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"MATHEMATICS IS THE LOGIC OF THE INFINITE": ZERMELO'S PROJECT OF INFINITARY LOGIC

Abstract. In this paper I discuss Ernst Zermelo's ideas concerning the possibility of developing a system of infinitary logic that, in his opinion, should be suitable for mathematical inferences. The presentation of Zermelo's ideas is accompanied with some remarks concerning the development of infinitary logic. I also stress the fact that the second axiomatization of set theory provided by Zermelo in 1930 involved the use of extremal axioms of a very specific sort.[1]

Keywords: infinitary logic, axiomatic set theory, extremal axiom.

1. Introductory remarks

I am going to present here Zermelo's ideas behind his project of infinitary logic developed over eighty years ago. At the time, the project was innovative but drew little attention from logicians and mathematicians. Systematic research on infinitary logic was only to begin two decades later.

My considerations are based on Zermelo's original papers (including his *Nachlaß*) as well as the discussion of Zermelo's ideas published by other authors. The complete works of Ernst Zermelo were published quite recently in German, with an accompanying English translation; see Ebbinghaus, Fraser and Kanamori 2010. A few years ago I translated Zermelo's papers on the foundations of mathematics (from German to Polish). The translation, under the title *Matematyka jest logiką nieskończonego. Prace Ernsta Zermela z podstaw matematyki* (*Mathematics is the logic of the Infinite. Ernst Zermelo's works on the foundations of mathematics*) remains unpublished.

2. Ernst Zermelo: a few biographical remarks

Ernst Friedrich Ferdinand Zermelo (born 27 July 1871 in Berlin, died 21 May 1953 in Freiburg i. Br.) was one of the most prominent mathematicians of the twentieth century. He is known primarily as the author of the first axiomatization of set theory. Other topics he covered include the calculus of variations, applications of mathematics in physics, navigation problems, and the application of set theory to the game of chess. He discovered Russell's antinomy before Russell and was also a translator of mathematical as well as literary works (fragments of *Odyssey*). He was the editor of Georg Cantor's collected works.

Zermelo worked at the universities in Göttingen, Zürich and Freiburg, and presented the first course entitled mathematical logic. He was involved in the investigations concerning the foundations of mathematics in two periods: in Göttingen (1899–1910), working with David Hilbert, and in Freiburg (1921–1935). His project of infinitary logic was initiated in the second of these periods.

Ernst Zermelo's *Collected works* were published quite recently, in two volumes, with the original German texts and English translations; see Ebbinghaus, Fraser and Kanamori 2010. The first volume (*Set Theory, Miscellanea*) contains Zermelo's works on the foundations of mathematics. The book Ebbinghaus 2007 contains a very detailed biography of Ernst Zermelo together with an elaborated analysis of his works.

3. Zermelo's first axiomatization of set theory

By the turn of the twentieth century, mathematicians (Peano, Dedekind, Veblen, Huntington, and Hilbert, among others) had been working intensively on providing axiomatic foundations for several fundamental mathematical theories: arithmetic, algebra, geometry. Set theory, proposed by Georg Cantor, has also been given an axiomatic treatment. Its first axiomatization was given by Ernst Zermelo in 1908, its goal being not only to secure set theory from the danger of antinomies, but also to provide a solid background for the proof that any set can be well ordered. As is known, Zermelo gave two such proofs, and it was the second one that was based on his axioms. Zermelo's axiomatization has been discussed in detail in many places, for instance: Fraenkel, Bar Hillel and Levy 1973, Hallet 1984, Kanamori 1996, 2004, Moore 1980, 1982, Ebbinghaus, Fraser and Kanamori 2010. Other axiomatic systems for set theory were proposed in the presence of the one given by Zermelo in 1908 (for instance in the works of Fraenkel, Bernays, von Neumann, Gödel, and Quine). The same applies, of course, to additions to Zermelo's original system (by Skolem, Fraenkel, and Mirimanoff). Set theory had been developing for about forty years as a mathematical theory, and only later did investigations into models of this theory begin.

It seems that there were two reasons for the predicate $\in$ being transferred from the domain of logic into a separate mathematical domain. The first of them was the growing interest in set theory and the fact the it could be axiomatized, as shown by Zermelo in 1908. The second reason, I believe, derived from the consequences of the Löwenheim-Skolem theorem (Löwenheim 1915, Skolem 1919, 1920, 1922), which revealed the relativity of certain set theoretical concepts (such as cardinality).

In this paper I am interested mainly in Zermelo's ideas concerning infinitary logic, which were evidently influenced by his work in set theory. His first axiomatization of set theory was created in accordance with the first phase of Hilbert's program; let us bear in mind that Zermelo was collaborating with Hilbert in Göttingen at that time. But in his works from the nineteen twenties and thirties, Zermelo does not accept the finitistic point of view in mathematics. One must also remember that Zermelo's ideas concerning infinitary logic appeared at the time of *interregnum* in mathematical logic and foundations of mathematics, that is between the paradigms of *Principia Mathematica* and *Grundlagen der Mathematik*, and they did not belong to the main stream of investigations in these domains. This was also a period when certain fundamental metalogical concepts, such as categoricity and completeness, were emerging.

The axioms for set theory proposed in Zermelo 1908 are well known. However, it could be interesting to compare this list with the second axiomatization, proposed in Zermelo 1930 and which I shall discuss a little later. I am using the English translation proposed in Ebbinghaus, Fraser and Kanamori 2010, 193–201:

1. *Axiom of extensionality.* If every element of a set M is also an element of N and vice versa, if, therefore $M \subseteq N$ and $N \subseteq M$, then always $M = N$; or, more briefly: Every set is determined by its elements.
2. *Axiom of elementary sets.* There exists a (fictitious) set, the "*null set*" 0, that contains no element at all. If a is any object of the domain, there exists a set $\{a\}$ containing a and only a as element; if a and b are any two objects of the domain, there always exists a set $\{a, b\}$ containing as elements a and b but no object x distinct from both.
3. *Axiom of separation.* Whenever the propositional function $\mathfrak{C}(x)$ is definite for all elements of a set M, M possesses a subset $M_{\mathfrak{C}}$ containing as elements precisely those elements x of M for which $\mathfrak{C}(x)$ is true.
4. *Axiom of the power set.* To every set T there corresponds another set $\mathfrak{U}T$, the "*power set*" of T, that contains as elements precisely all subsets of T.
5. *Axiom of the union.* To every set T there corresponds another set $\mathfrak{S}T$, the "*union*" of T, that contains as elements precisely all elements of the elements of T.

6. *Axiom of choice.* If T is a set whose elements all are sets that are different from 0 and mutually disjoint, its union $\mathfrak{S}T$ includes at least one subset S_1 having one and only one element in common with each element of T.
7. *Axiom of infinity.* There exists in the domain at least one set Z that contains the null set as an element and is so constituted that to each of its elements a there corresponds a further element of the form $\{a\}$, in other words, that with each of its elements a it also contains the corresponding set $\{a\}$ as an element.

The careful reader may note that these formulations of the axioms are in principle the same as their modern formulations. The only exception is the axiom of separation, in which Zermelo uses the term "definite property". The axiom of replacement and the axiom of foundation, absent on the above list, were added later.

4. Zermelo on the concept of *Definitheit*

Many mathematicians attempted to define precisely the notion of "definite property" (Zermelo, Fraenkel, Weyl, Skolem, von Neumann, and others). Below I limit myself to the presentation of a solution given by Zermelo. In the paper "Über den Begriff der Definitheit in der Axiomatik" (Zermelo 1929) Zermelo seeks to characterize this notion axiomatically. Definite property is a notion that is crucial in the formulation of the axiom of separation in set theory. Zermelo makes use of his axiomatization of set theory from 1908 and considers three options:

1. The concept could be treated as useless, which is not a position taken by Zermelo. It is important to add that Zermelo does not want to analyze this concept in logic itself, thus outside of set theory.
2. A general characterization of this notion could be omitted, while focusing on specification of the shape of formulas occurring in the axiom of separation. This was done for instance by Fraenkel in the second edition (Fraenkel 1923) of his *Einleitung in die Mengenlehre*. Zermelo rejects this solution, because it presupposes the concept of a natural number, while Zermelo considers the latter secondary with respect to pure set theoretical concepts.
3. An axiomatical characterization of the notion in question could be attempted. This is exactly the position taken by Zermelo. He pays attention to another solution of this kind, namely the one proposed by von Neumann in 1925 which has the notion of a function (rather than a set) as a primitive notion of set theory (von Neumann 1925).

It should perhaps be added that Zermelo does not comment on the solution proposed a few years earlier by Skolem. Zermelo's first remarks about Skolem can be found in a footnote to the paper Zermelo 1930. Skolem himself wrote as follows on Zermelo's solution:

> Dabei fällt es mir besonders auf, daß er meinen Helsingforser Vortrag vom Jahre 1922 nicht zu kennen scheint, worin ich genau dieselbe Idee zur Verschärfung jenes Begriffes ausgesprochen habe wie Zermelo auf Seite 342 in seiner Arbeit. (Skolem 1930, 275)

After some preliminary explanations (concerning axiomatic systems, models, consistency, categoricity, etc.), Zermelo claims that his understanding of the notion of a definite property should be summarized as follows:

> "Definite" is thus what is *decided in every* single *model*, but may be decided differently in different models; "decidedness" refers to the individual *model*, whereas "definiteness" itself refers to the *relation* under consideration and to the entire *system*. (Zermelo 1929, 341–342; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 361–363)

According to the above, by definite property one should understand any property on which it can be decided whether the elements of any model of the system of axioms in question do or do not possess this property. From the point of view of contemporary logical standards such a formulation is obviously imprecise. Zermelo does not formulate explicitly in which formal language one should express such properties. Moreover, these properties are characterized with respect to their content and not syntax. As examples of properties which are *not* definite in his sense, Zermelo himself gives such properties as: *to be painted in green* or *to be an irrational number which can not be defined in finitely many words in any Indoeuropean language.*

It should be clear from the above that by definite properties (for a given axiomatic system) Zermelo understands properties that are *essential*, *natural*, or *meaningful*. Zermelo insists that the following characterization of the notion of definite property is *not* adequate:

> *A proposition is called "definite" for a given system if it is constructed from the fundamental relations of the system only by virtue of the logical elementary operations of negation, conjunction and disjunction, as well as quantification, all these operations in arbitrary yet finite repetition and composition.* (Zermelo 1929, 342; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 363)

The reason for the rejection of such characterization lies, according to Zermelo, in the fact that the definition of this form does not refer to the sentences themselves but to the methods of their construction, and the latter presupposes the notion of *an arbitrary natural number*, while such procedure was not acceptable to Zermelo.

Finally, Zermelo gives his axiomatic characterization of the notion of a definite property, in his opinion the only adequate description. For a given domain B and fundamental relation defined on B and forming a system R Zermelo proposes three axioms characterizing the meaning of "the sentence p is definite with respect to R", in symbols Dp. Firstly (Axiom I), all expressions of the form $r(x, y, z, \ldots)$, for any relation r from R and any combination of variables from B, are definite sentences. Secondly (Axiom II), sentences formed from definite sentences with the help of negation, conjunction, disjunction and quantification (over B) are definite sentences. Furthermore, Zermelo also admits second order quantification:

> If $DF(f)$ holds for all definite functors $f = f(x, y, z, \ldots)$, then $D\bigcap_f F(f)$ and $D\bigcup_f F(f)$ hold as well. (Zermelo 1929, 343; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 365)

(this is exactly the condition which was criticized by Skolem in Skolem 1930).

Zermelo notices that the first two of the above axioms only say which sentences are definite and provide no information regarding which sentences are not definite. For this purpose he adds the following axiom:

> Axiom III) *If P is the system of all "definite" propositions, or, more generally, any system of propositions p of the constitution Dp, then it has no proper subsystem P_1 that, on the one hand, contains all fundamental relations from R, in accordance with I and II, while already including, on the other hand, all negations, conjunctions, disjunctions and quantifications of its own propositions and propositional functions.* (Zermelo 1929: 344; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 367)

Zermelo is aware that the above formulation implicitly refers to iterations of a syntactic operation *finitely many times*, but claims that he is able to get rid of this restriction, promising the details in a later work (probably in Zermelo 1935, his last printed text).

5. Zermelo's second axiomatization of set theory

Zermelo's very important paper "Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre" (Zermelo 1930) can be considered in two aspects. For a start, it portrays the development of Zermelo's views concerning set theory. I believe there are three stages to this development: the papers Zermelo 1904 and Zermelo 1908, the paper Zermelo 1930, and notes from *Nachlaß*. Secondly, it is a turning point in the evolution of Zermelo's views concerning the foundations of logic and mathematics: it seems that up to 1930 Zermelo accepts the finitary character of syntactic constructions, while after that date he begins work on his proposal of infinitary logic which could be suitable for mathematical inferences.

In this section I discuss the first of these aspects. As can be seen from his correspondence with Baer, Zermelo is already familiar with Skolem 1922 and thus is aware of the set-theoretical relativism observed by Skolem. He accepts the necessity of adding the axiom of replacement, uses the representation of ordinal numbers proposed by von Neumann, and makes an essential use of strongly inaccessible cardinals, determining those levels in the hierarchy of his normal domains which are natural models of the axioms of set theory. He proves theorems that characterize up to isomorphism his normal domains. The world of set theory is thought of in this paper as a transfinite hierarchy of normal domains. Zermelo rejects the idea that this world can be represented by a single model, and sets aside metatheoretical questions concerning set theory for a possible later work.

5.1. Axioms of the system from Zermelo 1930

The system from 1930 has the following axioms (citing the translation of Zermelo 1930 in Ebbinghaus, Fraser and Kanamori 2010, 403):

- B) *Axiom of extensionality*: Every set is determined by its elements, provided that it has any elements at all.
- A) *Axiom of separation*: Every propositional function $\mathfrak{f}(x)$ separates from every set m a subset $m_{\mathfrak{f}}$ containing all those elements x for which $\mathfrak{f}(x)$ is true. Or: To each part of a set there in turn corresponds a set containing all elements of this part.
- P) *Axiom of pairing*: If a and b are any two elements, then there is a set that contains both of them as its elements.
- U) *Axiom of the power set*: To every set m there corresponds a set $\mathfrak{U}m$ that contains as elements all subsets of m, including the null set and m itself. Here, an arbitrarily chosen "urelement" $\mathfrak{u}_0$ takes the place of the "null set".
- V) *Axiom of the union*: To every set m there corresponds a set $\mathfrak{S}m$ that contains the elements of its elements.
- E) *Axiom of replacement*: If the elements x of a set m are replaced in a unique way by arbitrary elements x' of the domain, then the domain contains also a set m' that has as its elements all these elements x'.

F) *Axiom of foundation*: Every (decreasing) chain of elements, in which each term is an element of the preceding one, terminates with finite index at an urelement. Or, what amounts to the same thing: Every partial domain T contains at least one element t_0 that has no element in T.

There are similarities as well as differences between this system and the system from Zermelo 1908. Here Zermelo considers set theory which, besides sets, also takes into account "indecomposable" elements or atoms, with unknown structure (urelements, *Urelemente*). Their totality is denoted by U.

The separation axiom [A] looks as if it were formulated in a second-order language, thus admitting quantification over propositional functions. Zermelo himself does not specify the language in which properties of sets should be expressed. He only writes that an arbitrary propositional function can determine a subset of a given set; moreover, to each part (*Teil*) of a given set there corresponds a set that has as its elements all elements of this part. In addition, no limitations are imposed on the form of propositional functions occurring in the axiom schema of replacement.

The axiom of foundation is new (with respect to the system from Zermelo 1908). Zermelo writes that this axiom is required in order to exclude circular and non-founded (*„zirkelhafte" und „abgründige" Mengen*) sets. The axiom of infinity is not assumed here as (in Zermelo's own words) not belonging to general set theory (*„allgemeinen" Mengenlehre*). He shows that a normal domain consisting of all finite sets satisfies all the axioms of such general theory. The power set axiom states the existence of the family of all subsets of a given set. The axiom of choice is assumed as a logical principle and is not included in proper axioms of the system. Zermelo writes that this axiom cannot be used for the limitation of the domains in question (*Abgrenzung der Bereiche*). He adds that in all further considerations the fact that any set can be well ordered is of the utmost importance.

5.2. Normal domains and isomorphism theorems

Systems consisting of sets and atoms with the fundamental relation $a \in b$ (meaning: a is an element of b) which satisfy the axioms BAPUVEF are called by Zermelo the normal domains (*Normalbereiche*).

One can apply the same operations on normal domains as those which are applicable to sets (unions, intersections); one can also investigate the relationships between such domains (for instance the relation of being a subdomain). Particular normal domains are not sets in any absolute sense: if a domain P_1 is an element of a domain P_2, then P_1 is a set in P_2. Domains that are sets in this sense are called closed. Domains that are not closed are called open.

Zermelo introduces his *Grenzzahlen* (in modern terminology: strongly inaccessible numbers) as fixed points of a certain function defined by transfinite induction. Zermelo represents the set theoretical universe as a hierarchy of normal domains. The internal structure of these domains is characterized by three *development theorems*, the first of which is formulated (with Zermelo’s commentary) as follows:

> *First development theorem. Each normal domain P of characteristic π can be decomposed into a well-ordered [[sequence]] of type π of non-empty and disjoint “layers” Q_α, so that each layer Q_α includes all elements of P which occur in no earlier layer and whose elements belong to the corresponding “segment” P_α, that is, to the sum of preceding layers. The first layer Q_0 includes all the urelements.*
>
> For the partial domains, or “segments”, P_α are defined by transfinite induction by virtue of the following stipulations:
>
> 1. $P_1 = Q_0 = Q$ shall include the whole basis, the totality of urelements.
> 2. $P_{\alpha+1} = P_\alpha + Q_\alpha$ shall contain all sets of P that are “rooted” in P_α, that is all those sets whose elements lie in P_α.
> 3. If α is a limit number, then P_α shall be the sum or union of all preceding P_β with smaller indices $\beta < \alpha$.
>
> (Zermelo 1930, 36; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 413)

Further, Zermelo establishes the cardinalities of the particular levels of this hierarchy (*Second development theorem*). Finally, he proves that each normal domain can be represented in some canonical form (*Third development theorem (theorem on “canonical” development)*).

If α is a strongly inaccessible cardinal, then P_α satisfies all the axioms of the system BAPUVEF. The basis of a normal domain is the totality of its atoms, and the characteristic of a normal domain equals the least ordinal number that is not a set in it. Zermelo proves the following theorems characterizing normal domains up to isomorphism with respect to two parameters taken into account, namely the cardinality of the basis and the characteristic of the domain:

> *First isomorphism theorem. Two normal domains with the same characteristic and with equivalent bases are isomorphic. In fact, the isomorphic mapping of the two domains onto one another is uniquely determined by the mapping of their bases.*
>
> *Second isomorphism theorem. Given two normal domains with equivalent bases and different boundary numbers π, π', one is always isomorphic to a canonical segment of the other.*

> *Third isomorphism theorem. Given two normal domains with the same characteristic, one is always isomorphic to a (proper or improper) subdomain of the other.* (Zermelo 1930, 36–39; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 421–423)

At the end of the paper Zermelo includes some remarks concerning models of set theory. He also presents the following argumentation for the necessity of considering a transfinite hierarchy of strongly inaccessible numbers:

> Let us now put forth the general hypothesis that every *categorically determined domain can also be conceived as a "set" in one way or another*; that is that it can occur as an element of a (suitably chosen) normal domain. It then follows that there corresponds to any normal domain a higher one with the same basis, to any unit domain a higher unit domain, and therefore also to any "boundary number" π a greater boundary number π'. [...] Once again, this is of course *not* "provable" on the basis of the ZF' axioms, since the asserted behavior leads us beyond any individual normal domain. Rather, we must postulate the *existence of an unlimited sequence of boundary numbers* as a new axiom for the "meta-theory of sets", where the question of "consistency" still requires closer examination. (Zermelo 1930, 46; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 429)

Zermelo stresses that one should not think of set theory as describing a single intended model. He claims that the fact that domains with different bases and characteristics are not isomorphic is a virtue of his system as far as the intuitions concerning the concept of a set are concerned.

5.3. A digression: extremal axioms

The term *extremal axiom* was introduced in the paper Carnap and Bachmann 1936. Intuitively speaking, extremal axioms are axioms that should uniquely characterize models of the underlying theory, and moreover as maximal or minimal with respect to their internal structure. Early examples of such axioms include Peano's axiom of induction in arithmetic and the axiom of completeness in Hilbert's system of geometry presented in Hilbert 1899. The latter is summarized as follows: *To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms.* An example of an early extremal axiom in set theory is Fraenkel's axiom of restriction, which says, roughly speaking, that only those sets exist whose existence can be explicitly proven in set theory. Neither Hilbert's axiom of completeness nor Fraenkel's axiom of restriction are statements formulated in the object language of the corresponding theory. Hilbert's axiom expresses the idea of maximality of the geometric universe, while Fraenkel's axiom should be understood as a requirement that the set theoretical universe should be as narrow as possible. Carnap and Bachmann tried to provide a general description of extremal axioms (maximal as well as minimal) in the formalism of the theory of types. They considered one maximal axiom (Hilbert's axiom of completeness) and two minimal axioms (Dedekind's characterization of the minimal infinite chain and Fraenkel's axiom of restriction). It is unclear, in my opinion at least, why they did not consider maximal axioms in set theory. Zermelo formulated such an axiom in Zermelo 1930 (page 46), which is clear from the citation at the end of the previous subsection. Postulating the existence of the transfinite hierarchy of strongly inaccessible numbers (boundary numbers in Zermelo's formulation) is exactly a certain condition expressing maximality of the set-theoretic universe. Moreover, this was the first time when such a maximality condition concerning this universe was formulated. Felix Hausdorff had earlier expressed the opinion that strongly inaccessible numbers were so large that they would never be needed in "normal" mathematics. Zermelo's maximality condition is in agreement with the contemporary views concerning the universe of set theory. Indeed, Kurt Gödel had already suggested that one needed to look for maximality conditions in set theory, similar to the Hilbert's axiom in geometry. And he did so just after introducing his axiom of constructibility, a kind of restriction axiom, being used in Gödel's proof of consistency of axiom of choice and the continuum hypothesis with the remaining axioms of Zermelo-Fraenkel set theory.

Further examples of restriction axioms in set theory are John von Neumann's axiom of the limitation of size (von Neumann 1925) and Roman Suszko's axiom of canonicity (Suszko 1951). A sharp critique of restriction axioms was presented in Fraenkel, Bar-Hillel and Levy 1973. Modern set theory focuses attention rather on maximal axioms, namely axioms of the existence of large cardinal numbers. Therefore I dare to claim that Zermelo was prophetic in postulating the existence of the transfinite hierarchy of strongly inaccessible numbers as the principle that should govern the set-theoretic universe. The recent book Pogonowski 2019 discusses the issue of extremal axioms in mathematics, including the circumstances of their origin and their consequences.

6. Zermelo's anti-Skolemism

Zermelo became acquainted with Skolem's works on set theory rather late, as attested to in his correspondence (see Ebbinghaus 2001). He criticized Skolem's remarks on his axiomatization from Zermelo 1930 (the dispute concerned Zermelo's notion of *Definitheit* described in Zermelo 1929). One may presume, in my opinion, that Zermelo's critical attitude towards Skolemism and "finitistic prejudicies" (Zermelo's own term) were among the inspirations for his project of infinitary logic.

Let me recall that the Löwenheim-Skolem theorem (formulated in contemporary terminology) states that if a theory in first-order language has an infinite model, then it has a countable model. This theorem has the following seemingly paradoxical consequences: firstly, in first-order languages, in which we have only countably many closed terms (which can name the elements of the domain of a model), we are unable to give names to almost all elements in uncountable domains; secondly, Cantor's theorem in set theory states that no set is equinumerous with its power set – and when applied to a countably infinite set it has as a consequence that its power set is uncountable. The Löwenheim-Skolem theorem applied to set theory formulated in a first-order language together with this result, in the presence of the axiom of infinity (stating the existence of at least one infinite set), may seem paradoxical: how could it be possible for a countable model of set theory to admit the existence of an uncountable element in its domain? This phenomenon is usually called Skolem's paradox.

Explications of Skolem's paradox are well known. As a matter of fact, it is not a genuine paradox but rather an effect (of using first-order languages). See for instance Benacerraf and Wright 1985, Bays 2000, van Dalen and Ebbinghaus 2000, George 1985, Klenk 1976, McCarthy and Tennant 1987, Moore 1985, Myhill 1951, 1952, Putnam 1980, Quine 1966, Resnik 1966, 1969, Shapiro 1996, Suszko 1951, Thomas 1968, 1971, and Wang 1955, as well as the textbooks DeLong 1970, Mostowski 1948, Hunter 1971, and Wang 1962. Briefly speaking, one has to remember that an infinite set is uncountable if and only if there exists no bijection between it and the set of all natural numbers. Now, if a given model does not include enough bijections, then inside the model there may occur elements that are uncountable *from the point of view of this model.* The Löwenheim-Skolem theorem for systems of logic different from the first-order logic is discussed for instance in Barwise and Feferman 1985, and Shapiro 1996.

Zermelo's anti-Skolemism can be summarized as follows. Firstly, Zermelo did not endorse the idea that the properties taken into account in the axiom of separation should be formulated in a first-order language. Secondly, Zermelo did not endorse the idea that set theory should be connected with a single model.

Skolemism was, in Zermelo's opinion, a completely inadequate approach to set theory. He considered the reduction of the world of sets to one countable model as a deformation of the Cantorian essence of this theory.

Skolem did not consider set theory a good candidate for the basis of all of mathematics, contrary to Zermelo. In turn, Zermelo claimed that the proof techniques used in mathematics should not be restricted by considering only one finitistic system.

In his short note "Relativism in set theory and the so-called Skolem theorem" Zermelo tried to show the inadequacy of the early set-theoretical relativism in a purely mathematical way, attempting to prove that the assumption that there may exist "bigger" and "smaller" sets playing the role of continuum leads to a contradiction.[2] However, his argument misses the point. Dirk van Dalen and Heinz-Dieter Ebbinghaus show that the main fault lies in an unwarranted assumption concerning closures with respect to arbitrary unions and intersections, and they conclude:

> So altogether Zermelo's refutation amounts to a proof of the set-theoretical statement saying that, given a denumerable set M, any subset of the powerset K of M that is closed under arbitrary unions and intersections (and under complements with respect to M) and whose union is M, either is finite or of the same cardinality as K. (van Dalen, Ebbinghaus 2000, 156)

These authors also write that Zermelo's faulty argumentation might have been caused by his epistemological position with respect to set theory, because Zermelo believed in the existence of an absolute hierarchy of sets.

7. Zermelo's infinitary logic

In my opinion the following factors are essential to Zermelo's motivation for his project of infinitary logic:

1. Zermelo's views on the nature of infinity: he claimed that *Mathematics is the logic of the Infinite.*
2. Zermelo's belief that his set theory is fundamental to the entirety of mathematics. In particular, he conceived formulas and proofs as well-founded constructs from the cumulative hierarchy of sets.
3. Zermelo's rejection of Skolemism. Zermelo believed that investigations into set theory should not be restricted to its single model and that

one should not accept the finitistic standpoint in mathematical proof procedures.

4. Zermelo's belief that no restriction should be imposed on formulas occurring in the axiom of separation (in particular the restriction to a first-order language is not justified).

Zermelo was primarily a genuine mathematician and only rarely did he express philosophical opinions, always stressing the fact that he was mainly interested in mathematical aspects of the investigated problems. His "Thesen über das Unendliche in der Mathematik", a short note (included in his *Nachlaß*) prepared in 1921 for his lectures in Warsaw in Spring 1929, was analyzed in Taylor 2002 and van Dalen, Ebbinghaus 2000. The text runs as follows:

> 17 July 1921
>
> Theses concerning the infinite in mathematics
>
> I) Every genuinely mathematical proposition is "infinitary" in character, that is, is concerned with an *infinite* domain and is to be considered a collection of infinitely many "elementary propositions".
>
> II) The infinite is not given to us physically or psychologically in reality, it must be grasped as "idea" in Plato's sense and "posited".
>
> III) Since infinitary propositions can never be derived from finitary ones, the "axioms" of any mathematical theory, too, must be infinitary, and the "consistency" of such a theory can be "proved" by no other means than the presentation of a corresponding consistent system of infinitely many elementary propositions.
>
> IV) Traditional "Aristotelian" logic is, according to its nature, finitary, and hence not suited for the foundation of mathematical science. Whence the necessity of an extended "infinitary" or "Platonic" logic that rests on some kind of infinitary "intuition" – as, e.g., in connection with the question of the "axiom of choice" – but which, paradoxically, is rejected by the "intuitionists" by force of habit.
>
> V) Every mathematical proposition must be considered a collection of (infinitely many) elementary propositions, the "fundamental relations", by means of conjunction, disjunction and negation, and every deductio of a proposition from other propositions, in particular every "proof", is nothing but a "regrouping" of the elementary propositions.
>
> (Zermelo 1921; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 307)

How are these claims to be understood? One should bear in mind that Zermelo never used formalized languages in the modern sense of the term. The following passage is very often cited:

> At the time, a universally acknowledged "mathematical logic" on which I could have relied did not exist – nor does it exist today when every foundational

> researcher has his own logistic. (Zermelo 1929, 340; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 359)

Zermelo used a metaphorical description of mathematics as the logic of the Infinite – compare the following note from *Nachlaß*, related to the third (*Finite and infinite domains*) out of nine his lectures presented in Warsaw between 27 May and 8 June 1929:

> A purely "finitistic" mathematics, in which nothing really requires proof since everything is already verifiable by use of the finite model, would no longer be mathematics in the true sense of the word. Rather, true mathematics is infinitistic according to its nature and rests on the assumption of infinite domains; it may even be called the "logic of the infinite". (Zermelo 1929a; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 383)

In my opinion the most important aspect of the "Thesen über das Unendliche in der Mathematik" is the ontological claim that mathematics deals primarily with infinite structures. Any attempts at expressing mathematical dependencies with the help of finitary linguistic tools were considered by Zermelo mere approximations.

According to Thesis I, the nature of mathematical statements is related to the fact that they concern infinite domains and should be conceived of as collections of infinitely many elementary statements.

Infinity should be understood in the Platonic sense, and it is neither accessible physically or psychically: this is the content of Thesis II.

Zermelo argues in Thesis III that because from finitary sentences one cannot infer statements of infinitary character (and such are, according to Thesis I, all mathematical statements), then the axioms of any mathematical theory should have an infinitary character. The consistency of such theories can be proved only by providing an infinite set of elementary statements free of contradictions.

Traditional (Aristotelian) logic has a finitary character and is not suitable as a basis of mathematical knowledge; such is Zermelo's claim in Thesis IV. An infinitary, Platonic logic appropriate for considerations of an infinitary character, as for instance those connected with the axiom of choice, needs to be constructed.

Every mathematical statement is a collection of infinitely many elementary statements connected by negations, conjunctions and disjunctions. Every inference of a mathematical statement from other such statements is nothing other than a recombination of the elementary statements taken into account. This is the content of Thesis V.

The last paper published by Zermelo (Zermelo 1935) contains proposals for applying the results from Zermelo 1930 to the reconstruction of fundamental logical concepts (to *Beweistheorie*). The starting point is the relation of entailment:

> We say "p follows from $a, b, c, \ldots$" and write $a, b, c, \ldots \rightarrow p$ if along with the truth of $a, b, c, \ldots$ also that of p is supposed to be posited. And we call the complex $a, b, c \ldots$ the "ground", the proposition p the "consequence", and the sense inherent to this formula the "justification" of the proposition p. (Zermelo 1935, 139; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 587–589)

Zermelo proves that if the relation introduced above is well-founded, then it preserves validity. Then he gives a definition of the notion of (mathematical) proof, based on this relation:

> A (direct) *proof* is a system of propositions well-founded by means of inference (justification) to which the proposition to be proved belongs and whose basis consists only of true propositions forming the presupposition of the proposition. (Zermelo 1935, 140; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 589)

A typical example of such a proof is an inference based on the principle of mathematical induction as explained in Zermelo 1935 on page 140. The crucial point in this explanation refers to the fact that such proof consists of a well-ordered sequence of propositions.

A detailed analysis of Zermelo's views concerning his infinitary logic was presented in Taylor 2002. In what follows I am going to use the notation from that paper.

Every mathematical theory, writes Zermelo, refers to some (in general, infinite) domain of elements such as numbers, points, and so on. This domain is called the fundamental domain and denoted by D. Between elements of D several relations may hold, for instance: $a < b$, $a+b = c$, a lies on the straight line determined by b and c.

Zermelo calls such relations basic relations (*Grundrelationen*) and denotes their totality by R. From these relations one obtains further relations by applying to them the operations of negation and arbitrary (also infinite) conjunctions and disjunctions. Let the totality of relations obtained in this way be denoted by C_R. Elements of C_R are called fundamental relations.

Zermelo implicitly assumes that the relation of being a subexpression (part of a given expression) is well-defined and that it enables us to introduce a well-founded ordering in the totality of all fundamental relations.

Thus, the totality of expressions forms a hierarchy similar to the hierarchies of sets investigated in Zermelo 1930. Let this hierarchy be denoted by $H_{D,R}$. Some subdomains of this hierarchy may be sets (closed domains) and some may be open domains. This dichotomy reflects the differences between mathematical theories. The former case concerns arithmetic, real and complex analysis, and geometry, while the latter is related to the “*general* theory of fields”, *general* geometry or *general* set theory extended to *all* normal domains (Zermelo 1935, 142).

To sum up – at this moment the initial steps in the formation of Zermelo’s system of infinitary logic are based on the following constructions:

1. Basic relations on D: $R = \{R_1^{n_1}, \ldots, R_j^{n_j}\}$.
2. Fundamental relations on D: the collection C_R of atomic sentences of the form $R_i^{n_i} a_1 \ldots a_{n_i}$ (for $R_i^{n_i}$ from R and $a_1, \ldots, a_{n_i}$ from D).
3. Hierarchy of expressions: $H_{D,R}$ (where C_R is the basis).

The next constructions introduce semantic notions. If any dichotomic partition of the collection C_R is given, then it can be extended to a partition of all expressions from the hierarchy $H_{D,R}$:

1. We use the standard conditions concerning negations, conjunctions and disjunctions.
2. Suppose that there exist sentences in $H_{D,R}$ with non-determined logical value. They should then form a subdomain, say S. In S there must exist the first element (which followsfrom the well-ordering of the hierarchy of expressions), say t_0. This element does not belong to C_R and is constructed from expressions belonging to $H_{D,R} - S$ only (by using negations, conjunctions and disjunctions). But these expressions already have determined logical values and hence we obtain a contradiction. Therefore, all expressions in $H_{D,R}$ have uniquely determined logical values (modulo the given partition of the sentences from C_R).

Let Π denote the totality of all dichotomic partitions of C_R (that is partitions into true and false sentences). Two sentences are logically equivalent if and only if for any partition of the basic expressions they belong to exactly the same classes in the partition of the entire hierarchy of expressions. Zermelo shows that the De Morgan laws hold in his system. Logical entailment is defined with the use of partitions from Π. The class of all sentences from $H_{D,R}$ which are true under the given partition is a subdomain of $H_{D,R}$. For a given sentence s, the intersection $V^*(s)$ of all such classes (indexed by the elements of Π) which contain the sentence s contains also all sentences that are true if s is true. Thus, this intersection corresponds to the class of all sentences that logically follow from s.

Similarly, the intersection $U^*(s)$ of all classes (subdomains of $H_{D,R}$) consisting of false sentences and containing the sentence s contains all sentences from which s logically follows. In Zermelo's own terminology $V^*(s)$ is called *der Folgebereich von* s, and $U^*(s)$ is called *der Ursprungsbereich von* s.

Sentences logically equivalent with s form the following class:

$$A^*(s) = V^*(s) \cap U^*(s)$$

Zermelo gives the following example:

> If, e.g., s is a "system of axioms" of a mathematical theory that rests on the fundamental relations of, e.g., arithmetic or Euclidean geometry, then $A^*(s)$ comprises all equivalent axiom systems, and $V^*(s)$ comprises all propositions of the theory that follow from s, and in particular all *more general* axiom systems, and $U^*(s)$ comprises all *more particular* axiom systems. (Zermelo 1935, 144; citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 595)

The next passage reflects Zermelo's views about the connections between the notions introduced above, mathematical proofs, and the notion of provability in general. In order to avoid misunderstanding, it would be better to cite verbatim Zermelo's own formulation:

> If a proposition t lies in the consequence domain $V^*(s)$, that is, if it "follows" from s, then it is also "syllogistically derivable" from s. In particular, it is so derivable already within the common "root domain", or "definition domain" $W(s,t)$ of s and t. This is the intersection of all "root domains" W containing s and t, namely of all those which along with every derived proposition contained in them also contain all of its "roots", that is, along with every negation $\overline{a}$ the negated proposition a, and along with every conjunction or disjunction all of their terms. There corresponds then to any arbitrary truth distribution of the entire system also one such of $W(s,t)$, and every intersection V' with a truth domain V either fails to contain s or contains it along with t. The domain W contains all intermediate propositions required for the derivation of t, and their "valuation", their truth distribution, proceeds in it according to the syllogistic rules.
>
> Accordingly, every proposition that follows from s would therefore be "provable" as well, but, at first, only in the absolute, "infinitistic" sense. Such a "proof" for the most part contains *infinitely many* intermediate propositions, and it has yet to be determined to what extent and by what means it can be rendered evident to our *finite* mind. *Every* mathematical proof, such as the inference method of mathematical induction, is actually quite "infinitistic", and yet we are capable of grasping it. Firm limits on comprehensibility do not seem to exist here. (Zermelo 1935, 144 citing the translation in Ebbinghaus, Fraser and Kanamori 2010, 595–597)

It can be seen from these passages that Zermelo wanted to understand the notion of mathematical proof in an absolute sense, as a notion determined by semantic dependencies only. Proofs known in mathematical research practice are only certain approximations of proofs in such an absolute sense.

According to Zermelo, a mathematical theory is a system of a form:

$$T = \langle D, C_R, \Pi \rangle.$$

Let Val_Π be the class of all sentences from $H_{D,R}$ which are Π-valid, that is true under all partitions from Π. Then both Val_Π and $H_{D,R}$ are open domains. If α is strongly inaccessible, then the domain $Val_\Pi \cap P_\alpha$ is an approximation of T (here P_α is the level with index α of the hierarchy of sets whose basis is C_R). It is understood that the hierarchy $H_{D,R}$ is a subdomain in the hierarchy of all sets.

Zermelo's sentence $\bigwedge(Val_\Pi \cap P_\alpha)$ is an example of a true but not provable sentence in the system $\langle P_\alpha, \prec \rangle$. Here $\prec$ is a syntactic relation corresponding to the relation of being a subexpression. It is understood that this relation has a set-theoretical representation.

8. Zermelo and Gödel

Correspondence between Zermelo and Gödel has been analyzed by several authors, for instance Grattan-Guinness 1979, Dawson 1985, Buldt et al. 2002, Ebbinghaus, Fraser, Kanamori 2010. Zermelo and Gödel met each other in Bad Elster, where they both gave lectures on September 15, 1931. Gödel's lecture, under the title "Über die Existenz unentscheidbarer arithmetischer Sätze in den formalen Systemen der Mathematik" was not published in the reports from the meeting, while the text of Zermelo's lecture "Über Stufen der Quantifikation und die Logik des Unendlichen" was published in these reports, that is in *Jahresbericht der Deutschen Mathematiker-Vereinigung* 41, 1932, 85–92. We know from Zermelo's correspondence (see Peckhaus 1990) that the first fragment of his text was not read and was added only in print. In it, Zermelo expresses his condemnation of Skolemism, of "finitistic prejudicies", and the idea that set theory may be represented by a countable model. He writes:

> Mathematics, generally speaking, is *not* really concerned with "combinations of signs", as some assume, but with *conceptually ideal relations* among the elements of a conceptually posited *infinite manifold*. Our systems of signs are but

> *imperfect expedients* of our *finite* mind, which we, adapting them to the circumstances at hand, employ in order to at least gradually get a hold on the infinite, which we can *neither* "survey" *nor* grasp *immediately* and *intuitively.* (Zermelo 1932, 85; citing Ebbinghaus, Fraser and Kanamori 2010, 543–545)

In what follows Zermelo presents in brief his proposal of the understanding of the notion of mathematical proof discussed in the previous section. Mathematical proof is for him a well-founded system of propositions. The term "levels of quantification" (*Stufen der Quantifikation*) used in this text is explicated by a reference to the indices of the levels of the set-theoretical hierarchy. Zermelo compares his proposal with Gödel's results concerning incompleteness:

> From our point of view, every "true" proposition is therefore also "*provable*", and every proposition "definable" by means of a well-founded propositional system S is also "*decidable*" *without* ascent to a higher level of quantification being necessary. There are no (objectively) "*undecidable*" propositions. On the other hand, Mr. *Gödel* (Wiener Monatshefte Bd. 38, S. 173) has tried to prove the *opposite*. In order to do so, he tried to derive a proposition A, given a "*PM*-system" of limited (namely *finite*) level of quantification, which is supposed to be *demonstrably* (at least in this system) *undecidable*. But the only reason Gödel's proof works is because he applies the "finitistic" restriction only to the "provable" propositions of the system and not to *all* propositions belonging to the system. Thus only the former, but not the latter, form a *countable* set, and of course, when understood in *this* sense, there must be "undecidable" propositions. But, as G. himself states, the very proposition constructed by him as an example of an "undecidable" proposition later turns out to be "decidable" after all, even not in the sense of the original definition. This whole argument, in my opinion, only serves as evidence for the inadequacy of *any* "finitistic" proof theory without, however, providing the means to remove this ill. Such relativistic considerations in no way touch on the real question as to whether there are absolutely undecidable propositions or absolutely unsolvable problems in mathematics. (Zermelo 1932, 87; citing Ebbinghaus, Fraser and Kanamori 2010, 547)

Zermelo and Gödel corresponded with each other in 1931, see Gödel's *Collected Works*, volume IV, 418–431. Zermelo writes in his letter from 21 September 1931 that he has found a gap in Gödel's proof of the existence of true but unprovable sentences in systems like that of *Principia Mathematica*. Gödel's answer from 12 October 1931 contains an explanation concerning the flaw in Zermelo's argument (shortly speaking: Zermelo did not distinguish between formulas and names of formulas). It may be added that the letter contains Gödel's remark about non-definability of the truth predicate (the fact usually related to the name of Tarski). Gödel is also interested in the paper Zermelo 1930. In Zermelo’s opinion Gödel was one of the few people who could fully appreciate the results of that paper.

Zermelo’s answer from 29 October 1931, contained in the letter that was to be the last in this exchange, is rather short. The following passage from it is important:

> What do we mean by “proof”? Generally speaking, by “proof” we mean a system of propositions so constituted that, under the assumption of the premisses, the validity of the claim can be made *obvious*. The only remaining question is, What all is considered “obvious”? Certainly *not merely*, as you yourself have shown after all, the propositions of some finitistic schema, which, in your case, too, can always be *extended*. But then we would actually agree, except that I, at the very outset, proceed from a *more general* schema, which does not *need* to be extended first. And in *this* system now really *any* proposition whatsoever is decidable! (From Zermelo’s letter to Gödel [29 X 1931]; citing Ebbinghaus, Fraser and Kanamori 2010, 501).

Zermelo adds that sentences undecidable in Gödel’s sense in some system are decidable in a stronger system (Gödel himself was of course aware of this fact), but this stronger system is obtained not by the addition of new sentences but by expansion of the proof possibilities.

9. A few remarks about infinitary logic

At the time of preparation of Zermelo’s project of infinitary logic, the project itself stood little chance of being developed by others. Systematic investigations in infinitary logic only came about two decades later.

Zermelo was frustrated by there being practically no response to his project of infinitary logic. He wrote on 1 October 1941 to Bernays, thanking him for his birthday congratulations:

> For as one finds oneself growing more and more lonely, one is all the more grateful for anyone cherishing one’s memory. [...] Every mention of my name is invariably connected *only* with the “principle of choice”, to which I have *never* laid any special claim. [...] And I recall that my presentation on propositional systems had already been excluded from a discussion session during the meeting of mathematicians in Bad Elster due to a plot engineered by the Vienna Circle represented by Hahn and Gödel. Ever since, I have lost all interest in speaking publicly on foundational matters. This apparently is the lot of anyone who has no “school” or clique behind him. But perhaps a time will come when even my papers will be rediscovered and read again. (citing Ebbinghaus, Fraser and Kanamori 2010, 609)

Several authors (for instance G.H. Moore, H.D. Ebbinghaus, and J. Ferreirós) have pointed out a very interesting fact from the history of mathematical logic: investigations into infinitary logic initially conducted in a non-systematic manner in the nineteen twenties and thirties were quickly set aside, and only two decades later was there a revival of interest in them. This is a topic that surely deserves more comprehensive treatment.

In what follows I am going to discuss very briefly some facts from the history of infinitary logic, using information contained in Moore 1995, Bell 2016, Keisler and Knight 2004, Barwise and Feferman 1985.

If we call a logic an infinitary logic, then it may refer either to the fact that the formal language taken into account admits infinitely long formulas or that infinitary rules of inference are allowed and consequently also infinitely long proofs are admissible (or both, of course). These properties have a syntactic character. They presuppose a formal representation that is adequate for talking about infinitary constructions. The best candidate in this respect is set theory.

9.1. Infinitary logic: prehistory

According to G. H. Moore (Moore 1995, 109), the first uses of infinitary formulas in logic can be found in *Mathematical Analysis of Logic* (Boole 1847), where arbitrary Boolean functions are developed into formal MacLaurin series. Boole used such constructions also in his *Laws of Thought* (Boole 1854). Peirce used infinitary formulas in 1885, when he introduced symbolism for quantifiers:

> Here ... we may use $\sum$ for *some*, suggesting a sum, and $\prod$ for *all*, suggesting a product. Thus $\sum_i x_i$ means that x is true of some of the individuals denoted by i or $\sum_i x_i = x_i + x_j + x_k +$ etc. In the same way, $\prod_i x_i$ means that x is true of all these individuals, or $\prod_i x_i = x_i x_j x_k$ etc. ... $\sum_i x_i$ and $\prod_i x_i$ are only *similar* to a sum and product; they are not strictly of that nature, because the individuals of the universe may be innumerable. (Peirce 1885, 194–195; citing Moore 1995, 110)

Ernst Schröder in his monumental *Vorlesungen über die Algebra der Logik* (Schröder 1885) also used infinitely long formulas. Schröder's notes were elaborated by Müller and published in 1910 as *Abriss der Algebra der Logik* (Schröder 1910). There one can find statements about equivalence of formulas with quantifiers with infinitely long conjunctions and disjunctions.

Leopold Löwenheim (Löwenheim 1915) and Thoralf Skolem (Skolem 1919, 1920, 1922) wrote in the algebraic tradition, going back to Peirce and Schröder. Löwenheim in 1915 used not only infinite conjunctions and disjunctions but also infinitely long quantifier prefixes. It seems justified to claim that he also used infinitary rules of inference, when he wrote that if an infinite number of sentences is true, then also its (infinitary) conjunction is true.

Löwenheim's logical protosystem thus used tools typical for second-order infinitary logic and the same may be claimed about Skolem's works from 1919 and 1920; in contemporary terminology the formal languages in question are $L_{\omega_1\omega}$ and $L_{\omega_1\omega_1}$. Namely, Skolem allowed formulas in normal forms with a finite number of general quantifiers followed by a countable number of existential quantifiers. Skolem's departure from infinitary logic is his article Skolem 1922 in which he formulated for the first time the axioms of set theory solely in the first-order language. That paper also contains critical remarks about Zermelo's system of set theory.

According to G. H. Moore some works by Hilbert and Lewis also belong to the prehistory of infinitary logic. Hilbert used infinitary expressions in 1905, but he eliminated them in the late nineteen twenties. Nevertheless, he introduced an infinitary rule of inference in 1931. Lewis represented quantified formulas as equivalent to infinite conjunctions and disjunctions and was aware that his infinite expressions may also contain uncountably many symbols (Lewis 1918).

Around 1940 infinitary systems were considered by Carnap, Nowikow, and Bochwar, among others. Carnap considered infinitary rules of inference. Also Ajdukiewicz in his *Główne zasady metodologji nauk i logiki formalnej* (Ajdukiewicz 1928) wrote about the ω-rule. Remarks about Carnap's proposals can be found in Robinson 1951. In 1939 Nowikow investigated systems of logic with countable conjunctions and disjunctions. Bochwar was interested in the metalogical properties of such systems.

When investigating definable ordering types, Kazimierz Kuratowski used quantifiers of the form *there exists a natural number* n *such that* $\varphi_n(x)$, where $\varphi_n(x)$ is a formula of the first-order predicate calculus, and claimed that such an expression is equivalent to a countable disjunction (Kuratowski 1937). Tarski modified Kuratowski's approach by eliminating infinite disjunctions and introducing instead a predicate $P(x, n)$ with two variables, equivalent to the formula $\varphi_n(x)$.

Like Zermelo, Helmer also conducted his research of infinitary logic in isolation (Helmer 1938). He took into account infinitely long formulas (being well-ordered sequences of symbols of the ordering type less than ω^2) and infinitely long numerical expressions (coding real numbers). He used the principle of mathematical induction and a rule corresponding to the Dedekind's continuity axiom, and also formulated a theorem resembling Gödel's incompleteness theorem. Helmer was considering two kinds of incompleteness: deductive and definitional.

Around 1940 investigations into infinitary logic came to a halt. It is likely that the decisive factor for that was the propagation of the standard of the first-order (finitary) logic.

9.2. Infinitary logic: a few historical remarks

Systematic investigations into infinitary logic began around the nineteen fifties. Alfred Tarski, who initiated these investigations, quoted the works Jordan 1949 and Krasner 1938 as preceding his own research. Let us bear in mind that Tarski had previously rejected the possibility of using infinitary formulas when he was introducing mathematical foundations of semantics in the nineteen thirties.

Inspirations for the systematic investigations of infinitary logic were primarily mathematical. Logical systems were being sought with sufficiently strong expressive power for characterizing fundamental mathematical concepts. In 1951 Abraham Robinson used infinite conjunctions and disjunctions in his algebraic works (the axiom of Archimedes, for instance, can be formulated as an infinite disjunction). Infinitary formulas were applied in the investigation of the problem as to whether the first strongly inaccessible number was or was not measurable.

According to Moore (Moore 1995, pages 107 and 121) the borderline between the prehistory and history of infinitary logic is determined by the works Henkin 1955 and Robinson 1957. The algebraic inspiration for Henkin's work may be traced as follows:

1. Stone proved in 1934 his representation theorem for Boolean algebras: every Boolean algebra is a homomorphic image of some field of sets.
2. Loomis extended Stone's result in 1947 by proving that every σ-complete Boolean algebra is a σ-homomorphic image of some σ-field of sets.
3. Tarski, Chang and Scott proved further results concerning representations of Boolean algebras in the fifties.
4. At the same time Tarski inspired Henkin to work on generalizations of his earlier results concerning cylindric algebras. One of Tarski's main results was a representation theorem for locally finite infinitely dimensional cylindric algebras. Such algebras are algebraic counterparts of first-order logic. Henkin worked on generalizations of the representation theorem for ω-dimensional cylindric algebras and was looking for extensions of first-order logic being their counterparts. He used predicates with an infinite number of arguments and obtained some metalogical results concerning infinitary logic.

In the late fifties Tarski and Scott investigated propositional calculi with conjunctions and disjunctions of length less than an arbitrary infinite regular cardinal number. In particular, they obtained completeness theorems for such calculi (Scott and Tarski 1957, 1958). At that time Tarski was considering systems of logic which in contemporary terminology are known as $L_{\omega_1\omega_1}$. Systematic research of languages with formulas of infinite length was conducted by Carol Karp, who was a doctoral student of Leon Henkin.

The first monographs devoted entirely to infinitary logic are Karp 1964 and Dickmann 1975. The monograph Keisler 1971 was also very influential. One should of course also mention the monumental work Barwise and Feferman 1985.

9.3. Infinitary logic: a few recent results

Let me finish this section with a few remarks concerning the present state of knowledge in the domain of infinitary logic. It may be interesting to compare recent findings with the goal declared in Zermelo's project.

9.3.1. How strong does expressive power need to be in logic?

What are the main methodological reasons why the first-order logic (FOL) is – or should be – recognized as the standard system of logic? I believe that at least the following items should be distinguished:

1. FOL is sound and complete: the sets of its theses (in an axiomatic approach) and tautologies are equal.
2. FOL is sound and complete also in a strong sense: syntactic derivability reflects semantic entailment.
3. The same can be said about other proof systems proposed for FOL, such as analytic tableaux, resolution, natural deduction, sequent calculi, and so on.
4. FOL is consistent and compact.
5. FOL satisfies the theorem on the neutrality of non-logical predicates, function symbols and individual constants.
6. FOL satisfies the Löwenheim-Skolem-Tarski theorem, that is, roughly speaking, FOL does not distinguish infinite powers.
7. Zermelo-Fraenkel set theory can be expressed in FOL.
8. According to the Lindström theorem, any logic that is compact and satisfies the Löwenheim-Skolem theorem (under some natural assumptions) is equivalent to FOL.

These properties are considered methodological ideals by logicians, and they are connected to the finitary character of the process of deduction. They also reflect the universal character of FOL. For these reasons, many logicians accept the First-Order Thesis claiming that FOL is *the* logic, the standard and natural logical system (see for instance Woleński 2004).

On the other hand, the assumptions concerning syntax and inferences in first-order language have important consequences for the expressive power of such language: it appears to be very weak. This means, among others, that many fundamental mathematical concepts are not expressible in a first-order language by a single formula, for instance: infinity, continuity, a set of measure zero, or the Archimedean axiom. Categorical characterizations of mathematical structures are impossible in first-order languages. The famous limitative theorems (Gödel, Tarski, Rosser, Church) show incompleteness and non-definability phenomena typical of theories formulated in such languages.

Extensions of FOL that have greater expressive power are interesting for both mathematicians and logicians. There are several possibilities for such extensions:

1. New logical constants can be added to the standard collection, the latter including truth connectives, quantifiers, and the identity predicate. In this way we obtain several sorts of modal logics or logics with generalized quantifiers.
2. Non-finitary syntactic constructions, for example infinite conjunctions, disjunctions or quantifier prefixes, can be allowed.
3. New rules of inference, in particular infinitary rules (as for instance the ω-rule), can be added.

All these possibilities have been exploited and the metatheoretical properties of such new logical systems have been investigated. Allow me to add in the margins that there have also been situations of some predicates being expelled from logic; this was the case of the relation $\in$ which became a non-logical relation after the development of set theory. The mereological relation of being a part introduced in Leśniewski's systems is no longer counted as a logical constant, at least in the mainstream of logical investigations. The problem of what constitutes a logical notion has been investigated by several researchers; see for instance Lindenbaum and Tarski 1936, Tarski 1986, Woleński 1997.

Andrzej Mostowski introduced generalized quantifiers in 1957 (Mostowski 1957). The quantifier Q_α has the following intended meaning: there exist (in the domain of the model) at least $\aleph_\alpha$ objects (with a property expressed by a formula of this extended language). Henkin introduced branching quantifiers (Henkin 1957). Lindström's works from the nineteen sixties and Barwise's works from the nineteen seventies created a whole new domain of research, called abstract model theory (sometimes also called soft model theory); see for example Lindström 1966, 1969, Barwise 1974. Detailed information can be found for instance in Barwise and Feferman 1985, Shapiro 1996, Westerståhl 1989, Krynicki, Mostowski and Szczerba 1995.

9.3.2. Metalogic for infinitary logic

Interpreters of Zermelo's works differ in their opinions regarding what kind of contemporary infinitary logic could be the closest to his semi-formal proposals. As candidates in this respect one can consider: logic $L_{\infty\infty}$; logic $L_{\kappa\kappa}$, where κ is a strongly inaccessible cardinal number; logic $L_{\infty\omega}$; and second order logic. These systems of logic have different expressive power, which is evident from the following examples:

1. The standard model of Peano arithmetic can be characterized in $L_{\omega_1\omega}$; the same concerns the class of all finite sets.
2. The theory of ordered Archimedean fields is finitely axiomatizable in $L_{\omega_1\omega}$.
3. The truth predicate for a language with a countable number of symbols is definable in $L_{\omega_1\omega}$.
4. The notion of well-ordering is not definable in $L_{\omega_1\omega}$, but it is definable (by a single formula) in $L_{\omega_1\omega_1}$.

Infinitary logics with infinite quantifier prefixes are close to second-order logic and hence they do not satisfy the completeness theorem. In $L_{\omega_1\omega_1}$ we have Scott's theorem about non-definability of the truth predicate in the language of this logic. A distinguished place among infinitary logics is occupied by $L_{\omega_1\omega}$. It satisfies the completeness theorem, if the infinitary rule which gives the conclusion $\bigwedge \Phi$ (the infinite conjunction) from the set of premises Φ satisfies the condition that Φ is at most countable. This condition is essential: there exists an uncountable set of sentences of $L_{\omega_1\omega}$ which does not have a model but whose every countable subset has a model. This example shows that the compactness theorem does not hold in either $L_{\omega_1\omega}$ or in any of $L_{\alpha\beta}$, where $\alpha \geqslant \omega_1$. Nevertheless, one can define suitable generalizations of the compactness theorem which are appropriate in such cases (they are related to the existence of large cardinal numbers).

Any countable structure with a countable number of predicates can be characterized up to isomorphism in $L_{\omega_1\omega}$, as shown in Scott's theorem. The semantic properties of models of $L_{\alpha\omega}$ and $L_{\infty\omega}$ can be characterized in algebraic terms (Karp's theorem about partial isomorphisms).

It is not the goal of this paper to systematically discuss the metatheoretical properties of recently-investigated infinitary logics. One may wonder what Zermelo's reaction would be to the fact that the downward Löwenheim-Skolem theorem has its counterpart in $L_{\omega_1\omega}$ and, in general, in all infinitary logics. As the reader probably remembers, Zermelo did not accept that theorem. The upward Löwenheim-Skolem-Tarski theorem does not hold in infinitary logics.

9.3.3. Admissible sets and generalized recursion

Formulas of the first-order logic $L_{\omega\omega}$ allow coding by natural numbers or, equivalently, by hereditary finite sets, that is, elements of $H(\omega)$. In turn, formulas of the logic $L_{\omega_1\omega}$ allow coding by elements of $H(\omega_1)$, that is, by hereditary countable sets. Proofs in this logic can also be coded by elements of $H(\omega_1)$. Such proofs have countable length.

One can give an example of a set of sentences Γ and a sentence σ from the language of $L_{\omega_1\omega}$ such that $\Gamma \models \sigma$, but there does not exist a proof of σ from Γ in $L_{\omega_1\omega}$. The set Γ can be chosen in such a way that it is Σ_1 in $H(\omega_1)$; see for instance Bell 2016.

The set $H(\omega_1)$ is closed with respect to the operations of forming countable subsets and sequences. But the fact mentioned immediately above shows that it is not closed with respect to the operation of coding proofs from Σ_1 in $H(\omega_1)$ sets of sentences. It is thus natural to look for such sets A replacing the set $H(\omega_1)$ that would be closed with respect to the operation of coding proofs and consideration of only such sentences which have codes in A. This was one of the motivations for investigating the so-called admissible fragments L_A of the logic $L_{\omega_1\omega}$.

Barwise discovered that there exist countable sets (*admissible sets*) $A \subseteq H(\omega_1)$ which do satisfy these conditions. They are thus generalizations of the hereditary finite sets which allow a reasonable recursion theory and proof theory. He proved his famous compactness theorem: if A is a countable admissible set, then every set of formulas of the language L_A which is Σ_1 in A and such that its every subset (being at the same time an element of A) has a model, then the entire set in question also has a model. This theorem has many applications; for instance one can prove that any countable transitive model of ZFC has a proper end extension. Barwise's work is a subtle combination of research in model theory, recursion theory and set theory.

The investigations of admissible sets can be conducted very smoothly in the special set theory KP, proposed in the nineteen sixties by Kripke and Platek. This is an elementary theory with $\in$ as its non-logical constant, which is a certain weakening of Zermelo-Fraenkel set theory. It does not assume the full power set axiom and a special role is played by the axioms of Δ_0-*separation* and Δ_0-*collection* (counterparts of the axiom schemas of separation and replacement) in which formulas of the class Δ_0 occur. Transitive sets A such that $(A, \in)$ is a model of KP are called *admissible sets*. Another theory considersed is KPU, that is, KP with urelements. A complete exposition of the theory of admissible sets can be found in Barwise 1975. A brief and accessible exposition of the recent state of the theory is for instance Keisler and Knight 2004.

10. Final remarks

Is any of the recently-considered systems of infinitary logic close to Zermelo’s original ideas? In particular, could KPU be a counterpart of Zermelo’s semi-formal constructions? Or perhaps systems of second-order logic (or systems between first- and second-order logic) would be more appropriate in this respect? Zermelo himself was interested primarily in purely mathematical reasoning, while logical reflection was of secondary importance for him. I think his position is expressed soundly in his own *dictum*: mathematics is the logic of the Infinite.

The axiomatic systems of set theory, proposed by Zermelo in 1908 and 1930 (and improved by others), gained the acceptance of mathematicians working in all domains right from the very beginning. Set theory is still considered a firm foundation of all mathematical theories. There exist other foundational systems, for instance category theory whose formalism is useful in algebra and algebraic topology, but the role of set theory is still dominant throughout mathematics. This is above all the legacy of Ernst Zermelo.

“Normal” mathematicians, that is those who are not preoccupied with foundational research in set theory and mathematical logic, seem to be perfectly satisfied with the present shape of set theory. It is believed that any mathematical reasoning can in principle be fully formalized in some system of logic, although mathematicians seldom stress this fact in everyday research practice. The job of formalization is handed over to logicians. I dare to say that many mathematicians share Zermelo’s view on logical dependencies between mathematical statements.

Systems of logic are evaluated differently from the point of view of logicians and that of mathematicians. Logicians are primarily interested in deductive aspects of systems of logic, while mathematicians seem to put more stress on their expressive power (obviously not forgetting the deductive properties). Ion Barwise writes:

> But if you think of logic as the mathematicians in the street, then the logic in a given concept is what it is, and if there is no set of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with. (Barwise 1985, 7)

It should be added that the work Zermelo 1930 inspired logicians' efforts to develop set theory within the formalism of modern second-order logic; see for instance Uzquiano 1999.

I am well aware that this paper is only a very sketchy presentation of Zermelo's ideas concerning his system of infinitary logic. Much more information about this issue can be found in the monumental work Ebbinghaus, Fraser and Kanamori 2010. All of Zermelo's texts included in this work are accompanied by introductory notes written by prominent contemporary logicians.

NOTES

[1] The starting point for this paper was Pogonowski 2006, an article in Polish, published in a journal of local coverage and related to a more comprehensive but still unpublished work *Infinitarna Logika Ernsta Zermela* (*The Infinitary Logic of Ernst Zermelo*) written for the research grant KBN 2H01A 00725 *Metody nieskończonościowe w teorii definicji* (*Infinitary methods in the theory of definitions*) headed by Professor Janusz Czelakowski at the Institute of Mathematics and Information Science of the University of Opole, Poland. The final version of this paper contains essential improvements of the works mentioned above. The work on this paper has been sponsored by the National Scientific Center research grant 2015/17/B/HS1/02232 *Extremal axioms: logical, mathematical and cognitive aspects.*

[2] The original text, under the title "Der Relativismus in der Mengenlehre und der sogenannte Skolem'sche Satz" is included in Zermelo's *Nachlaß* in the library of the Universität Freiburg; it has been published together with the English translation in the paper van Dalen, Ebbinghaus 2000, 145–161, and in Ebbinghaus, Fraser, Kanamori 2010, 602–605.

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