ON DECOMPOSITION OF GENERALIZED CONTINUITY

SADIK BAYHAN — ALEV KANIBIR — IVAN L. REILLY

ABSTRACT. The aim of this paper is to obtain some decompositions of generalized continuity by providing some new variants of generalized continuous functions. Our partial success is an indication of significant differences between the theory of topological spaces and that of generalized topological spaces.

1. Introduction

In [8], A. Császár introduced the notions of generalized topological spaces and generalized neighborhood systems. He also introduced generalized continuous functions by using these concepts and studied some of the properties of these functions. Furthermore, his study of generalized open sets in generalized topological spaces [6]–[10] has led to others defining new concepts and deriving new results. After Császár's foundational ideas, several papers have provided new aspects of the theory of generalized continuous functions. The reader may refer (for example) to [4], [5], [17], [18], [20], [24]–[28]. In this paper, we give decompositions of generalized continuity and of g - α -continuity by providing some new types of generalized continuous functions. We adapt the approach for topological continuity using generalized open sets described by Reilly [32].

In the field of Mathematical Psychology, the notion of knowledge space was introduced by Doignon and Falmagne [11]. These spaces provide a rigorous mathematical foundation for various practical systems of knowledge assessment. Their theory is enunciated in the book [12] by Doignon and Falmagne, where the mathematics is largely combinatorial and stochastic. It turns out that a knowledge space is exactly a strong generalized topological space in the sense

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of Császár [8]. Knowledge spaces are certainly the most important current example and application of generalized topological spaces, outside of topology.

In the topological category, the morphisms are continuous functions between topological spaces. Over the years, many generalizations of continuous functions have been introduced. Two of the oldest and most significant are nearly continuous functions and quasi-continuous functions. In 1922, Blumberg [3] defined near continuity for real-valued functions on Euclidean space. He used the term *densely approached*. This concept was generalized later to functions between general topological spaces by Pták [29] who used the term *nearly continuous*, and by Husain [16] who called this notion *almost continuity*. Mashhour, Abd El-Monsef and El-Deeb [23] have used the term *pre-continuity* for nearly continuity. The concepts of nearly continuous and nearly open functions are significant in functional analysis, especially in the context of open mapping theorems and closed graph theorems. Even older is the notion of quasi-continuity, which is intimately connected with the question of separate and joint continuity of functions from $\mathbb{R}^2$ to $\mathbb{R}$. According to Baire [1], it was Volterra who observed that if $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is separately continuous, then for each point $(a, b) \in \mathbb{R} \times \mathbb{R}$, for each disc D with center (a, b) and for each $\varepsilon > 0$, there is a disc D_1 contained in D such that

$$|f(x, y) - f(a, b)| < \varepsilon$$

for all points (x, y) in D_1 . This property of separately continuous functions was later called quasi-continuity by Kempisty [19]. This property of functions between topological spaces has subsequently appeared under different names in the literature, for example, the term *neighborly* was used by Bledsoe [2], *almost continuous* by Frolík [13], and *semicontinuous* by Levine [21].

A decomposition of continuity is a pair of properties of functions between topological spaces each of which is weaker than continuity, and which are together equivalent to continuity. One member of the pair is called a continuity dual of the other. In this paper, we shall attempt to obtain “generalized continuity duals”, or decompositions of generalized continuity. In particular, we shall try to obtain generalized topological space versions of the two important decompositions of continuity proved by Ganster and Reilly [14], [15]. These papers provided continuity duals of near continuity and of quasi-continuity, respectively.

It turns out, rather surprisingly, that we are able to obtain a generalized topological space analogue for just one of these two fundamental decompositions of continuity. In fact, we show that the decomposition of generalized continuity corresponding to that of continuity, where one continuity dual is quasi-continuity (see [15]) holds. But the other classical decomposition of continuity, where one continuity dual is near continuity (see [14]) has no parallel result for generalized

continuity. We provide a counter-example for the anticipated theorem (conjecture). So, the behaviour of functions between generalized topological spaces differs from the behaviour of functions between topological spaces in a significant fashion.

2. Preliminaries

If a non-empty set X is given with the power set $\exp X$, a subset g_X of $\exp X$ is said to be a generalized topology (briefly *GT*) on X if and only if $\emptyset \in g_X$ and an arbitrary union of elements of g_X belongs to g_X (see [8]). If $X \in g_X$, then g_X is a strong *GTS*. If g_X is a *GT* in X , the elements of g_X are called g_X -open sets and their complements are called g_X -closed sets. The generalized interior of a subset A of X , denoted by $i_{g_X}(A)$, is the union of the collection of all generalized open sets contained in A , i.e., the largest g_X -open set contained in A . The generalized closure of a subset A of X , denoted by $c_{g_X}(A)$, is the intersection of the collection of all generalized closed sets containing A , i.e., the smallest g_X -closed set containing A (see, e.g., [9]). A set $A \subset X$ is g_X -semiopen [7] if and only if $A \subset c_{g_X}(i_{g_X}(A))$, g_X -preopen [9] if and only if $A \subset i_{g_X}(c_{g_X}(A))$, g_X - α -open [9] if and only if $A \subset i_{g_X}(c_{g_X}(i_{g_X}(A)))$. The classes of all g_X -semiopen sets, all g_X -preopen sets, all g_X - α -open sets on X are denoted by $\sigma(g_X), \pi(g_X), \alpha(g_X)$ (briefly σ, π, α), respectively.

DEFINITION 2.1. Let (X, g_X) and (Y, g_Y) be generalized topological spaces. Then, a function $f : X \rightarrow Y$ is said to be:

- (1) (g_X, g_Y) -continuous [8] if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X -open in X ,
- (2) (α, g_Y) -continuous [28] if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X - α -open in X ,
- (3) (σ, g_Y) -continuous [28] if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X -semiopen in X ,
- (4) (π, g_Y) -continuous [28] if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X -preopen in X ,
- (5) weakly (g_X, g_Y) -continuous [25] if for each $x \in X$ and each g_Y -open set V containing $f(x)$, there exists a g_X -open set U containing x such that $f(U) \subset c_{g_Y}(V)$.

The terms g -continuous, g - α -continuous, g -semicontinuous, g -precontinuous and weakly g -continuous are sometimes used for the properties we have denoted by (g_X, g_Y) -continuous, (α, g_Y) -continuous, (σ, g_Y) -continuous, (π, g_Y) -continuous and weakly (g_X, g_Y) -continuous, respectively. We observe that (σ, g_Y) -continuity is the generalized version of the classical notion of quasi-continuity

of functions between topological spaces, while (π, g_Y) -continuity is the generalized analogue of nearly continuous functions.

We now recall some notions defined in [10].

Let g_X be a GT on X and $f: X_0 \rightarrow X$ be a function. Then $\{f^{-1}(G): G \in g_X\}$ constitutes the inverse image $g_{X_0} = f^{-1}(g_X)$ of g_X .

PROPOSITION 2.2 ([10]). *If g_X is a GT on X , then $f^{-1}(g_X)$ is a GT on X_0 .*

The g_{X_0} -closed sets are those of the form $f^{-1}(F)$, where F is g_X -closed. In the particular case when $X_0 \subset X$ and $f(x) = x$ for $x \in X_0$, then g_{X_0} is the restriction of $g_X \rightarrow X_0$. It is denoted by $g_{X_0} = g_X|_{X_0}$. In this case $f^{-1}(G) = G \cap X_0$ for $G \subset X$. So G is g_{X_0} -closed if and only if $G = F \cap X_0$, where F is g_X -closed.

Let $\beta \subset \exp Y$ be arbitrary. Then,

LEMMA 2.3 ([10]). *The family $g_Y \subset \exp Y$ composed of $\emptyset$ and all sets $N \subset Y$ of the form $N = \cup_{i \in I} B_i$, where $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary, is a GT on Y .*

Hence we say that the base β generates the GT g_Y .

LEMMA 2.4 ([10]). *Let g_X be a GT on X and the GT g_Y on Y be generated by the base β . Then a map $f: X \rightarrow Y$ is (g_X, g_Y) -continuous if and only if $f^{-1}(B) \in g_X$ for each $B \in \beta$.*

3. Decomposition of generalized continuity

In [15], Ganster and Reilly introduced the concept of interior-closed subsets (*ic* sets) of a topological space. We adapt their definition for generalized topological spaces in Definition 3.1. Note that a set H is g_A -closed in $A \subset (X, g_X)$ if $H = F \cap A$ for some g_X -closed subset F of (X, g_X) .

DEFINITION 3.1. Let (X, g_X) be a generalized topological space. A set $A \subset X$ is called g_X -interiorclosed (briefly g_X -*ic*) if $i_{g_X}(A)$ is g_A -closed in A .

It is clear that any g_X -open set is g_X -*ic* but the converse is not true. A simple example for the case of topological space may be given (see Example 1 of [15]).

Now, Definition 3.1 allows us to obtain a theorem analogous to one of the main theorems of [15]. However, our method of proof is not entirely parallel to the corresponding proof in [15].

THEOREM 3.2. *Let (X, g_X) be a generalized topological space and $A \subset X$. Then A is g_X -open if and only if A is g_X -*ic* and g_X -semiopen.*

Proof. Sufficiency is obvious. If A is g_X - ic , then $i_{g_X}(A) = F \cap A$, where F is g_X -closed. Since $c_{g_X}(i_{g_X}(A))$ is the smallest g_X -closed subset containing $i_{g_X}(A)$, we have $i_{g_X}(A) = c_{g_X}(i_{g_X}(A)) \cap A$. If A is g_X -semiopen, then $A \subset c_{g_X}(i_{g_X}(A))$, so that $i_{g_X}(A) = c_{g_X}(i_{g_X}(A)) \cap A \supset A$. Hence, A is g_X -open. $\square$

Again, following the pattern established by Ganster and Reilly [14], we now define a new class of functions between generalized topological spaces.

Examples from [15] can be used to distinguish between functions satisfying Definition 3.3 and functions not having this property.

DEFINITION 3.3. Let (X, g_X) and (Y, g_Y) be generalized topological spaces. Then a function $f: X \rightarrow Y$ is called (ic, g_Y) -continuous (or briefly g - icc) if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X - ic in X .

The proof of the following decomposition of (g_X, g_Y) -continuity follows from Theorem 3.2 and Definitions 3.1, 3.3, and 2.1.

THEOREM 3.4. Let $f: (X, g_X) \rightarrow (Y, g_Y)$ be a function. Then f is (g_X, g_Y) -continuous if and only if f is (ic, g_Y) -continuous and (σ, g_Y) -continuous.

In 1985, Reilly and Vamanamurthy [31] proved for any topological space (X, τ) that a subset A of X is α -open if and only if A is semi-open and preopen. Twenty years later, Császár [9] obtained the corresponding result for generalized topologies. It is stated as our next theorem.

THEOREM 3.5 ([9]). If g_X is a GT on X , then $\alpha(g_X) = \sigma(g_X) \cap \pi(g_X)$.

We have a proof of the following decomposition for (α, g_Y) -continuity from Theorem 3.5 and Definition 2.1.

THEOREM 3.6. Let $f: (X, g_X) \rightarrow (Y, g_Y)$ be a function. Then f is (α, g_Y) -continuous if and only if f is (σ, g_Y) -continuous and (π, g_Y) -continuous.

4. Differences from decomposition of continuity

There are two important decompositions of continuity between (ordinary) topological spaces. Let X and Y be topological spaces and $f: X \rightarrow Y$ be a function. Then,

- A. f is continuous if and only if f is nearly continuous and LC -continuous [14], and
- B. f is continuous if and only if f is quasi-continuous and ic -continuous [15].

In this section we show the surprising result that decomposition A does not have an analogous result for generalized topological spaces, while decomposition B has an exactly parallel decomposition for generalized continuity (namely Theorem 3.5). This dichotomy is unexpected.

Recall that a subset R of the generalized topological space (X, g_X) is defined to be g_X -regular open (or $g_X r$ -open) if $R = i_{g_X}(c_{g_X}(R))$. The complements of g_X -regular open sets are called g_X -regular closed.

DEFINITION 4.1 ([30]). Let (X, g_X) be a generalized topological space and $A \subset X$. Then, A is said to be:

- (1) g_X -locally closed if $A = U \cap F$, where U is g_X -open and F is g_X -closed,
- (2) a g_X - $\mathcal{A}$ -set if $A = U \cap F$, where U is g_X -open and F is g_X -regular closed.

We shall denote the collections of g_X -locally closed and g_X - $\mathcal{A}$ -sets by $LC(g_X)$ and $\mathcal{A}(g_X)$, respectively. Similarly to the classical case, a subset A of (X, g_X) is g_X -locally closed if $A = U \cap c_{g_X}(A)$ for some g_X -open set U .

In topological spaces, it is the case that every open subset is locally closed and that every closed subset is locally closed, see G anster and Re illy [14]. The corresponding situation in generalized topological spaces is quite different. In a generalized topological space, every g_X -open set is g_X -locally closed but every g_X -closed set is not g_X -locally closed in general. Furthermore, every g_X - $\mathcal{A}$ -set is g_X -locally closed but the converse is not true.

EXAMPLE 4.2. Let $X = \{a, b, c, d\}$ and $g_X = \{\emptyset, \{b\}, \{b, c, d\}\}$. Then g_X is a GT on X . Furthermore, the g_X -closed subsets are $\{X, \{a, c, d\}, \{a\}\}$ and $LC(g_X) = g_X \cup \{\{c, d\}\}$, and $\mathcal{A}(g_X) = g_X$. Then the subset $\{a, c, d\}$ is g_X -closed but it is not g_X -locally closed. Also $\{c, d\}$ is g_X -locally closed but it is not a g_X - $\mathcal{A}$ -set.

DEFINITION 4.3. Let (X, g_X) and (Y, g_Y) be generalized topological spaces. Then a function $f: X \rightarrow Y$ is said to be:

- (1) (LC, g_Y) -continuous [22], or g - LC -continuous, if for each g_Y -open set U in Y , $f^{-1}(U)$ is g_X -locally closed in X ,
- (2) $(\mathcal{A}, g_Y)$ -continuous, or g - $\mathcal{A}$ -continuous, if for each g_Y -open set U in Y , $f^{-1}(U)$ is a g_X - $\mathcal{A}$ -set in X .

It is clear that $(\mathcal{A}, g_Y)$ -continuity implies (LC, g_Y) -continuity but the converse relation may not be true in general. A simple example for the case of topological spaces may be given (see [14, Example 1]).

In [14], the following theorem was proved for topological spaces. But we observe from Example 4.5 below that the generalized analogue of Theorem 4.4 does not hold in generalized topological spaces. One reason for that difference is

because in generalized topological spaces we do not necessarily have the preservation of the family of g_X -open sets under finite intersection.

THEOREM 4.4 ([14]). *Let $f: X \longrightarrow Y$ be a function between topological spaces X and Y . Then,*

- (1) *f is continuous if and only if f is precontinuous and LC -continuous,*
- (2) *f is continuous if and only if f is precontinuous and $\mathcal{A}$ -continuous,*
- (3) *f is continuous if and only if f is α -continuous and LC -continuous,*
- (4) *f is $\mathcal{A}$ -continuous if and only if f is semicontinuous and LC -continuous.*

EXAMPLE 4.5. Let $X = \{a, b, c, d, e\}$ and $g_X = \{\emptyset, \{a\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{c, e\}, \{a, c, e\}, \{a, b, c, e\}, X\}$. Then, g_X is a GT on X . The subset $\{a, b\}$ is g_X - α -open since $\{a, b\} \subset i_{g_X}(c_{g_X}(i_{g_X}(\{a, b\})))$. Furthermore, $\{a, b\}$ is g_X -locally closed since $\{a, b\} = \{a, b, c\} \cap \{a, b, d\}$, where $\{a, b, c\}$ is a g_X -open set and $\{a, b, d\}$ is a g_X -closed set.

Now, let $Y = X$ and let $g_Y = \{\emptyset, \{a, b\}\}$. Then, g_Y is a GT on Y .

Let $f: (X, g_X) \longrightarrow (Y, g_Y)$ be the identity function defined by $f(x) = x$ for each $x \in X$. Since $f^{-1}(\{a, b\}) = \{a, b\} \notin g_X$, f is not (g_X, g_Y) -continuous. On the other hand, since $\{a, b\}$ is g_X -locally closed, g_X - α -open and g_X - $\mathcal{A}$ -set, f is (LC, g_Y) -continuous, (α, g_Y) -continuous and $(\mathcal{A}, g_Y)$ -continuous.

We can extend this example to show that Theorem 4.4 (4) does not hold in general for generalized topological spaces. Let $Z = X$ and define $g_Z = \{\emptyset, \{c\}\}$. Then g_Z is a GT on Z . Let $h: (X, g_X) \longrightarrow (Z, g_Z)$ be the identity function. Now, the g_X - $\mathcal{A}$ -sets are $g_X \cup \{\{a, b\}, \{c\}\}$. Consider the set $\{c\}$. It is a g_X - $\mathcal{A}$ -set but it is not g_X -semiopen. Hence h is $(\mathcal{A}, g_Z)$ -continuous, but h is not (σ, g_Z) -continuous.

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Sadik Bayhan
Department of Mathematics
Mehmet Akif Ersoy University
15030, Örtülü, Burdur
TURKEY
E-mail: bayhan@mehmetakif.edu.tr

Alev Kanibir
Department of Mathematics
Hacettepe University
06532 Beytepe, Ankara
TURKEY
E-mail: kanibir@hacettepe.edu.tr

Ivan L. Reilly
Department of Mathematics
University of Auckland
P.B. 92019, Auckland
NEW ZEALAND
E-mail: i.reilly@auckland.ac.nz