EXISTENCE OF THE ASYMPTOTICALLY PERIODIC SOLUTION TO THE SYSTEM OF NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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ABSTRACT. The system of nonlinear neutral difference equations with delays in the form

$$\begin{cases} \Delta(y_i(n) + p_i(n)y_i(n - \tau_i)) = a_i(n)f_i(y_{i+1}(n)) + g_i(n), \\ \Delta(y_m(n) + p_m(n)y_m(n - \tau_m)) = a_m(n)f_m(y_1(n)) + g_m(n), \end{cases}$$

for $i = 1, \dots, m-1$, $m \geq 2$, is studied. The sufficient conditions for the existence of an asymptotically periodic solution of the above system, are established. Here sequences $(p_i(n))$, $i = 1, \dots, m$, are bounded away from -1. The presented results are illustrated by theoretical and numerical examples.

1. Introduction

The aim of this paper is to prove the existence theorem for a nonlinear difference system of m ($m \geq 2$) equations given below

$$\begin{cases} \Delta(y_i(n) + p_i(n)y_i(n - \tau_i)) = a_i(n)f_i(y_{i+1}(n)) + g_i(n), \\ \Delta(y_m(n) + p_m(n)y_m(n - \tau_m)) = a_m(n)f_m(y_1(n)) + g_m(n), \end{cases} \quad (1)$$

where $i \in \{1, \dots, m-1\}$ and Δ stands for the forward difference operator, i.e., $\Delta u(n) = u(n+1) - u(n)$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\tau_i \in \mathbb{N}$, where $i = 1, \dots, m$. Set $n_0 = \max\{\tau_1, \dots, \tau_m\}$ and $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, \dots\}$. We consider system (1) for $n \in \mathbb{N}_{n_0}$. Moreover, $a_i, g_i, p_i: \mathbb{N}_{n_0} \rightarrow \mathbb{R}$, $i \in \{1, \dots, m\}$ are given real

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sequences, $y_i: \mathbb{N} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are unknown real sequences and functions $f_i: \mathbb{R} \rightarrow \mathbb{R}$. Throughout this paper, $Y = (y_1, \dots, y_m)$ and $Y(n)$ denotes $(y_1(n), \dots, y_m(n)) \in \mathbb{R}^m$.

The term 'neutral' was introduced by Kamenskii in the mid-20th century in the continuous case (see [13]). In 1955, the classification of equations as retarded, neutral, or advanced was used by Myshkis [18]. An equation of neutral type represents a system in which the present rate of change of a quantity depends on the past rate of change as well as the past and present values of the quantity. Experience in mathematical modelling has shown that the evolution equation of actual processes retardation are often neutral type equations. Besides its theoretical interest, strong interest in the study of the asymptotic and periodic behaviour of solutions of neutral type equations is motivated by the fact that they arise in many areas of applied mathematics, such as population dynamics, transport problems, and dynamical behaviour of delayed network systems. So, neutral equations have long played an important role in the history of theoretical population dynamics and there is no doubt that they will continue to serve as indispensable tools in future investigations.

Asymptotic properties of nonlinear neutral difference equations have been widely studied in many papers, for instance in article by Jankowski and Schmeidel [11, 12], Migda et al. [15, 16], Wang and Sun [26]. Wang and Young [25] investigated the existence of positive periodic solutions to one equation

$$\Delta(y(n) - cy(n - m)) = a(n)y(n) + f(n; y(n - m)),$$

where $a(n) = a(n + \omega)$, $f(n + \omega; u) = f(n; u)$, c is the positive constant and ω, m are positive integers. Philos and Purnaras [21] examined the asymptotic behaviour, the nonoscillation and the stability of the solutions of the equation in the form

$$\Delta\left(y_n + \sum_{i \in I} c_i y_{n-\sigma_i}\right) = a(n)y_n + \sum_{j \in J} b_j(n)y_{n-\tau_j},$$

for c_i - real numbers and a, b_j - real periodic sequences. Positive periodic solutions were the aim of the study of Nalini and Banu [19]. They considered the system

$$\begin{aligned} u_i(t+1) = & a_i(t)u_i(t) + \sum_{j=1}^{\infty} b_{ij}(t)f_j(u_j(t)) + \sum_{j=1}^{\infty} c_{ij}(t)\Delta(u_j(t-\tau)) \\ & + \sum_{j=1}^{\infty} d_{ij}(t)f_j\left[\sum_{\eta_1}^{\infty} h_j(\eta)u_j(t-\eta)\right] + I_i(t), \quad i \in \{1, \dots, m\}. \end{aligned}$$

Ren and Siegmund [22] investigated the difference equation with variable delay

$$y_{n+1} - cy_{n+1-\tau(n+1)} = a_n y_n + \lambda h_n f(y_{n-\tau(n)}),$$

where $n \in \mathbb{Z}$, a_n, h_n are ω -periodic positive sequences, $\tau(n)$ is an ω -periodic integer sequence, $f \in C([0; +\infty); [0; +\infty))$, $\lambda > 0$ and $0 < c < 1$. Boundedness and existence of nonoscillatory solutions of some classes of neutral difference equations were also studied by Thandapani, Karunakaran and Arockiasamy in [24] and by Zhou et al. [27, 28]. Oscillation of solutions of nonlinear neutral difference equations was investigated, for example, in papers by Agarwal et al. [2, 3], Bolat and Akin [5], and Parhi and Tripathy [20]. In [17], the authors established the sufficient conditions in order to get the existence theorem of a bounded solution of (1). Here, conditions for the existence of asymptotically periodic solution are presented.

Asymptotically periodic solutions were investigated among others for Volterra difference equations in [1, 6–10] as well as in [4, 14] for the rational difference equations.

Since any periodic sequence is bounded, then the conditions ensuring existence of a periodic solution of (1) are sufficient for existence of bounded solution to system (1). Therefore, results presented here are complementary to the results obtained in [17].

Let us recall some basic facts.

DEFINITION 1. Let τ be a positive integer. The sequence $y: \mathbb{N} \rightarrow \mathbb{R}$ is called τ -periodic, if $y(n + \tau) = y(n)$ for all $n \in \mathbb{N}$. The sequence y is called asymptotically τ -periodic, if there exist two sequences $u, v: \mathbb{N} \rightarrow \mathbb{R}$ such that u is τ -periodic, $\lim_{n \rightarrow \infty} v(n) = 0$ and $y(n) = u(n) + v(n)$ for all $n \in \mathbb{N}$.

As a solution of system (1) we treat a vector Y such that its components $y_1, \dots, y_m$ satisfy system (1) for sufficiently large n . Let $\tau = (\tau_1, \dots, \tau_m)$. Sequence Y that satisfies (1) is called τ -periodic if

$$(y_1(n), \dots, y_m(n)) = (y_1(n + \tau_1), \dots, y_m(n + \tau_m)).$$

It means that the solution Y of (1) is periodic, if all its components are periodic. Solution Y of (1) is asymptotically periodic, if all its components are asymptotically periodic.

By $(\mathcal{B}, \|\cdot\|)$ we denote the Banach space of all bounded sequences in $\mathbb{R}^m$ with the supremum norm, i.e.

$$\mathcal{B} = \left\{ Y: \mathbb{N} \rightarrow \mathbb{R}^m: \|Y\| = \sup_{n \in \mathbb{N}} |Y(n)| < \infty \right\},$$

where

$$|Y| = \max_{i \in \{1, 2, \dots, m\}} |y_i|.$$

Let r be the fixed positive real number. Define a set

$$S := \{Y \in \mathcal{B} : \|Y\| \leq r \text{ and } Y \text{ is asymptotically } \tau\text{-periodic}\}.$$

Theorems given below will be used in the next paragraphs.

THEOREM 1 (Tychonoff's Theorem, [23]). *The product of any non-empty class of compact spaces is compact.*

LEMMA 1. *S is convex and compact subset of $\mathcal{B}$.*

Proof. It is easy to see that S is convex subset of $\mathcal{B}$. Indeed, let $Y, Z \in S$ and $\alpha \in [0, 1]$. If

$$-r \leq y_i(n), z_i(n) \leq r,$$

for $i = 1, \dots, m$, $n \in \mathbb{N}$, then

$$-r\alpha \leq \alpha y_i(n) \leq r\alpha,$$

$$-r(1 - \alpha) \leq (1 - \alpha)z_i(n) \leq r(1 - \alpha),$$

and

$$-r \leq \alpha y_i(n) + (1 - \alpha)z_i(n) \leq r \quad \text{for } i = 1, \dots, m, n \in \mathbb{N}$$

which means that S is convex.

Now, we prove that S is also compact. By virtue of Tychonoff's Theorem 1, in order to prove that S is compact, it is enough to prove that the set (for $i = 1, \dots, m$)

$$B_i := \left\{ y_i : \mathbb{N} \rightarrow \mathbb{R} : \sup_{n \in \mathbb{N}} |y_i(n)| \leq r \text{ and } y_i \text{ is asymptotically } \tau_i\text{-periodic} \right\},$$

is compact. Since y_i is asymptotically τ_i -periodic for $i = 1, \dots, m$, there exist sequences $u_i, v_i : \mathbb{N} \rightarrow \mathbb{R}$ such that $y_i(n) = u_i(n) + v_i(n)$ for all $n \in \mathbb{N}$ and $u_i(n + \tau_i) = u_i(n)$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} v_i(n) = 0$. Hence, the terms of the sequence (u_i) are as follows $\{C_{i1}, C_{i2}, \dots, C_{i\tau_i}, C_{i1}, C_{i2}, \dots\}$, where $C_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, \tau_i$.

Let us take $\varepsilon_i > 0$. There exists $n^* \in \mathbb{N}$ such that $|y_i(n) - C_{ij}| < \varepsilon_i$ for some $j = 1, \dots, \tau_i$ and $n > n^*$. Denote $\rho_i := \lfloor \frac{r}{\varepsilon_i} \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function.

Then the ε_i -neighbourhood of the points

$$(\delta_{i1}, \delta_{i2}, \dots, \delta_{in^*}, C_{i(j+1)}, \dots, C_{i\tau_i}, C_{i1}, C_{i2}, \dots, C_{i\tau_i}, \dots)$$

where $\delta_{ij} \in \{0, \pm\varepsilon_i, \pm 2\varepsilon_i, \dots, \pm\rho_i\varepsilon_i\}$, $j = 1, \dots, n^*$, is the ε_i -cover of the set B_i . Cardinality of this ε_i -cover is finite and equal to $(2\rho_i + 1)^{n^*}$. Hence, the set B_i is compact. $\square$

2. Major outputs

We can now formulate theorem, which gives the sufficient condition for the existence an asymptotically periodic sequence satisfying (1).

THEOREM 2. *Let $i = 1, \dots, m$. If*

$$\sum_{s=n_0}^{\infty} |a_i(s)| < \infty, \quad (2)$$

$$(g_i(n)) \text{ are } \tau_i\text{-periodic sequences such that } \sum_{s=0}^{\tau_i-1} g_i(s) = 0, \quad (3)$$

$$p_i(n) \neq -1 \text{ for each } n \in \mathbb{N}, \text{ and } (p_i(n)) \text{ are } \tau_i\text{-periodic sequences,} \quad (4)$$

$$f_i: \mathbb{R} \rightarrow \mathbb{R} \text{ are continuous functions,} \quad (5)$$

then system (1) has an asymptotically τ -periodic solution.

PROOF. The series $\sum_{s=0}^{\infty} g_i(s)$, $i = 1 \dots, m$, by assumption (3), are - in general - divergent but each subtotal is always bounded. Hence, there exists the positive constant

$$c_g := \max_{\substack{n \in \{0, \dots, \tau_i-1\} \\ i=1, \dots, m}} \left| \sum_{s=0}^n g_i(s) \right|.$$

Condition (4) implies existence of the positive constant

$$c_p := \sup_{\substack{n \in \mathbb{N} \\ i=1, \dots, m}} \frac{1}{|1 + p_i(n)|}.$$

From now on we make the assumption that

$$r \geq 3c_p c_g.$$

Since condition (5) is satisfied, we have

$$M := \max_{i=1, \dots, m} \{|f_i(t)| \text{ for } |t| \leq r\}. \quad (6)$$

Let us take $n_1 \geq n_0$ such that

$$\sum_{s=n_1}^{\infty} |a_i(s)| < \frac{r}{3Mc_p} \quad \text{for } i = 1, \dots, m. \quad (7)$$

For $y_i \in S$, $i = 1 \dots, m$, we have

$$\begin{aligned} y_i(n - \tau_i) &= u_i(n - \tau_i) + v_i(n - \tau_i) \\ &= u_i(n) + v_i(n - \tau_i) = y_i(n) - v_i(n) + v_i(n - \tau_i). \end{aligned} \quad (8)$$

Moreover, since $\lim_{n \rightarrow \infty} (v_i(n - \tau_i) - v_i(n)) = 0$ and p_i is periodic for $i = 1, \dots, m$, we can write

$$\left| \frac{p_i(n)}{1 + p_i(n)} \right| |v_i(n - \tau_i) - v_i(n)| \leq \frac{r}{3}, \quad n \geq n_2, \quad (9)$$

for some $n_2 \geq n_0$.

Let $n_3 = \max\{n_1, n_2\}$. We define the mapping $T: S \rightarrow \mathcal{B}$, $T = (T_1, \dots, T_m)$ in such a way

$$(T_i Y)(n) = \begin{cases} (T_i Y)(n_3), & 0 \leq n < n_3, \\ \frac{1}{1+p_i(n)} \left(- \sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s)) \right. \\ \left. + \sum_{s=n_3}^{n-1} g_i(s) - p_i(n)(v_i(n - \tau_i) - v_i(n)) \right), & n \geq n_3, \end{cases} \quad (10)$$

where $i \in \{1, \dots, m-1\}$ and

$$(T_m Y)(n) = \begin{cases} (T_m Y)(n_3), & 0 \leq n < n_3, \\ \frac{1}{1+p_m(n)} \left(- \sum_{s=n}^{\infty} a_m(s) f_m(y_1(s)) \right. \\ \left. + \sum_{s=n_3}^{n-1} g_m(s) - p_m(n)(v_m(n - \tau_m) - v_m(n)) \right), & n \geq n_3, \end{cases} \quad (11)$$

where v_i is part of y_i such that $\lim_{n \rightarrow \infty} v_i(n) = 0$.

We will show that T satisfies the assumptions of Schauder's Fixed Point Theorem. We claim that, if $Y \in S$, then $TY \in S$. For $n \geq n_3$, $i \neq m$, we have

$$\begin{aligned} |(T_i Y)(n)| &= \left| \frac{1}{1+p_i(n)} \left(- \sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s)) \right. \right. \\ &\quad \left. \left. + \sum_{s=n_3}^{n-1} g_i(s) - p_i(n)(v_i(n - \tau_i) - v_i(n)) \right) \right| \\ &\leq \frac{1}{|1+p_i(n)|} \sum_{s=n}^{\infty} |a_i(s)| |f_i(y_{i+1}(s))| \\ &\quad + \frac{1}{|1+p_i(n)|} \left| \sum_{s=n_3}^{n-1} g_i(s) \right| + \left| \frac{p_i(n)}{1+p_i(n)} \right| |v_i(n - \tau_i) - v_i(n)| \\ &\leq c_p \frac{r}{3Mc_p} M + c_p c_g + \frac{r}{3} \leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r \end{aligned}$$

since (3), (6), (7) and (9). Estimations for $i = m$ hold analogously. Moreover, let us notice that $\frac{1}{1+p_i(n)} \sum_{s=n_3}^{n-1} g_i(s)$ is τ_i -periodic, whereas

$$\frac{p_i(n)}{1+p_i(n)} (v_i(n - \tau_i) - v_i(n)) \quad \text{and} \quad \frac{1}{1+p_i(n)} \sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s))$$

tend to 0, if $n \rightarrow \infty$.

In order to prove continuity of T , we set $Y_j = (y_{j,1}, \dots, y_{j,m}) \in S$ for $j \in \mathbb{N}$ and $i \in \{1, \dots, m\}$ there is $y_{j,i}(n) \rightarrow y_i(n)$ when $j \rightarrow \infty$. The set S is closed.

Therefore, $Y = (y_1, \dots, y_m) \in S$. By (3) and (5) we receive

$$\begin{aligned}
 & |(T_i Y_j)(n) - (T_i Y)(n)| \\
 & \leq \frac{1}{|p_i(n) + 1|} \sum_{s=n}^{\infty} |a_i(s)| |f_i(y_{j,i+1}(s)) - f_i(y_{i+1}(s))| \\
 & \quad + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n - \tau_i) - v_i(n - \tau_i)| + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n) - v_i(n)| \\
 & \leq \sup_{s \geq n} |f_i(y_{j,i+1}(s)) - f_i(y_{i+1}(s))| \frac{1}{|p_i(n) + 1|} \sum_{s=n}^{\infty} |a_i(s)| \\
 & \quad + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n - \tau_i) - v_i(n - \tau_i)| + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n) - v_i(n)| \\
 & \leq c_p \cdot \frac{r}{3Mc_p} \sup_{s \geq n} |f_i(y_{j,i+1}(s)) - f_i(y_{i+1}(s))| \\
 & \quad + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n - \tau_i) - v_i(n - \tau_i)| \\
 & \quad + \left| \frac{p_i(n)}{p_i(n) + 1} \right| |v_{j,i}(n) - v_i(n)| \rightarrow 0 \quad \text{if } j \rightarrow \infty, \quad i = 1, \dots, m-1,
 \end{aligned}$$

since f is continuous function and $(y(n))$ is an asymptotically τ -periodic sequence. For $i = m$ the estimation is analogous, then omitted. Therefore,

$$\|(TY_j) - (TY)\| \rightarrow 0 \quad \text{if } j \rightarrow \infty$$

and T is the continuous mapping.

By Schauder's Fixed Point Theorem, there exists Y such that $(TY)(n) = Y(n)$. Final step is to verify if Y satisfies system (1), where $n \geq n_3$. We have ($i = 1, \dots, m-1$)

$$\begin{aligned}
 y_i(n) &= \frac{1}{1 + p_i(n)} \left(-\sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s)) + \sum_{s=n_3}^{n-1} g_i(s) - p_i(n) (v_i(n - \tau_i) - v_i(n)) \right), \\
 (1 + p_i(n)) y_i(n) &= -\sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s)) + \sum_{s=n_3}^{n-1} g_i(s) - p_i(n) (v_i(n - \tau_i) - v_i(n)), \\
 y_i(n) + p_i(n) (y_i(n) + (v_i(n - \tau_i) - v_i(n))) &= -\sum_{s=n}^{\infty} a_i(s) f_i(y_{i+1}(s)) + \sum_{s=n_3}^{n-1} g_i(s).
 \end{aligned}$$

Hence, because of (8), we obtain

$$y_i(n) + p_i(n)y_i(n - \tau_i) = -\sum_{s=n}^{\infty} a_i(s)f_i(y_{i+1}(s)) + \sum_{s=n_3}^{n-1} g_i(s), \quad i = 1, \dots, m-1,$$

$$\Delta(y_i(n) + p_i(n)y_i(n - \tau_i)) = a_i(n)f_i(y_{i+1}(n) + g_i(n)), \quad i = 1, \dots, m-1.$$

The same method as above works if $i = m$. The proof is complete. $\square$

Note that for $p_i(n) = 0$, $i \in \{1, \dots, m\}$, $n \geq n_3$, system (1) is not of the neutral type, but Theorem 2 is still true.

Remark 1. Let us notice that all results obtained for system (1) are also true in case of one equality $\Delta(y(n) + p(n)y(n - \tau)) = a(n)f(y(n)) + g(n)$.

EXAMPLE 1. Consider the example of system (1)

$$\begin{cases} \Delta(y_1(k) + p_1(k)y_1(k-3)) = a_1(k)y_2(k) + g_1(k), \\ \Delta(y_2(k) + p_2(k)y_2(k-2)) = a_2(k)y_1(k) + g_2(k). \end{cases}$$

Here $\tau = (\tau_1, \tau_2) = (3, 2)$, $f_1(t) = t$, $f_2(t) = t$ and for $k \geq n_0 = 3$

$$p_1(k) = \begin{cases} -2 & \text{for } k \equiv 0 \pmod{3}, \\ 1 & \text{for } k \equiv 1 \pmod{3}, \\ 5 & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$p_2(k) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod{2}, \\ 7 & \text{for } k \equiv 1 \pmod{2}, \end{cases}$$

$$a_1(k) = \begin{cases} \frac{39}{2^{k+1}-2} & \text{for } k \equiv 0 \pmod{6}, \\ \frac{23}{2^{k+2}-2} & \text{for } k \equiv 1 \pmod{6}, \\ -\frac{97}{2^{k+1}-2} & \text{for } k \equiv 2 \pmod{6}, \\ \frac{39}{2^{k+2}-2} & \text{for } k \equiv 3 \pmod{6}, \\ \frac{23}{2^{k+1}-2} & \text{for } k \equiv 4 \pmod{6}, \\ -\frac{97}{2^{k+2}-2} & \text{for } k \equiv 5 \pmod{6}, \end{cases}$$

$$a_2(k) = \begin{cases} -\frac{19}{2^{k+1}+2} & \text{for } k \equiv 0 \pmod{6}, \\ \frac{53}{2^{k+2}+2} & \text{for } k \equiv 1 \pmod{6}, \\ -\frac{19}{3 \cdot 2^{k+1}+2} & \text{for } k \equiv 2 \pmod{6}, \\ \frac{53}{2^{k+1}+2} & \text{for } k \equiv 3 \pmod{6}, \\ -\frac{19}{2^{k+2}+2} & \text{for } k \equiv 4 \pmod{6}, \\ \frac{53}{3 \cdot 2^{k+1}+2} & \text{for } k \equiv 5 \pmod{6}, \end{cases}$$

$$g_1(k) = \begin{cases} 5 & \text{for } k \equiv 0 \pmod{3}, \\ 14 & \text{for } k \equiv 1 \pmod{3}, \\ -19 & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$g_2(k) = \begin{cases} 14 & \text{for } k \equiv 0 \pmod{2}, \\ -14 & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

One of the solutions is

$$y_1(k) = \begin{cases} 1 + \frac{1}{2^k} & \text{for } k \equiv 0 \pmod{3}, \\ 2 + \frac{1}{2^k} & \text{for } k \equiv 1 \pmod{3}, \\ 3 + \frac{1}{2^k} & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$y_2(k) = \begin{cases} 1 - \frac{1}{2^k} & \text{for } k \equiv 0 \pmod{2}, \\ 2 - \frac{1}{2^k} & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

The following numerical example shows the system of the form (1) having both: periodic and nonperiodic solutions depending on chosen initial values.

EXAMPLE 2. Consider the system

$$\begin{cases} \Delta(y_1(k) + \frac{(-1)^k}{10} y_1(k-1)) = \frac{1}{k^2} (y_2(k))^2 + (-1)^k, \\ \Delta(y_1(k) + \frac{(-1)^k}{10} y_1(k-1)) = \frac{1}{k^2+1} (y_1(k))^2 + (-1)^k. \end{cases} \quad (12)$$

All assumptions of Theorem 2 are satisfied and hence system (12) has an asymptotically 2-periodic solution. It is hard to find the solution of (12) in the analytical form since system (12) is nonlinear. Due to these problems we treat this example as the numerical illustration to Theorem 2. Figure 1 and Figure 2 show the trajectory of y_1 and y_2 received for initial conditions

$$y_1(1) = y_2(1) = 0, \quad y_1(2) = y_2(2) = -1.$$

Both plots suggest that solutions y_1 and y_2 are asymptotically 2-periodic.

FIGURE 1. Trajectory of y_1 .

FIGURE 2. Trajectory of y_2 .

FIGURE 3. Trajectory of y_1 .

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FIGURE 4. Trajectory of y_2 .

System (12) has also solutions that are not asymptotically 2-periodic. In the following two figures (Fig.3 and Fig.4), there are trajectories of y_1 and y_2 for initial conditions

$$y_1(1) = y_2(1) = 1, \quad y_1(2) = y_2(2) = 1.8.$$

The values of y_1 and y_2 tend to infinity as k tends to infinity.

The following result may be proved in the same way as Theorem 2:

THEOREM 3. *Let $i = 1, \dots, m$. If conditions (2), (3) and (5) hold,*

$$p_i(n) \neq -1 \text{ for each } n \in \mathbb{N}, \text{ and } \lim_{n \rightarrow \infty} p_i(n) = \beta \neq -1,$$

then system (1) has an asymptotically τ -periodic solution.

The next examples refer to Theorem 3.

EXAMPLE 3. Consider the system below for $k > n_0 = 2$

$$\begin{cases} \Delta(y_1(k) + (1 + \frac{1}{k^2}) y_1(k-2)) = \frac{2k^2+2k+1}{(1+k)^2(2k^2+1)} y_2(k) + 4 \cdot (-1)^{k+1}, \\ \Delta(y_2(k) + y_2(k-2)) = \left(\frac{1}{(k-2)^2} + \frac{1}{(k-1)^2} + \frac{1}{k^2} + \frac{1}{(k+1)^2} \right) y_1(k) + 8 \cdot (-1)^k. \end{cases}$$

One of the solutions is $y_1(k) = (-1)^k$ and $y_2(k) = (2 + \frac{1}{k^2}) \cdot (-1)^{k+1}$.

EXAMPLE 4. Let us consider the system

$$\begin{cases} \Delta(y_1(k) + p_1(k) y_1(k-2)) = a_1(k) y_2^2(k) + g_1(k), \\ \Delta(y_2(k) + p_2(k) y_2(k-3)) = a_2(k) y_1(k) + g_2(k). \end{cases}$$

Here we have

$$\tau = (\tau_1, \tau_2) = (2, 3), \quad f_1(t) = t^2, \quad f_2(t) = t;$$

and for $k > n_0 = 3$,

$$p_1(k) = 4 + \frac{1}{k^4}, \quad p_2(k) = \frac{1}{k^4},$$

$$\begin{aligned}
a_1(k) &= \begin{cases} \frac{-5k^8-16k^7-19k^6-21k^5-4k^4+2k^3+k^2-2k-1}{36k^{10}+48k^9-131k^8-263k^7-40k^6+186k^5+133k^4+29k^3+2k^2}, & k \equiv 0 \pmod{6}, \\ \frac{-5k^8-16k^7-21k^6-19k^5-2k^4+8k^3-7k-3}{4k^{10}+8k^9-11k^8-39k^7-24k^6+18k^5+29k^4+13k^3+2k^2}, & k \equiv 1 \pmod{6}, \\ \frac{-5k^8-16k^7-19k^6-21k^5-4k^4+2k^3+k^2-2k-1}{49k^{10}+63k^9-181k^8-349k^7-39k^6+253k^5+169k^4+33k^3+2k^2}, & k \equiv 2 \pmod{6}, \\ \frac{-5k^8-16k^7-21k^6-19k^5-2k^4+8k^3-7k-3}{36k^{10}+48k^9-131k^8-263k^7-40k^6+186k^5+133k^4+29k^3+2k^2}, & k \equiv 3 \pmod{6}, \\ \frac{-5k^8-16k^7-19k^6-21k^5-4k^4+2k^3+k^2-2k-1}{4k^{10}+8k^9-11k^8-39k^7-24k^6+18k^5+29k^4+13k^3+2k^2}, & k \equiv 4 \pmod{6}, \\ \frac{-5k^8-16k^7-21k^6-19k^5-2k^4+8k^3-7k-3}{49k^{10}+63k^9-181k^8-349k^7-39k^6+253k^5+169k^4+33k^3+2k^2}, & k \equiv 5 \pmod{6}, \end{cases} \\
a_2(k) &= \begin{cases} \frac{-k^8+2k^7+2k^6-8k^5+42k^4+8k^3-94k^2-107k-34}{k^{10}-9k^8-10k^7+15k^6+36k^5+25k^4+6k^3}, & k \equiv 0 \pmod{6}, \\ \frac{-k^8+2k^7+11k^6-37k^5+40k^4-26k^2-31k-10}{2k^{10}-k^9-17k^8-12k^7+32k^6+55k^5+31k^4+6k^3}, & k \equiv 1 \pmod{6}, \\ \frac{-k^8+2k^7+5k^6-27k^5+74k^4+10k^3-111k^2-126k-40}{k^{10}-9k^8-10k^7+15k^6+36k^5+25k^4+6k^3}, & k \equiv 2 \pmod{6}, \\ \frac{-k^8+2k^7+2k^6-8k^5+42k^4+8k^3-94k^2-107k-34}{2k^{10}-k^9-17k^8-12k^7+32k^6+55k^5+31k^4+6k^3}, & k \equiv 3 \pmod{6}, \\ \frac{-k^8+2k^7+11k^6-37k^5+40k^4-26k^2-31k-10}{k^{10}-9k^8-10k^7+15k^6+36k^5+25k^4+6k^3}, & k \equiv 4 \pmod{6}, \\ \frac{-k^8+2k^7+5k^6-27k^5+74k^4+10k^3-111k^2-126k-40}{2k^{10}-k^9-17k^8-12k^7+32k^6+55k^5+31k^4+6k^3}, & k \equiv 5 \pmod{6}, \end{cases}
\end{aligned}$$

$$g_1(k) = \begin{cases} 5, & k \equiv 0 \pmod{2}, \\ -5, & k \equiv 1 \pmod{2}, \end{cases}$$

$$g_2(k) = \begin{cases} -4, & k \equiv 0 \pmod{3}, \\ 5, & k \equiv 1 \pmod{3}, \\ -1, & k \equiv 2 \pmod{3}. \end{cases}$$

One of the solutions is

$$y_1(k) = \begin{cases} 1 + \frac{1}{k}, & k \equiv 0 \pmod{2}, \\ 2 + \frac{1}{k}, & k \equiv 1 \pmod{2}, \end{cases}$$

$$y_2(k) = \begin{cases} 6 + \frac{1}{k}, & k \equiv 0 \pmod{3}, \\ 2 + \frac{1}{k}, & k \equiv 1 \pmod{3}, \\ 7 + \frac{1}{k}, & k \equiv 2 \pmod{3}. \end{cases}$$

EXISTENCE OF THE ASYMPTOTICALLY PERIODIC SOLUTION

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