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TRAVEL DEMAND ESTIMATION IN URBAN ROAD NETWORKS AS INVERSE TRAFFIC ASSIGNMENT PROBLEM

***A. Krylatov*[1,2]*, A. Raevskaya*[1]*, V. Zakharov*[1]**

[1]*Saint Petersburg State University, Saint Petersburg, Russia,*
[2] *Institute of Transport Problems of the Russian Academy of Sciences, Saint Petersburg, Russia*
aykrylatov@yandex.ru

Nowadays, traffic engineers employ a variety of intelligent tools for decision support in the field of transportation planning and management. However, not a one available tool is useful without precise travel demand information which is actually the key input data in simulation models used for traffic prediction in urban road areas. Thus, it is no wonder that the problem of estimation of travel demand values between intersections in a road network is a challenge of high urgency. The present paper is devoted to this urgent problem and investigates its properties from computational and mathematical perspectives. We rigorously define the travel demand estimation problem as directly inverse to traffic assignment in a form of a bi-level optimization program avoiding usage of any pre-given (a priori) information on trips. The computational study of the obtained optimization program demonstrates that generally it has no clear descent direction, while the mathematical study advances our understanding on rigor existence and uniqueness conditions of its solution. We prove that once a traffic engineer recognizes the travel demand locations, then their values in the road network can be found uniquely. On the contrary, we discover a non-continuous dependence between the travel demand locations and absolute difference of observed and modeled traffic values. Therefore, the results of the present paper reveal that the actual problem to be solved when dealing with travel demand estimation is the problem of recognition of travel demand locations. The obtained findings contribute in the theory of travel demand estimation and give fresh managerial insights for traffic engineers.

Keywords: travel demand estimation; OD-matrix estimation; congestion; traffic assignment problem; primal and inverse problems

1. Introduction

Modern large cities demonstrate increasing dynamics of motorization which leads to various negative outcomes such as congestions, accidents, decreasing of average speed, traffic jams and noise, lack of parking space, inconveniences for pedestrians, pollution and environmental damage. The efficient traffic management is appeared to be the only way for coping with these problems since capacities of the actual road networks today are often close to their limits and already cannot be increased. However, the actual road networks are extremely huge and traffic engineers have to use the intelligent systems for decision-making support when planning or managing transportation processes. The reliability of such decisions highly depends on precision of travel demand information (e.g. Heydecker *et al.*, 2007; Kitamura and Susilo, 2005; Lessan and Fu, 2019). On the other hand, travel demand estimation is a highly sophisticated problem itself. Indeed, the problem of travel demand estimation has been investigated by researchers during the last 40 years but there are still practical and theoretical gaps to be fulfilled (e.g. Frederix *et al.*, 2013; Hernandez *et al.*, 2019).

1.1. Approaches and techniques

Researchers deal with travel demand estimation (or origin-destination (OD) matrix estimation) by virtue of different approaches and techniques. From the mathematical point of view the simplest approach may refer to the gravity methods, which have been well described by Isard (1960) in context of spatial regional analysis. Another large class of approaches can be associated with a priori economic theorizing (see, for instance, Fisher (1962)). Such basic concepts have been developed and implemented in various ways by scientists from various fields. Moreover, development of technical facilities and devices, such as traffic counters or plate scanning sensors has not completely solved the above-mentioned problem, but has initiated investigations of emerging problem issues. Thus, Makowski and Sinha (1976) have developed a technique which could possibly help to overcome the incompleteness of data gathered by plate scanners. In fact, even back then, incomplete data was a routine input condition when solving the travel demand estimation problem.

Carey *et al.* (1981) minimized the difference between observations and model estimates in that regard. Zuylen and Willumsen (1980) and Bell (1983) proposed the methods to solve the problem by maximizing the entropy-based objective function while maintaining consistency between a priori and model estimates, where a priori estimates had been believed to be observed by traffic counters. Cascetta (1984), as well as McNeil and Hendrickson (1985) also contributed to this area of study by proposing several techniques based on the least square estimator. In turn, Spiess (1987) made some refinements to the maximum likelihood model, while Brenninger-Gothe *et al.* (1989) expanded an entropy-based model through multi-objective formulation. Furthermore, Bell (1991) continued investigation of estimation techniques based on least squares, whereas Watling (1994) contributed in techniques based on maximum likelihood estimation.

First bi-level formulations of the OD-matrix estimation problem employing the user equilibrium assignment principle were made by Nguyen (1977) and Fisk (1988). Yang *et al.* (1992) reinforced this approach when clearly demonstrated various alternative bi-level problems with user equilibrium assignment (in lower level), which reflected the travel demand estimation process. Since those bi-level programs which had appeared, were actually NP-hard, Bell *et al.* (1997) proposed a single-level optimization model based on the stochastic user equilibrium concept to estimate path flows and hence, the travel demand. In turn, Bar-Gera (2006) presented a method for estimating the set of routes and their flows under the static user-equilibrium assignment principle. Lundgren and Peterson (2008) offered to cope with complication of the bi-level problem by means of heuristic methodology. Moreover, Shen and Wynter (2012) offered convex approximation of the bi-level program by single-level optimization. The stochastic user equilibrium principle was also employed by Wei and Asakura (2013) in order to avoid bi-level formulation of the travel demand estimation problem.

Statistical model of the transport system with Poisson distributed OD-flows was given by Hazelton (2000) and Hazelton (2001). Gunn (2001) implemented the transferability analysis in the important problems concerning travel demand. First consideration of the travel demand estimation problem as an inverse general traffic assignment problem was made by Bierlaire (2002). The techniques for OD-matrix estimation based on a dynamic change of traffic conditions were offered by Sherali, and Park (2001), Li, and Moor (2002) and Zhou and Mahmassani (2007). Further development of entropy-based approaches and approaches based on the least squares was made by Nie *et al.* (2005), Xie *et al.* (2010) and Xie *et al.* (2011). Parry and Hazelton (2012) contributed in likelihood-based approaches adapted for travel demand estimation. Moreover, in this regard let us also mention the researches made by Cheng *et al.* (2014) and De Grange *et al.* (2017). We also refer to the current works of Yang *et al.*, (2019) and Cantelmo *et al.*, (2019) as these are the most recent researches on the topic.

Models and methods for travel demand estimation also appeared to be fruitful in several areas of study related to traffic flows. Ohazulike *et al.* (2013) showed how employment of OD-data results in social benefits via the mechanism of toll pricing. Wang *et al.* (2018) developed this idea by creating the set of practical extensions and improvements. Moreover, the developed approaches appeared to be useful for geographic distribution of air travel demand. Indeed, Lia and Wanb (2019) implemented the bi-level optimization model for estimation of emerging air travel demand and its geographic distribution in airports.

1.2. Traffic data collection

Generally, the travel demand estimation problem implies that travel demand values are unknown variables to be found through the traffic values observed in a road network. Hence, the way of solving this problem highly depends on technical devices available for traffic observation and collection of traffic data. Quandt and Baumol (1966) were one of the first researchers who pointed out the problems concerning traffic data collection. Nowadays, the traffic data collection issue is clearly understood by researchers as a very important one (e.g. Yang and Fan, 2015, Yang *et al.*, 2018). Thus, let us itemize the main groups of devices which are the most frequently mentioned by researchers with regard to gathering traffic data.

Counters. A traffic counter calculates the precise amount of vehicles crossing it. In other words, a traffic engineer can collect precise traffic data on roads (arcs of graph of a presented road network) with pre-installed counters. The key advantages of traffic counters are their simplicity and cheapness. Moreover, the counters collect information on amount of vehicles and do not collect any personal data on their owners. Many researchers base their investigations of the travel demand estimation problem on traffic counters as a source of initial data (e.g.

Yang *et al.*, 1991; Yang *et al.*, 1992; Yang and Zhou, 1998; Chootinan *et al.*, 2005; Ehlert *et al.*, 2006; Gan *et al.*, 2005; Eisenman *et al.*, 2006; Viti *et al.*, 2014). The auxiliary problem raised here is a search for optimal location of such sensors (e.g. Yang and Zhou, 1998; Hu *et al.*, 2009; Ng 2012; Simonelli *et al.*, 2012; Ng, 2013; Bianco *et al.*, 2014).

Plate scanning sensors. A plate scanning sensor recognizes plate numbers of vehicles crossing it. On the one hand, a traffic engineer can collect the precise traffic data on roads (arcs of graph of a presented road network) with pre-installed plate scanning sensors. On the other hand, a traffic engineer can reconstruct actual routes by sets of roads which were crossed by the scanned vehicles. However, the plate scanning sensors collect both information on amount of vehicles and personal data on their owners which appears to be the key disadvantage of application of such devices. Nevertheless, large cities are already equipped with a big amount of plate scanning sensors which are mostly used for traffic regulations control. Thus, no wonder that many researchers base their investigations of the travel demand estimation problem on plate scanning sensors as a source of initial data (Makowski and Sinha, 1976; Watling, 1994; Castillo *et al.*, 2008; Castillo *et al.*, 2008; Minguez *et al.*, 2010; Li and Ouyang, 2011).

Combining devices. Some researchers combine data collected by both traffic counters and plate scanning sensors when solving the travel demand estimation problem or searching the optimal location of sensors (Medina *et al.*, 2002; Doblas and Benitez, 2005; Rajagopal and Varaiya, 2007; Castillo *et al.*, 2008; Zhou and List, 2010; Parry and Hazelton, 2012; Castillo *et al.*, 2013). No doubt that intelligent tools purposed for day-to-day solving of the travel demand estimation problem shall be able to process both types of initial data.

Online services. Modern services such as Google.Maps or Yandex.Maps provide information on traffic congestions online. Actually, due to such services it is possible to find the average speed of vehicles on any arc of a road network in the online mode. To the best of our knowledge, Krylatov, Shirokolobova, and Zakharov (2016) were first to reveal the application of online services as sources of initial data when solving the travel demand estimation problem.

1.3. Contribution of the present paper

In the present paper we investigate the travel demand estimation problem in a form of the bi-level optimization program which is directly inverse to equilibrium traffic assignment search. Important properties of obtained bi-level program by virtue of mathematical and computational analysis are revealed. First of all, we prove that the travel demand estimation problem has the unique solution if the origin-destination locations are known. Secondly, we show that if actual origin-destination pairs are unknown, then numerous travel demand patterns are able to approximate the traffic observed at the same level of accuracy. Therefore, the data on location of the travel demand pairs is appeared to be sufficient a priori information required in order to estimate travel demand in an urban area uniquely. Obtained findings allow traffic managers to avoid generation of a priori information about trips when searching travel demand but to concentrate only on recognition of actual origin-destination locations. Eventually, let us emphasize that the result of the traffic assignment search can be obtained both in link-based form or path-based form so, consequently, the inverse problem is well compatible with all the available sources of initial data such as traffic counters, plate scanning sensors or online services.

2. Travel Demand Values and Traffic Flow Assignment

Travel demand estimation and traffic assignment search are the tasks which are highly interrelated. Indeed, the solution variables of the travel demand estimation problem are natural input data for traffic assignment search, while the solution variables of the traffic assignment problem are natural input data for travel demand estimation. Moreover, one of the most popular approaches to cope with travel demand estimation is based on such a bi-level optimization program that exploits the traffic assignment problem at the lower level when it approximates the travel demand values by the traffic flows observed and by the a priori information on trips (see Yang *et al.*, 1992). Let us consider this type of approaches carefully.

2.1. Estimation of Travel Demand Values

Generally, the bi-level optimization program for travel demand estimation has the following form:

$$\min_{F}\left[Z_1(F,\bar{F})+Z_2(x,\bar{x})\right] \tag{1}$$

subject to

$$\Phi : F \to x, \tag{2}$$

Where F and x reflect the desired travel demand and corresponding traffic flows, while $\bar{F}$ and $\bar{x}$ are pre-given a priori travel demand and actually observed traffic flows respectively. Moreover, a flow assignment x is believed to be an image of the desired F by virtue of some given mapping Φ. Function Z_1 represents a measure of distance between a priori and desired travel demand, and function Z_2 represents a measure of distance between the observed traffic flows and the traffic flows corresponding to desired travel demand. Thus, when solving the bi-level program (1)–(2) one seeks to minimize the distance between *a priori* travel demand and desired travel demand, as well as the distance between observed traffic flows and traffic flows corresponding to the desired travel demand.

Let us consider a road network presented by a directed graph $G = (V, E)$ with the set of origin-destination pairs $W \subseteq V \times V$. For the directed graph any $w \in W$ can be associated with a non-zero travel demand value F^w and, hence, the travel demand F can be expressed by the vector of all non-zero travel demand values, i.e. $F = (\dots, F^w, \dots)^T$. Moreover, any arc $e \in E$ can be associated with the traffic flow x_e and so the traffic flows x can be expressed by the vector of all non-negative traffic flows, i.e. $x = (\dots, x_e, \dots)^T$. The traffic flows are believed to be depended on the travel demand by virtue of some mapping (2). For instance, in case of a non-congested network the flow does not affect the arc travel time and Φ is appeared to be such a linear function that $x = \mathrm{P}F$ with a fixed matrix P. However, in real road networks the influence of flows on the travel time seems to be highly important and shall be taken into account. Thus, the relation between x and F can be so sophisticated that in order to estimate the travel demand by virtue of the observed traffic flows $\bar{x}$, one need to exploit the vector of the pre-specified travel demand values $\bar{F}$. One of the most popular explicit forms of the bi-level program (1)–(2) is

$$\min_F \left[(\bar{F}-F)^{\mathrm{T}} \boldsymbol{Q}^{-1} (\bar{F}-F) + (\bar{x}-x)^{\mathrm{T}} \boldsymbol{U}^{-1} (\bar{F}-F) \right], \tag{3}$$

with the lower level of the following type:

$$\min_x \phi(x), \tag{4}$$

subject to

$$\sum_{e\in E_v^-} x_e^w - \sum_{e\in E_v^+} x_e^w = \begin{cases} F^w, & \text{if } v \text{ is origin of pair } w, \\ -F^w, & \text{if } v \text{ is destination of pair } w, \quad \forall v \in V, w \in W, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

$$x_e^w \ge 0 \quad \forall e \in E, w \in W, \tag{6}$$

and

$$x_e \sum_{w\in W} x_e^w \quad \forall e \in E, \tag{7}$$

where $\boldsymbol{Q}, \boldsymbol{U}$ are weight matrices, $E_v^- \subseteq E$ is a set of output edges, while $E_v^+ \subseteq E$ is a set of input edges for the node $v \in V$. Here, the condition (5) is a typical flow conservation condition, (6) requires the flows to be non-negative, while the goal function can be represented by the function of special type.

2.2. Traffic Assignment Search

The most frequently used lower-level optimization program for traffic assignment search has the following form:

$$\min_x \sum_{e\in E} \int_0^{x_e} t_e(u)\,du, \tag{8}$$

subject to

$$\sum_{r\in R^w} f_r^w = F^w \quad \forall w\in W, \tag{9}$$

$$f_r^w \geq 0 \quad \forall r\in R^w, w\in W, \tag{10}$$

with definitional constraints

$$x_e = \sum_{w\in W}\sum_{r\in R^w} f_r^w \delta_{e,r}^w \quad \forall e\in E, \tag{11}$$

where $t_e(x_e)$ is a smooth increasing function that models the travel time on the arc $e\in E$ depending on the traffic flow x_e, R^w determines the set of feasible routes between the origin-destination pair $w\in W$, f_r^w is introduced as the traffic flow through the route $r\in R^w$ between the OD-pair $w\in W$. Thus, the flow conservation condition (5) is replaced by the demand allocation rule (9) and non-negativity of the traffic flows is guaranteed by (10) and (11) since $\delta_{e,r}^w$ is such that

$$\delta_{e,r}^w = \begin{cases} 1, & \text{if the arc } e \text{ belongs to the route } r, \\ 0, & \text{otherwise,} \end{cases} \quad \forall e\in E, \forall r\in R, \forall w\in W.$$

The traffic assignment x^* obtained as a solution of the optimization problem (8)–(11) is called *user equilibrium* and proved to satisfy the following condition

$$\sum_{e\in E} t_e(x_e^*)\cdot\delta_{e,r}^w = \begin{cases} = t^w, & \text{if } f_r^{w*} > 0, \\ \geq t^w, & \text{if } f_r^{w*} = 0, \end{cases} \quad \forall r\in R, \forall w\in W, \tag{12}$$

subject to

$$x_e^* = \sum_{w\in W}\sum_{r\in R^w} f_r^{w*}\delta_{e,r}^w, \quad \forall e\in E,$$

where $t^w > 0$ is *equilibrium travel time* through any actually used route between the OD-pair $w\in W$.

Let us mention that (12) actually reflects the *user equilibrium principle* which was for the first time formulated by Wardrop (1952) as follows "the journey times in all actually used routes are equal and less than those that would be experienced by a single vehicle on any unused route". The first mathematical formulation of this behavioral principle was given by Beckman, McGuire, and Winsten (1956) and it appeared to be very fruitful for modeling route choices under traffic congestions (e.g. Sheffi, 1985; Patriksson, 1994; Krylatov *et al.*, 2020). Indeed, according to the principle of Wardrop, no driver can unilaterally reduce his/her travel costs by shifting to another route. Therefore, competitive equilibrium search seems to be the best way for assessment of selfish behavior of drivers in congested urban road areas. Hence, (8)–(11) helps a traffic engineer to predict congestions on arcs of the road network when the information on travel demand is available. We believe it seems natural to consider the travel demand information search as a directly inverse problem to prediction of congestions.

3. Inverse Traffic Assignment Problem and Travel Demand Estimation

The first consideration of the travel demand estimation problem as the inverse traffic assignment problem was made by Bierlaire (2002). However, he formulated the inverse problem in very general terms where the locations of OD-pairs as well as a matrix of route choice were believed to be given. One of the main purposes within the present paper is to eliminate the excessive use of pre-given and *a priori* information when estimating the travel demand. Indeed, the usage of an *a priori* OD-matrix in the objective function of the problem (3)–(7) leads the travel demand estimation problem to the standard deviation search: in fact, the problem of travel demand estimation is reduced to the problem of iterative updating of the travel demand information for a given set of origin-destination pairs.

We believe the assumption on existence of *a priori* OD-matrix is too strict when considering the travel demand estimation and we formulate this problem in such a way so we could avoid the pre-given or *a priori* information. First of all, let us introduce the set of all feasible travel demand patterns $\Psi = \left\{ F \middle| F^w \geq 0 \quad \forall w \in V \times V \right\}$ for the network G and the set of all feasible traffic assignment patterns $\aleph(F) = \left\{ x \middle| x_e = \sum_{w \in V \times V} \sum_{r \in R^w} f_r^w \delta_{e,r}^w \; \forall e \in E, \sum_{r \in R^w} f_r^w = F^w, f_r^w \geq 0, \forall r \in R^w, \forall w \in V \times V \right\}$ for any given $F \in \Psi$. Secondly, we define the mapping $\Im, \Im : \Psi \to \Re_+^{\mathrm{m}}$, where $\Re_+^{\mathrm{m}}$ is a nonnegative orthant of the vector space of dimension $\mathrm{m}, \mathrm{m} = |E|$, with the following function

$$\Im(F) = \arg \min_{x \in \aleph(F)} \sum_{e \in E} \int_0^{x_e} t_e(u) du.$$

One can see that the mapping $\Im$ establishes the correspondence between all the feasible equilibrium flow assignment patterns and all the feasible travel demand patterns in the network G. Thus, we can consider the *primal generalized* traffic assignment problem for the network G as the search of the image $\mathrm{Im}\Im$ of

$$\Im : \Psi \to \Re_+^{\mathrm{m}}, \quad \Im(F) = \arg \min_{x \in \aleph(F)} \sum_{e \in E} \int_0^{x_e} t_e(u) du, \tag{13}$$

while the *inverse generalized* traffic assignment problem for the transportation network G can be considered as the search of the function $\Xi(x)$ of the mapping

$$\Xi : \mathrm{Im}\Im \to \Psi. \tag{14}$$

The solutions of the inverse generalized traffic assignment problem for the networks of non-interfering routes were already obtained explicitly by Krylatov *et al.* (2016) under linear travel time functions and by Krylatov (2016) under non-linear travel time functions. Despite the fact that generally the function $\Xi(x)$ cannot be obtained explicitly, one can seek to minimize the following deviation

$$\min_{F \in \Psi} \left\| \Im(F) - \bar{x} \right\| \tag{15}$$

subject to

$$\Im(F) = \arg \min_{x \in \aleph(F)} \sum_{e \in E} \int_0^{x_e} t_e(u) du, \tag{16}$$

that is appeared to be the actual *travel demand estimation problem* which avoids usage of *a priori* data.

4. Computational Study of Travel Demand Estimation Problem

Let us study the bi-level optimization problem (15)–(16) from computational perspectives. First of all, we need to find a source of information on average congestions in actual road networks to obtain $\bar{x}$. Nowadays, online map services can be considered as such type of sources since they show traffic jams in real time. Indeed, Google or Yandex provide such GIS services which monitor and provide the data on daily dynamics of city traffic and display it on city maps. For instance, Figure 1 demonstrates traffic congestions in real-time in the Saint Petersburg road network taken from Yandex (https://yandex.com/maps/). Yandex associates the specific color tone with traffic intensity in each arc of a road network: green color means free travel, red means traffic jam. Moreover, each color tone corresponds to a specific speed value: green – the highest speed (60 km/h within city limits), red – the lowest speed (1-5 km/h).

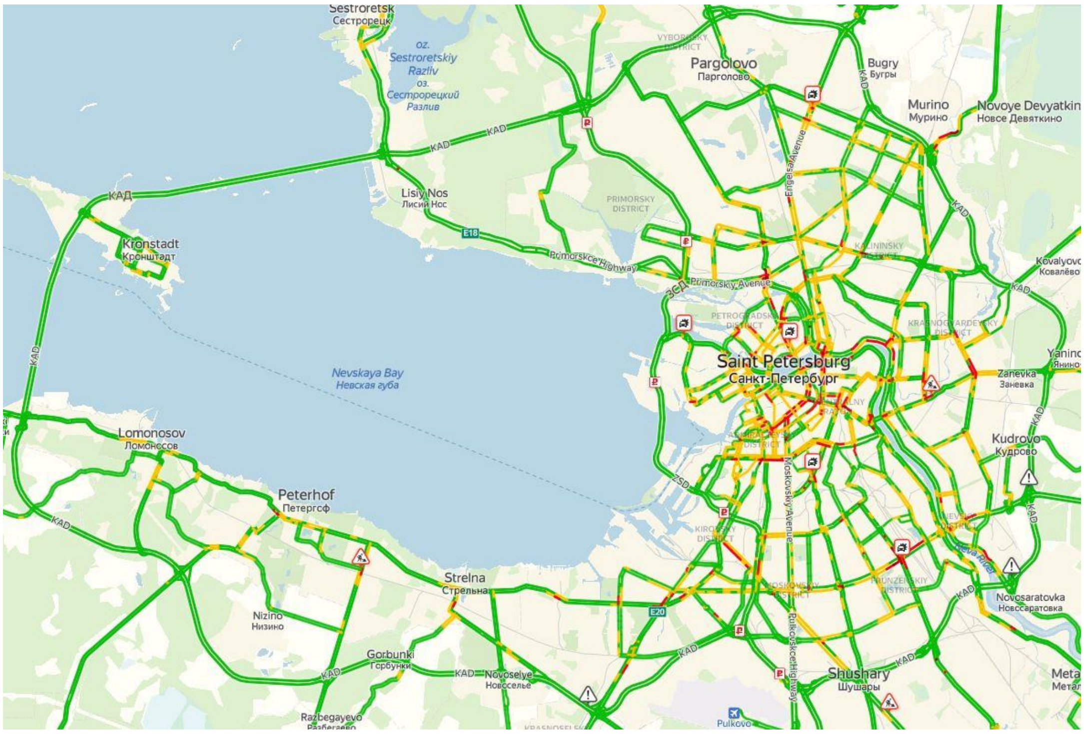

Figure 1. Congestions in the road network of Saint Petersburg in real time

Therefore, a traffic engineer is able to collect data on intensity (speed) in each arc of a road network. Once a traffic engineer gets to know the average travel speed in an arc and the length of this arc, he/she can immediately calculate the average travel time in this arc. Moreover, the travel time function, e.g. the most popular BPR (Bureau of Public Roads) function

$$t_e(x_e) = t_e^0 \left(1 + 0.15 \left(\frac{x_e}{c_e} \right)^4 \right) \quad \forall e \in E \tag{17}$$

with given t_e^0 and c_e as a free travel time and a capacity of the arc respectively, can, thus, allow the traffic engineer to reconstruct the average arc-flows $\bar{x}$ when the travel time $t_e(\bar{x}_e)$ is known for all $e \in E$. For more details on BPR-functions one can refer to Horowitz (1991).

Let us consider toy example of the Saint Petersburg road network presented by 253 nodes and 1012 arcs. Each node is associated with its exact geographical coordinates while each directed arc is characterized by a pair of nodes, its length and capacity. Moreover, for academic and research purposes only we obtained information on average speed in 1012 arcs of toy model of Saint Petersburg road network using the open databases of Yandex and then reconstructed the corresponding information on the average traffic arc-flows $\bar{x}$. All the input data gathered for our computational study is available at http://www.apmath.spbu.ru/ru/sta_/krylatov/_les/CongestedStPetersburg.csv.

We exploited the continuous optimization technique to solve the travel demand estimation problem (15)–(16) for a considered road network. Despite we call this network as toy model, its graph presented by 253 nodes and 1012 arcs is still quite large. We dealt with the problem under 5, 6, 7, 8 and 9 OD-pairs: the minimal achieved values of (15) for all these cases are given in the Table 1. Let us mention that the amount of OD-pairs influences the computational time drastically.

Table 1. Minimal values of the goal function (15) for different amount of OD-pairs

	5 OD-pairs	6 OD-pairs	7 OD-pairs	8 OD-pairs	9 OD-pairs
Deviation	3041.7	3042.2	3045.3	3047.1	3052.8

Continuous optimization implies that small changes of the variables' values lead to a small change of the goal function value. Therefore, our computational study (Table 1) demonstrates that the amount of OD-pairs does not influence directly on the value of the goal function (15) when solving the travel demand estimation problem. In other words, the descent direction of the problem (15)–(16) does not depend directly on increasing or decreasing of the amount of OD-pairs. Moreover, Figure 2 demonstrates four different solutions for 5 OD-pairs. As one can see, these solutions lead to very close values of the goal function (15) subject to highly diverse locations of OD-pairs. Thus, a solution of (15)–(16) is appeared to be totally depended on locations of OD-pairs at the initial iteration of an optimization algorithm.

The computational study we have conducted allows us to draw the two following conclusions which are important. The first one states that, in fact, the travel demand estimation problem (15)–(16) possesses two independent types of unknown variables such as locations of OD-pairs and travel demand values. Indeed, decreasing of the goal function can be achieved by changing of the travel demand values as well as by relocation of the OD-pairs. Moreover, both of these types of variables influence on the goal function equally critically. The second conclusion states that generally the travel demand estimation problem does not have a unique solution – different travel demand patterns can lead to close deviation values. Further mathematical investigation of these issues can reveal additional rigorous properties on the solution existence and its uniqueness for the travel demand estimation problem.

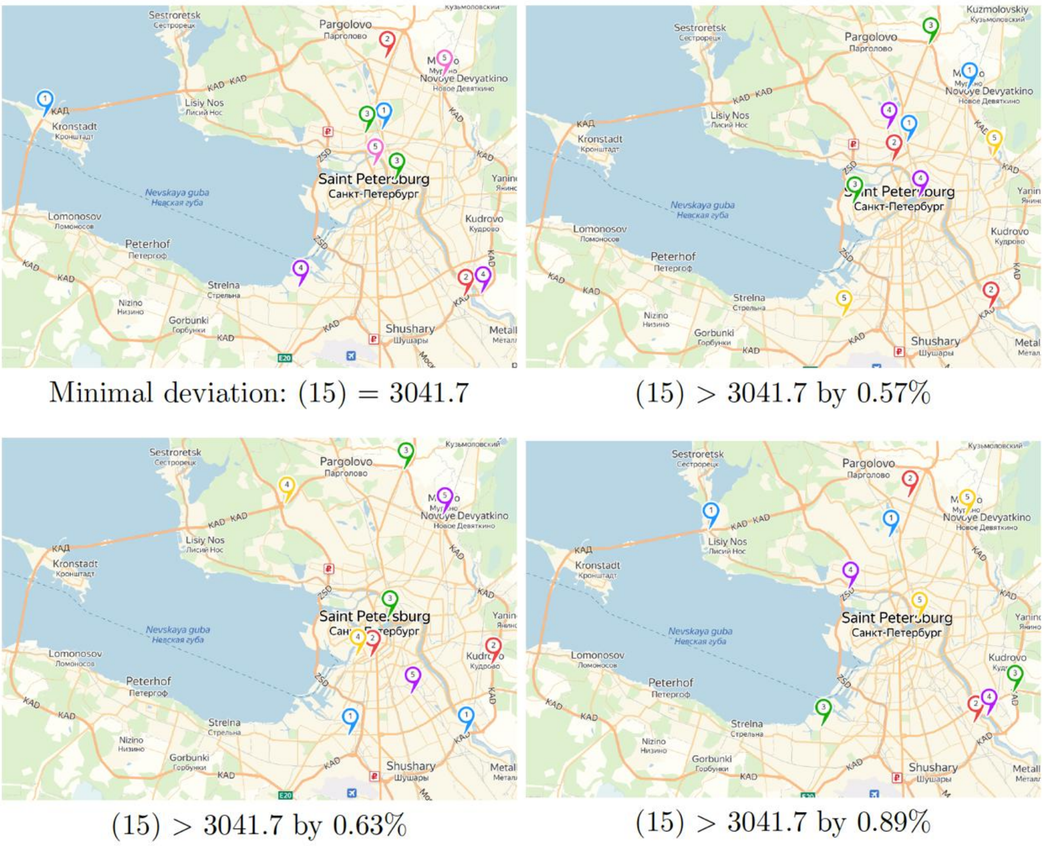

Figure 2. Four travel demand patterns with close goal function values under

5. Mathematical Study of Travel Demand Estimation as Inverse Equilibrium Traffic Assignment Problem

The previous section revealed a set of difficulties which appear when one seeks to solve the travel demand estimation problem directly by computational techniques. Within this section let us investigate the properties of the mappings $\Im$ and Ξ from mathematical perspectives in order to handle the revealed difficulties.

Lemma 5.1. *If the graph G is strongly connected, the functions $t_e(x_e)$ are continuous and $t_e(x_e) > 0$ for $x_e \geq 0, e \in E$, then the mapping Ξ is a surjection.*

Proof. Let us prove this Lemma by contradiction. Let us suppose that there is an element $\tilde{F} \in \Psi$, that is not an image of all $x \in \operatorname{Im}\Im$. Hence, according to the Theorem 2.4 from Patriksson (1994), if the graph G is strongly connected, and the functions $t_e(x_e)$ are continuous, and $t_e(x_e) > 0$ *for* $x_e \geq 0, e \in E$, then the solution of the problem (8)–(11) exists. In other words

$$\exists\, \tilde{x} \in \Re_+^{\mathrm{m}} \quad : \quad \tilde{x} = \Im(\tilde{F}) \quad \Rightarrow \quad \tilde{x} \in \operatorname{Im}\Im.$$

Thus, we come to a contradiction, and therefore the mapping Ξ is a surjection.

Remark 1. The surjectivity of the mapping Ξ guarantees the existence of a solution of the inverse problem to the traffic assignment problem. However, it does not guarantee the uniqueness.

Let us consider a transportation network presented by a directed graph with 4 nodes and 4 arcs (Fig. 3). According to Lemma 5.1, a solution to the problem inverse to the traffic assignment problem does exist. We suppose that there is a traffic flow on each arc, which is equal to 10. It is clear that there are many feasible solutions for the travel demand estimation problem: for example, $F^{(1,3)} = 10$ and $F^{(3,1)} = 10$ or $F^{(2,4)} = 10$ and $F^{(4,2)} = 10$ (Fig. 3).

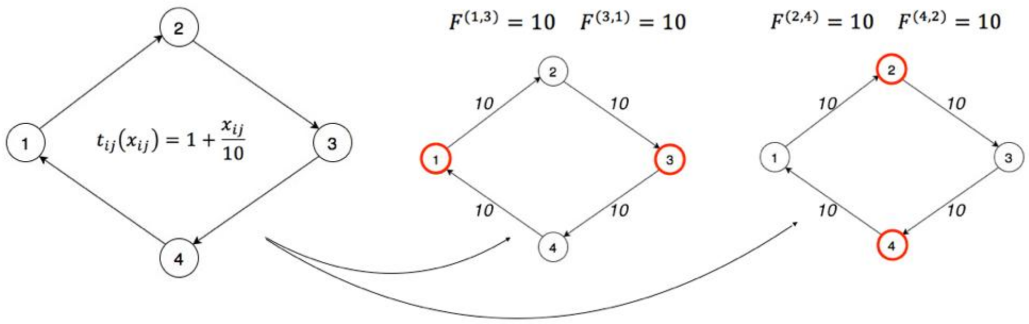

Figure 3. Transportation network of 4 nodes

Thus, one can see that generally the travel demand estimation problem has multiple feasible solutions. So, the mapping Ξ is neither injective, nor a fortiori bijective.

Theorem 5.2. *Let the graph G be strongly connected, the functions $t_e(x_e)$ be smooth and strictly increasing, $t_e(x_e) > 0$ for $x_e \geq 0, e \in E$, and $W \subseteq V \times V$ is a given set. If $\Psi = \left\{ F \middle| F^w > 0 \quad \forall w \in W, \quad F^w = 0 \quad \forall w \in V \times V \setminus W \right\}$, then the mapping Ξ is a bijection.*

Proof. The conditions specified in the theorem satisfy the Lemma 5.1, and the mapping Ξ is a surjection. So, we just need prove the injectivity of the mapping Ξ.

For the reasons of §convenience, let us represent the relations (9) in a matrix form:

$$\mathbf{F} = Af, \quad \text{where} \quad A = \begin{pmatrix} A_1 & \cdots & \mathrm{O} \\ \vdots & \ddots & \vdots \\ \mathrm{O} & \cdots & A_n \end{pmatrix}_{\mathrm{n}\times\mathrm{r}}, \tag{18}$$

where $|W| = \mathrm{n}$, $|R| = \left|\bigcup_{w\in W} R^w\right| = \mathrm{r}$, $A_w = (1,\ldots,1)_{1\times|R^w|}$ for all $w = \overline{1,\mathrm{n}}$, and $\mathbf{F} = (F^1,\ldots,F^{\mathrm{n}})^{\mathrm{T}}$, and $f = f_{\mathrm{r}\times 1}$. In turn, the relations (11) in a matrix form are

$$x = \Delta f, \quad \text{where} \quad \Delta = \begin{pmatrix} \delta^1_{1,1} & \cdots & \delta^n_{1,r} \\ \vdots & \ddots & \vdots \\ \delta^1_{m,1} & \cdots & \delta^n_{m,r} \end{pmatrix}_{m\times r}, \tag{19}$$

and $x = x_{m\times 1}$ while $f = \left(f^1,\ldots,f^n\right)$, where $f^w = \left(f_1^w,\ldots,f^w_{|R^w|}\right)_{1\times|R^w|}$ for all $w \in W$.

Since the set W is given, the primal generalized problem (13) is reduced to the traffic assignment problem (8)–(11) on the graph G . According to the Theorem 2.5 from Patriksson (1994), if the graph G is strongly connected, and the functions $t_e(x_e)$ are smooth and strictly increasing, $t_e(x_e) > 0$ for $x_e > 0, e \in E$, then the traffic assignment problem (8)–(11) has a unique solution. In other words, under given conditions, the image of $\Im$ is a single element of the set $R^m_+ : \operatorname{Im}\Im = x^*, x^* \in R^m_+$. Therefore, we just have to show that only one $F \in \Psi$ maps into $\operatorname{Im}\Im = x^*, x^* \in R^m_+$. We prove it by a contradiction.

Let us suppose that there are $F_1 \in \Psi$ and $F_2 \in \Psi$ such that $x^* = \Im(F_1) = \Im(F_2)$. We represent the matrices F_1 and F_2 as vectors $\mathbf{F}_1$ and $\mathbf{F}_2$. Hence, according to (18) and (19), the following equalities hold:

$$\begin{pmatrix} A \\ \Delta \end{pmatrix} f_1 = \begin{pmatrix} \mathbf{F}_1 \\ x^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A \\ \Delta \end{pmatrix} f_2 = \begin{pmatrix} \mathbf{F}_2 \\ x^* \end{pmatrix}.$$

Since x^* is a user equilibrium assignment for F_1 and F_2 , both systems are compatible, i.e. there exist two distinct non-zero vectors f_1 and f_2 which are the solutions of these systems of linear equations. Subtracting one system from another, we obtain the following expression:

$$\begin{pmatrix} A \\ \Delta \end{pmatrix} \left(f_1 - f_2\right) = \begin{pmatrix} \mathbf{F}_1 - \mathbf{F}_2 \\ x^* \end{pmatrix}.$$

Thus, there exists a non-zero route-flow assignment $f = \left(f_1 - f_2\right)$ which corresponds to the zero link-flow assignment $x = x^* - x^*$. Consequently, we come to a contradiction.

Remark 2. The bijectivity of the mapping Ξ guarantees the uniqueness of the solution of the inverse equilibrium traffic assignment problem. Therefore, if the set of OD-pairs is given, then the travel demand estimation problem has a unique solution.

6. Discussion

The paper is devoted to the travel demand estimation problem. This problem is formulated as an inverse problem to traffic assignment search. Two important points are revealed:

(1) generally speaking, the inverse equilibrium traffic assignment problem has no unique solutions;
(2) if the location the OD-pairs is known, then the inverse equilibrium traffic assignment problem has the unique solution.

Hence, once the travel demand locations are specified, then the obstacles concerning OD-pairs search are disappeared. Otherwise, the upper-level goal function demonstrates fuzzy behavior and continuous optimization cannot cope with the problem. Thus, the present paper contains the following set of theoretical results:

(1) a useful class of inverse equilibrium traffic assignment problems is introduced;
(2) the existence and uniqueness conditions of the inverse equilibrium traffic assignment problem are found;
(3) a bi-level model for travel demand estimation, which avoids a priori travel demand estimates, is given.

Therefore, the results of the present paper reveal that the actual problem to be solved when dealing with travel demand estimation is the problem of recognition of the travel demand locations. The obtained findings contribute in the theory of travel demand estimation and give fresh managerial insights for traffic engineers.

7. Conclusion

In the present paper we investigated the travel demand estimation problem in a form of a bi-level optimization program which is directly inverse to equilibrium traffic assignment search. The important properties of the bi-level program obtained by virtue of mathematical and computational analyses are revealed. First of all, we proved that the travel demand estimation problem has a unique solution if the origin-destination locations are known. Secondly, we showed that if actual origin-destination pairs are unknown then the numerous travel demand patterns are able to approximate the observed traffic at the same level of accuracy. Therefore, the location of travel demand pairs is appeared to be sufficient *a priori* information required in order to estimate the travel demand in an urban area uniquely. The findings obtained allow traffic managers to avoid generation of *a priori* information about trips when searching the travel demand but to concentrate only on recognition of actual origin-destination locations.

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