firms has been reached.

Indeed, in game-theoretic terms, it is β which rationalizes the submodularity or supermodularity of the aggregative Cournot game. Under submodularity, individual action and the action of the others lumped together, are strategic substitutes while under supermodularity, strategies complement each other, there is profitable imitation of the actions of others. Letting $\pi_i(q_i, Q_{-i})$ be the payoff function of the i th firm, and defining $h_i(q_i, Q_{-i}) = d\pi_i/dq_i + d\pi_i/dQ_{-i}$, the aggregative game is submodular (supermodular) if $dh_i/dQ_{-1} \leq 0$ (≥ 0). Now, we saw above that h_i is positive when products are heterogeneous, because $d^2\pi_i/dQ_{-1}^2 > 0$ as n increases and hence, $dh_i/dQ_{-1} > 0$. And, such products are associated with $\beta > 1$ while $\beta < 1$ is the case with product homogeneity, with $dh_i/dQ_{-1} < 0$. This is one of the two novelties of this paper, which along with the second one of not taking n to be fixed, but considering the equilibrium number of sellers to be shaped by the stability conditions coming out of the multivariate mean value theorem given β , have produced global stability for any type of quantity competition. Note that this version of the theorem does allow all potential players to play the equilibrium strategy, but β is one of the factors, the only one in the case of product homogeneity, which acts so as to sort out the equilibrium cluster right from the start of the game. When, for instance, the marginal utility curve crosses the quantity, horizontal axis at $q = \bar{q}$, only those firms that can produce $\bar{q}$ have a chance to form the equilibrium cluster in a stable fashion, and they do form it.

The nature of strategic substitutability or complementarity may be clarified further by looking into what best-reply behavior involves under the different types of product. When the product is heterogeneous, it might be argued that potential market entrants are inspired by heterogeneity imitation, whose limit is the technical possibility of such an imitation. There is clearly strategic complementarity, which in the context of Cournot's naive expectations, entails imitation of the sort that when a seller takes for granted the quantity produced by the others, s/he does not conjecture the same about the price at which this quantity is sold. Such a price-taking, Walrasian presumption is made by entrants only under product homogeneity, when trying to extract market share by incumbent firm(s) in the market through strategic substitutability. Price-taking behavior, deciding next-period profit-maximizing output by acting as a price taker, is a key component of "naive" strategic substitutability, but does not connote Walrasian behavior. The choice of Walrasian behavior, leading to a globally stable Walrasian equilibrium, can be motivated from the point of view of evolutionary game theory suffices to have the rate of change of marginal cost exceeding the sum of residual-market price sensitivities to output (Radi 2017). Here, in this paper, this rate is nil, and nil are the price sensitivities *ex hypothesi*.

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