DISTRIBUION OF LEADING DIGITS OF NUMBERS II

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ABSTRACT. In this paper, we study the sequence $(f(p_n))_{n \geq 1}$, where p_n is the n th prime number and f is a function of a class of slowly increasing functions including $f(x) = \log_b x^r$ and $f(x) = \log_b(x \log x)^r$, where $b \geq 2$ is an integer and $r > 0$ is a real number. We give upper bounds of the discrepancy $D_{N_i}^*(f(p_n), g)$ for a distribution function g and a sub-sequence $(N_i)_{i \geq 1}$ of the natural numbers. Especially for $f(x) = \log_b x^r$, we obtain the effective results for an upper bound of $D_{N_i}^*(f(p_n), g)$.

Communicated by Werner Georg Nowak

1. Introduction

In [8], we gave some results about the first digit problem in base $b \geq 2$ for the sequence $(n^r)_{n \geq 1}$ and for the sequence $(p_n^r)_{n \geq 1}$, where p_n is the n th prime number and $r > 0$ is a real number. In this paper, we generalize and sharpen them.

Throughout this paper, for $x \in \mathbb{R}$, let

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}, \quad \{x\} = x - \lfloor x \rfloor,$$

let c_E denote the characteristic function of the set $E \subset \mathbb{R}$, and let $\pi(x)$ be the number of primes not exceeding x .

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2010 Mathematics Subject Classification: 11K06, 11K31, 11K38.

Key words: prime numbers, Benford's law, distribution function, discrepancy, regularly varying function.

²Supported by Vega Project 2/0109/18.

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Let $(y_n)_{n \geq 1}$ be a sequence of real numbers. If there exists a strictly increasing sequence of positive integers $(N_i)_{i \geq 1}$ such that

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c_{[0,x)}(\{y_n\}) = g(x) \quad \text{for every } x \in [0, 1],$$

then g is called a distribution functions (d.f.) mod 1 of $(y_n)_{n \geq 1}$. Let $G(y_n \bmod 1)$ be the set of all d.f.'s of $(y_n)_{n \geq 1}$ (see [12]). If $G(y_n \bmod 1) = \{g(x) = x\}$, then the sequence $(y_n)_{n \geq 1}$ is said to be uniformly distributed mod 1 (abbreviating u.d.mod 1).

Suppose that g is a non-decreasing function on $[0, 1]$ with $g(0) = 0$, $g(1) = 1$. Then the extremal discrepancy $D_N(y_n, g)$ and the star discrepancy $D_N^*(y_n, g)$ of $(y_n)_{n \geq 1}$ with respect to g are defined by

$$D_N(y_n, g) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{N} \sum_{n=1}^N c_{[\alpha, \beta)}(\{y_n\}) - (g(\beta) - g(\alpha)) \right|$$

and

$$D_N^*(y_n, g) = \sup_{x \in [0, 1]} \left| \frac{1}{N} \sum_{n=1}^N c_{[0, x)}(\{y_n\}) - g(x) \right|,$$

respectively (see [12, 1.10.1]). If $g(x) = x$ for $0 \leq x \leq 1$, then g is omitted in $D_N^*(y_n, g)$ or $D_N(y_n, g)$.

Let $b \geq 2$ be an integer considered as a base for the development of a real number $x > 0$ and $M_b(x)$ be the mantissa of x defined by $x = M_b(x) \times b^{n(x)}$ such that $1 \leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $D = d_1 d_2 \cdots d_s$ be a positive integer expressed in the base b , that is

$$D = d_1 b^{s-1} + d_2 b^{s-2} + \cdots + d_{s-1} b + d_s,$$

where $d_1 \neq 0$ and at the same time $D = d_1 d_2 \cdots d_s$ is considered as an s -consecutive block of digits in the base b . Note that for x of the type $x = 0.00 \cdots 0 d_1 d_2 \cdots d_s \cdots$, $d_1 > 0$, we have $M_b(x) = d_1 d_2 \cdots d_s \cdots$ and the first zero digits are omitted. Thus a positive real number x has the first s -digits, starting a non-zero digit, that equal to $D = d_1 d_2 \cdots d_s$ if and only if

$$\frac{D}{b^{s-1}} = d_1 d_2 \cdots d_s \leq M_b(x) < d_1 d_2 \cdots (d_s + 1) = \frac{D + 1}{b^{s-1}}. \quad (1)$$

Let (x_n) be a sequence of positive number, and let $F(D, s, N, x_n)$ be the number of integers $1 \leq n \leq N$ such that the leading block of s digits (beginning with $\neq 0$) of x_n equals D . The sequence $(x_n)_{n \geq 1}$ is said to satisfy *Benford's law* (abbreviated by B.L.) in base b , if for every $s = 1, 2, \dots$ and every s -digits integer $D = d_1 d_2 \cdots d_s$

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$$\lim_{N \rightarrow \infty} \frac{F(D, s, N, x_n)}{N} = \log_b \left(\frac{D+1}{b^{s-1}} \right) - \log_b \left(\frac{D}{b^{s-1}} \right)$$

(see [3]). Since $\log_b M_b(x) = \{\log_b x\}$ by (1), the leading block of s digits (beginning with $\neq 0$) of x_n equals D if and only if

$$\log_b \left(\frac{D}{b^{s-1}} \right) \leq \{\log_b x_n\} < \log_b \left(\frac{D+1}{b^{s-1}} \right).$$

Hence, the sequence $(x_n)_{n \geq 1}$ satisfies Benford's law if and only if $(\log_b x_n)_{n \geq 1}$ is uniformly distributed mod 1 (see [3], [9]).

If $g \in G(\log_b x_n \bmod 1)$ and

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c_{[0,x)}(\{\log_b x_n\}) = g(x)$$

for $x \in [0, 1]$, then for every s -digits integer $D = d_1 d_2 \cdots d_s$, $s = 1, 2, \dots$,

$$\lim_{i \rightarrow \infty} \frac{F(D, s, N_i, x_n)}{N_i} = g \left(\log_b \left(\frac{D+1}{b^{s-1}} \right) \right) - g \left(\log_b \left(\frac{D}{b^{s-1}} \right) \right)$$

(see [9], [8]).

For $r > 0$, the sequence $(n^r)_{n \geq 1}$ and $(p_n^r)_{n \geq 1}$ do not satisfy Benford's law, that is, the sequence $(\log_b n^r)_{n \geq 1}$ and $(\log_b p_n^r)_{n \geq 1}$ are not u.d. mod 1. Eliahou, Massé, and Schneider [4] showed that for a positive integer r , the discrepancy of the sequence $(y_n) = (\log_{10} n^r)_{n \geq 1}$, $(\log_{10}(n \log n)^r)_{n \geq 1}$, or $(\log_{10} p_n^r)_{n \geq 1}$ satisfies $D_{\phi(r)}(y_n) = O(r^{-1})$ with $\phi(r) = \lfloor e^r \rfloor$. Thus, when $r \rightarrow \infty$, these sequences satisfy Benford's law in a sense.

It is also known that

$$G(\log_b n^r \bmod 1) = G(\log_b p_n^r \bmod 1) = \{g_w(x); 0 \leq w \leq 1\},$$

where

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \leq x \leq 1),$$

and if $\lim_{i \rightarrow \infty} \{\log_b p_{N_i}^r\} = w$, then for $x \in [0, 1]$

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c_{[0,x)}(\{\log_b p_n^r\}) = g_w(x)$$

(see [13], [6], [12, 2.12.1], and [7]), and so

$$\lim_{i \rightarrow \infty} \frac{F(D, s, N_i, p_n^r)}{N_i} = g_w \left(\log_b \left(\frac{D+1}{b^{s-1}} \right) \right) - g_w \left(\log_b \left(\frac{D}{b^{s-1}} \right) \right). \quad (2)$$

As examples of N_i and w , in [8] we gave the following:

- if $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$, then $\lim_{i \rightarrow \infty} \{\log_b p_{N_i}^r\} = w$,
- if $N_i = \pi(b^{\frac{i}{r}})$, then $\lim_{i \rightarrow \infty} \{\log_b p_{N_i}^r\} = 1$.

In [8], for rate of convergence of (2) we also proved that

$$\left| \frac{F(D, s, N_i, p_n^r)}{N_i} - \frac{(\frac{D+1}{b^{s-1}})^{\frac{1}{r}} - (\frac{D}{b^{s-1}})^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1} \right| \leq C_{b,r} \left(\frac{1}{\log p_{N_i}} \right), \quad (3)$$

for $N_i = \pi(b^{\frac{i}{r}})$, $i = 1, 2, \dots$ and a constant $C_{b,r} > 0$ depending on b and r .

In this paper, to refine (3), we consider a generalized sequence $(f(p_n))_{n \geq 1}$, where f is a function of a class of slowly increasing functions including

$$f(x) = \log_b x^r \quad \text{and} \quad f(x) = \log_b(x \log x)^r.$$

Since

$$\left| \frac{F(D, s, N, x_n)}{N} - \left(g\left(\log_b \frac{D+1}{b^{s-1}}\right) - g\left(\log_b \frac{D}{b^{s-1}}\right) \right) \right| \leq 2D_N^*(\log_b x_n, g),$$

an upper bound of $D_{N_i}^*(\log_b x_n, g)$ is the rate of convergence of $\frac{F(D, s, N_i, x_n)}{N_i}$. Especially, we obtain an effective result for an upper bound of $D_N^*(\log_b p_n^r, g)$.

2. Preliminaries

To characterize the class of slowly increasing functions, we need some definitions and results concerning regular variation.

DEFINITION 1 ([2], p.18). A function f is said to be regularly varying of index ρ if it is real-valued, positive and measurable on $[t_0, \infty)$ for some $t_0 > 0$, and if

$$\lim_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad (\forall \lambda > 0)$$

for some real number ρ . Then we write $f \in R_\rho$. If $f \in R_0$, then f is said to be slowly varying.

LEMMA 1 ([2], Theorem 1.3.1). *The function ℓ is slowly varying if and only if it may be written in the form*

$$\ell(x) = c(x) \exp \left\{ \int_a^x \varepsilon(u) du / u \right\} \quad (x \geq a)$$

for some $a > 0$, where $c(x)$ is measurable and $c(x) \rightarrow c \in (0, \infty)$, $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$.

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LEMMA 2. *If a function f is positive, strictly increasing, differentiable on $[t_0, \infty)$ for some $t_0 > 0$, and satisfies*

$$\lim_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} = \rho \neq 0,$$

then $f \in R_\rho$ and $f^{-1} \in R_{1/\rho}$.

P r o o f. Let $\ell(t) = t^{-\rho}f(t)$ for $t \geq t_0$. Then we have

$$\varepsilon(t) := \frac{t\ell'(t)}{\ell(t)} = -\rho + \frac{tf'(t)}{f(t)} \rightarrow 0 \quad (t \rightarrow \infty). \quad (4)$$

Since

$$\int_{t_0}^t \frac{\varepsilon(u)}{u} du = \int_{t_0}^t \frac{\ell'(u)}{\ell(u)} du = \int_{t_0}^t (\log \ell(u))' du = \log \ell(t) - \log \ell(t_0),$$

we have

$$\ell(t) = \ell(t_0) \exp \left(\int_{t_0}^t \frac{\varepsilon(u)}{u} du \right).$$

From Lemma 1, it follows that $\ell(t)$ is slowly varying. Hence $f(t) = t^\rho \ell(t)$ is regularly varying of index ρ , i.e., $f \in R_\rho$.

Let $h(t) = \log f(e^t)$. Then

$$h'(t) = \frac{f'(e^t)e^t}{f(e^t)} = \frac{\{\rho e^{(\rho-1)t}\ell(e^t) + e^{\rho t}\ell'(e^t)\}e^t}{e^{\rho t}\ell(e^t)} = \rho + \frac{e^t\ell'(e^t)}{\ell(e^t)} = \rho + \varepsilon(e^t).$$

Therefore by (4) we have

$$h'(t) \rightarrow \rho \quad (t \rightarrow \infty).$$

Hence

$$(h^{-1})'(t) = \frac{1}{h'(h^{-1}(t))} \rightarrow \frac{1}{\rho} \quad (t \rightarrow \infty). \quad (5)$$

Since $h^{-1}(t) = \log f^{-1}(e^t)$, we have

$$f^{-1}(t) = \exp(h^{-1}(\log t)).$$

Hence

$$\begin{aligned} \frac{f^{-1}(\lambda t)}{f^{-1}(t)} &= \frac{\exp(h^{-1}(\log \lambda + \log t))}{\exp(h^{-1}(\log t))} \\ &= \exp\{h^{-1}(\log \lambda + \log t) - h^{-1}(\log t)\} \\ &= \exp\{(\log \lambda)(h^{-1})'(\xi_t)\}, \end{aligned}$$

where $\log t \leq \xi_t \leq \log t + \log \lambda$. By this and (5) we have

$$\frac{f^{-1}(\lambda t)}{f^{-1}(t)} \rightarrow \exp\left(\frac{1}{\rho} \log \lambda\right) = \lambda^{1/\rho} \quad (t \rightarrow \infty).$$

Thus $f^{-1} \in R_{1/\rho}$. □

LEMMA 3 (Uniform Convergence Theorem: [2], [11]). *If $f \in R_\rho$, then for every $[a, b]$ with $0 < a < b < \infty$*

$$\lim_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{uniformly in } \lambda \in [a, b].$$

LEMMA 4. *If a real-valued function f is strictly increasing and differentiable on $[t_0, \infty)$ for some $t_0 > 0$, and satisfies $\lim_{t \rightarrow \infty} t f'(t) = \rho > 0$, then for $-\infty < a < b < \infty$*

$$\sup_{x \in [a, b]} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \rightarrow 0 \quad (t \rightarrow \infty).$$

Proof. Let $g(t) = \exp(f(t))$. From the assumption,

$$\frac{t g'(t)}{g(t)} = t f'(t) \rightarrow \rho \quad (t \rightarrow \infty).$$

By Lemma 2 we have $g^{-1} \in R_{1/\rho}$. Since $g^{-1}(y) = f^{-1}(\log y)$, Lemma 3 implies for $[a', b']$ with $0 < a' < b' < \infty$

$$\lim_{y \rightarrow \infty} \frac{f^{-1}(\log(\lambda y))}{f^{-1}(\log y)} = \lambda^{1/\rho} \quad \text{uniformly in } \lambda \in [a', b'],$$

and so

$$\lim_{t \rightarrow \infty} \frac{f^{-1}(t+x)}{f^{-1}(t)} = e^{x/\rho} \quad (t \rightarrow \infty)$$

uniformly for $x \in [a, b]$ ($-\infty < a < b < \infty$), that is,

$$\sup_{x \in [a, b]} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \rightarrow 0 \quad (t \rightarrow \infty). \quad \square$$

LEMMA 5. *If a real-valued function f is strictly increasing, differentiable on $[t_0, \infty)$, and satisfies $\lim_{t \rightarrow \infty} t f'(t) = \rho > 0$, then for $-\infty < a < b < \infty$*

$$\sup_{x \in [a, b]} \left| \frac{B(t+x)}{B(t)} - e^{x/\rho} \right| \rightarrow 0 \quad (t \rightarrow \infty),$$

where $B(t) = \frac{f^{-1}(t)}{\log f^{-1}(t)-1}$.

Proof. Since $\lim_{t \rightarrow \infty} \frac{f(t)}{\log t} = \lim_{t \rightarrow \infty} t f'(t) = \rho$, we have $\lim_{t \rightarrow \infty} f(t) = \infty$, and so $\lim_{t \rightarrow \infty} f^{-1}(t) = \infty$. By Lemma 4 we have

$$\left| \frac{\log f^{-1}(t) - 1}{\log f^{-1}(t+x) - 1} - 1 \right| = \frac{\log \frac{f^{-1}(t+x)}{f^{-1}(t)}}{\log f^{-1}(t+x) - 1} \leq \frac{\log \frac{f^{-1}(t+1)}{f^{-1}(t)}}{\log f^{-1}(t) - 1} \rightarrow 0 \quad (t \rightarrow \infty)$$

for $t \geq 0$ and $0 \leq x \leq 1$.

Therefore,

$$\begin{aligned}
 \sup_{x \in [a, b]} \left| \frac{B(t+x)}{B(t)} - e^{x/\rho} \right| &= \sup_{0 \leq x \leq 1} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} \cdot \frac{\log f^{-1}(t) - 1}{\log f^{-1}(t+x) - 1} - e^{x/\rho} \right| \\
 &\leq \sup_{0 \leq x \leq 1} \left(\frac{f^{-1}(t+x)}{f^{-1}(t)} \left| \frac{\log f^{-1}(t) - 1}{\log f^{-1}(t+x) - 1} - 1 \right| \right) + \sup_{0 \leq x \leq 1} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \\
 &\leq \frac{f^{-1}(t+1)}{f^{-1}(t)} \sup_{0 \leq x \leq 1} \left| \frac{\log f^{-1}(t) - 1}{\log f^{-1}(t+x) - 1} - 1 \right| + \sup_{0 \leq x \leq 1} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \\
 &\rightarrow 0 \quad (t \rightarrow \infty). \quad \square
 \end{aligned}$$

3. Estimate of discrepancy of $(f(p_n))$

In [8], we obtained an upper bound of $D_N^*(\log_b(p_n^r), g_w)$ for $w \in [0, 1]$. In slightly different fashion from [8], we have the following estimate of $D_N^*(f(p_n), g_w)$ for $w \in [0, 1]$, where f is a strictly increasing continuous function on $[X, \infty)$ for some $X > 0$.

THEOREM 1. *Suppose that f is a strictly increasing continuous function on $[p_{m+1}, \infty)$ for some integer $m \geq 0$, k_0 is an integer satisfying $f^{-1}(k_0) > \max(p_{m+1}, e)$, and ψ is an increasing positive function on $[0, 1]$ with $\psi(0) = 1$. Let*

$$B(x) = \frac{f^{-1}(x)}{\log f^{-1}(x) - 1}, \quad K_N = \lfloor f(p_N) \rfloor, \quad \text{and} \quad w_N = \{f(p_N)\}.$$

Then for $0 \leq w \leq 1$ the sequence $(f(p_{m+n}))_{n \geq 1}$, $n = 1, 2, \dots$ satisfies

$$\begin{aligned}
 D_N^*(f(p_{m+n}), g_w) &\leq \frac{2}{N} \sum_{k=k_0}^{K_N-1} B(k) \sup_{x \in [0, 1]} \left| \frac{B(k+x)}{B(k)} - \psi(x) \right| \\
 &\quad + (\psi(1) + 1) \frac{B(K_N)}{N} \sup_{x \in [0, 1]} \left| \frac{B(K_N+x)}{B(K_N)} - \psi(x) \right| \\
 &\quad + \left(\psi(1) \frac{B(K_N)}{N} + 1 \right) |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\
 &\quad + O \left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3} \right) + O \left(\frac{\psi(1)B(K_N)}{N(\log f^{-1}(K_N))^2} \right) \\
 &\quad + O \left(\frac{f^{-1}(k_0) + m}{N} \right) + O \left(\frac{\psi(1) \log f^{-1}(K_N)}{N(\log f^{-1}(K_N) - 1)} \right),
 \end{aligned}$$

where

$$g_w(x) = \frac{1}{\psi(w)} \frac{\psi(x) - 1}{\psi(1) - 1} + \frac{\min(\psi(x), \psi(w)) - 1}{\psi(w)} \quad (0 \leq x \leq 1).$$

Proof. Set

$$F_N(x) = \frac{\#\{1 \leq n \leq N; \{f(p_{m+n})\} \in [0, x)\}}{N} \quad \text{for } x \in [0, 1].$$

Then, for $0 \leq x \leq 1$

$$\begin{aligned} F_N(x) &= \frac{\#\{1 \leq n \leq N - m; \{f(p_{m+n})\} \in [0, x)\}}{N} \\ &\quad + \frac{\#\{N - m + 1 \leq n \leq N; \{f(p_{m+n})\} \in [0, x)\}}{N} \\ &= \frac{1}{N} \sum_{k=k_0}^{K_N-1} (A(k+x) - A(k)) \\ &\quad + \frac{\min(A(K_N+x), A(K_N+w_N)) - A(K_N)}{N} + \frac{O(f^{-1}(k_0) + m)}{N}, \quad (6) \end{aligned}$$

where

$$A(y) = \#\{1 \leq n \leq N : f(p_{m+n}) < y\} \quad \text{for } y \geq f(p_{m+1})$$

and

$$O(f^{-1}(k_0) + m) \leq f^{-1}(k_0) + m.$$

Then, for $y \geq f(p_{m+1})$

$$A(y) = \pi(f^{-1}(y)) - m + O(1).$$

Since

$$\pi(x) = \int_2^x \frac{1}{\log t} dt + O\left(\frac{x}{(\log x)^3}\right)$$

(see [5]) and that

$$\int_2^x \frac{1}{\log t} dt = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right),$$

we have

$$\pi(x) = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right).$$

Therefore, for $y \geq f(p_{m+1})$

$$A(y) = B(y) + O\left(\frac{f^{-1}(y)}{(\log f^{-1}(y))^3}\right), \quad (7)$$

and so,

$$A(k+x) - A(k) = B(k+x) - B(k) + O\left(\frac{f^{-1}(k+1)}{(\log f^{-1}(k+1))^3}\right) \quad (8)$$

for $k \geq k_0$ and $0 \leq x \leq 1$.

Since $B(x)$ and $f^{-1}(x)/(\log f^{-1}(x))^3$ are non-decreasing, by (7) and (8) we have

$$\min(A(K_N + x), A(K_N + w_N)) = \min(B(K_N + x), B(K_N + w_N)) + O\left(\frac{f^{-1}(K_N + 1)}{(\log f^{-1}(K_N + 1))^3}\right).$$

Hence, by (6) and (8) we have

$$F_N(x) = \frac{1}{N} \sum_{k=k_0}^{K_N-1} (B(k+x) - B(k)) + \frac{\min(B(K_N+x), B(K_N+w_N)) - B(K_N)}{N} + O\left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{\log f^{-1}(k))^3}\right) + \frac{O(f^{-1}(k_0) + m)}{N}. \quad (9)$$

We can express the first term in the right-hand side of (9) as

$$\frac{\sum_{k=k_0}^{K_N-1} (B(k+x) - B(k))}{\sum_{k=k_0}^{K_N-1} (B(k+1) - B(k))} \cdot \frac{B(K_N) - B(k_0)}{N},$$

and the second term as

$$\left(\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - 1\right) \frac{B(K_N)}{N}.$$

Hence we have for $x \in [0, 1]$

$$F_N(x) - g_w(x) \quad (10)$$

$$\begin{aligned} &= \frac{\sum_{k=k_0}^{K_N-1} (B(k+x) - B(k))}{\sum_{k=k_0}^{K_N-1} (B(k+1) - B(k))} \cdot \frac{B(K_N) - B(k_0)}{N} - \frac{\psi(x) - 1}{\psi(1) - 1} \cdot \frac{1}{\psi(w)} \\ &+ \left(\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - 1\right) \frac{B(K_N)}{N} - \frac{\min(\psi(x), \psi(w)) - 1}{\psi(w)} \\ &+ O\left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{\log f^{-1}(k))^3}\right) + \frac{O(f^{-1}(k_0) + m)}{N} \\ &= \left(\frac{\sum_{k=k_0}^{K_N-1} (B(k+x) - B(k))}{\sum_{k=k_0}^{K_N-1} (B(k+1) - B(k))} - \frac{\psi(x) - 1}{\psi(1) - 1}\right) \frac{B(K_N) - B(k_0)}{N} \end{aligned} \quad (11)$$

$$+ \left(\frac{B(K_N) - B(k_0)}{N} - \frac{1}{\psi(w)}\right) \frac{\psi(x) - 1}{\psi(1) - 1} \quad (12)$$

$$+ \left(\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - \min(\psi(x), \psi(w))\right) \frac{B(K_N)}{N} \quad (13)$$

$$\begin{aligned}
& + \left(\psi(u) - \frac{N}{B(K_N)} \right) \left(\frac{\min(\psi(x), \psi(w)) - 1}{\frac{N}{B(K_N)} \psi(w)} \right) \\
& + O \left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{\log f^{-1}(k)^3} \right) + \frac{O(f^{-1}(k_0) + m)}{N}.
\end{aligned} \tag{14}$$

In order to estimate (11), we set

$$S_j = \sum_{k=k_0}^{j-1} (B(k+x) - B(k)), \quad T_j = \sum_{k=k_0}^{j-1} (B(k+1) - B(k))$$

for $j \geq k_0 + 1$, $S_{k_0} = T_{k_0} = 0$, and $U = \frac{\psi(x)-1}{\psi(1)-1}$. Then we have, successively,

$$\begin{aligned}
& \frac{\sum_{k=k_0}^{K_N-1} (B(k+x) - B(k))}{\sum_{k=k_0}^{K_N-1} (B(k+1) - B(k))} - \frac{\psi(x) - 1}{\psi(1) - 1} = \frac{S_{K_N}}{T_{K_N}} - U \\
& = \frac{1}{T_{K_N}} \sum_{k=k_0}^{K_N-1} (S_{k+1} - S_k) - \frac{U}{T_{K_N}} \sum_{k=k_0}^{K_N-1} (T_{k+1} - T_k) \\
& = \frac{1}{T_{K_N}} \sum_{k=k_0}^{K_N-1} (T_{k+1} - T_k) \left(\frac{S_{k+1} - S_k}{T_{k+1} - T_k} - U \right) \\
& = \frac{1}{B(K) - B(k_0)} \times \sum_{k=k_0}^{K_N-1} (B(k+1) - B(k)) \left(\frac{B(k+x) - B(k)}{B(k+1) - B(k)} - \frac{\psi(x) - 1}{\psi(1) - 1} \right).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
|(11)| & \leq \frac{1}{N} \sum_{k=k_0}^{K_N-1} (B(k+1) - B(k)) \left| \frac{B(k+x) - B(k)}{B(k+1) - B(k)} - \frac{\psi(x) - 1}{\psi(1) - 1} \right| \\
& = \frac{1}{N} \sum_{k=k_0}^{K_N-1} (B(k+1) - B(k)) \left| \frac{\Psi_k(x) - 1}{\Psi_k(1) - 1} - \frac{\psi(x) - 1}{\psi(1) - 1} \right| \\
& \leq \frac{1}{N} \sum_{k=k_0}^{K_N-1} (B(k+1) - B(k)) \\
& \quad \times \left| \frac{(\Psi_k(x) - \psi(x))(\psi(1) - 1) - (\Psi_k(1) - \psi(1))(\psi(1) - 1)}{(\Psi_k(1) - 1)(\psi(1) - 1)} \right| \\
& \leq \frac{1}{N} \sum_{k=k_0}^{K_N-1} \frac{B(k+1) - B(k)}{\Psi_k(1) - 1} (|\Psi_k(x) - \psi(x)| + |\Psi_k(1) - \psi(1)|)
\end{aligned}$$

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$$\leq \frac{2}{N} \sum_{k=k_0}^{K_N-1} B(k) \sup_{x \in [0,1]} |\Psi_k(x) - \psi(x)|, \quad (15)$$

where $\Psi_k(x) = \frac{B(k+x)}{B(k)}$.

Every second factor in (12), (13) and (14) is bonded. In fact, for (12)

$$0 \leq \frac{\psi(x) - 1}{\psi(1) - 1} \leq 1 \quad \text{for } x \in [0, 1]. \quad (16)$$

Since

$$N = A(K_N + w_N) + m + 1 = B(K_N + w_N) + O\left(\frac{f^{-1}(K_N + w_N)}{(\log f^{-1}(K_N + w_N))^3}\right) + m + 1, \quad (17)$$

we have

$$\frac{N}{B(K_N)} = \Psi_{K_N}(w_N) + R(N), \quad (18)$$

where

$$R(N) = O\left(\frac{1}{(\log f^{-1}(K_N))^2}\right) + O\left(\frac{(m+1) \log f^{-1}(K_N)}{f^{-1}(K_N)}\right) \rightarrow 0 \quad (N \rightarrow \infty). \quad (19)$$

By (16), we have

$$\begin{aligned} |(12)| &\leq \left| \frac{B(K_N) - B(k_0)}{N} - \frac{1}{\psi(w)} \right| \\ &\leq \left| \frac{1}{\frac{N}{B(K_N)}} - \frac{1}{\psi(w_N)} \right| + \left| \frac{1}{\psi(w_N)} - \frac{1}{\psi(w)} \right| + \frac{B(k_0)}{N} \\ &= \left| \frac{1}{\Psi_{K_N}(w_N) + R(N)} - \frac{1}{\psi(w_N)} \right| + \frac{|\psi(w) - \psi(w_N)|}{\psi(w_N)\psi(w)} + \frac{B(k_0)}{N} \\ &\leq \frac{|\psi(w_N) - \Psi_{K_N}(w_N)| + R(N)}{\psi(w_N)(\Psi_{K_N}(w_N) + R(N))} + |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\ &\leq \frac{B(K_N)}{N} (|\psi(w_N) - \Psi_{K_N}(w_N)| + R(N)) + |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\ &\leq \frac{B(K_N)}{N} \sup_{x \in [0,1]} |\psi(x) - \Psi_{K_N}(x)| + |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\ &\quad + O\left(\frac{f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N) - 1)}\right) \\ &\quad + O\left(\frac{\log f^{-1}(K_N)}{N(\log f^{-1}(K_N) - 1)}\right). \end{aligned} \quad (20)$$

We can rewrite (13) to the form

$$|(13)| \leq \frac{B(K_N)}{N} \left(\left| \min \left(\frac{B(K_N + x)}{B(K_N)}, \frac{B(K_N + w_N)}{B(K_N)} \right) - \min(\psi(x), \psi(w_N)) \right| + \left| \min(\psi(x), \psi(w_N)) - \min(\psi(x), \psi(w)) \right| \right).$$

If $x < w_N$, then $B(K_N + x) < B(K_N + w_N)$, then $\psi(x) < \psi(w_N)$ and then

$$\left| \min \left(\frac{B(K_N + x)}{B(K_N)}, \frac{B(K_N + w_N)}{B(K_N)} \right) - \min(\psi(x), \psi(w_N)) \right| = |\Psi_{K_N}(x) - \psi(x)| \leq \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)|.$$

If $x < w_N$ and $x < w$, then $\min(\psi(x), \psi(w_N)) - \min(\psi(x), \psi(w)) = 0$. If $w < x < w_N$, then $|\min(\psi(x), \psi(w_N)) - \min(\psi(x), \psi(w))| = |\psi(x) - \psi(w)| \leq |\psi(w_N) - \psi(w)|$. Similarly, in all other cases. Thus we have

$$|(13)| \leq \frac{B(K_N)}{N} \left(\sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| + |\psi(w_N) - \psi(w)| \right). \quad (21)$$

By (19) we obtain

$$\begin{aligned} |(14)| &\leq \frac{B(K_N)}{N} (\psi(1) - 1) \left(|\psi(w) - \psi(w_N)| + \left| \psi(w_N) - \frac{N}{B(K_N)} \right| \right) \\ &= \frac{B(K_N)}{N} (\psi(1) - 1) \left(|\psi(w) - \psi(w_N)| + |\psi(w_N) - \Psi_{K_N}(w_N)| + R(N) \right) \\ &\leq \frac{B(K_N)}{N} (\psi(1) - 1) \left(|\psi(w) - \psi(w_N)| + \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| + \right. \\ &\quad \left. O \left(\frac{1}{(\log f^{-1}(K_N))^2} \right) + O \left(\frac{\log f^{-1}(K_N)}{f^{-1}(K_N)} \right) \right) \\ &\leq (\psi(1) - 1) \left(|\psi(w) - \psi(w_N)| + \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| \right) \frac{B(K_N)}{N} \\ &\quad + O \left(\frac{(\psi(1) - 1) f^{-1}(K_N)}{N (\log f^{-1}(K_N))^2 (\log f^{-1}(K_N) - 1)} \right) \\ &\quad + O \left(\frac{(\psi(1) - 1) \log f^{-1}(K_N)}{N (\log f^{-1}(K_N) - 1)} \right). \end{aligned} \quad (22)$$

Combining (10), (15), (20), (21), and (22), we have

$$\begin{aligned}
 & |F_N(x) - g_w(x)| \\
 & \leq \frac{2}{N} \sum_{k=k_0}^{K_N-1} B(k) \sup_{x \in [0,1]} |\Psi_k(x) - \psi(x)| + \frac{B(K_N)}{N} \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| \\
 & \quad + |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} + O\left(\frac{f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N) - 1)}\right) \\
 & \quad + O\left(\frac{\log f^{-1}(K_N)}{N(\log f^{-1}(K_N) - 1)}\right) \\
 & \quad + \frac{B(K_N)}{N} \left(\sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| + |\psi(w_N) - \psi(w)| \right) \\
 & \quad + (\psi(1) - 1) \frac{B(K_N)}{N} \left(|\psi(w) - \psi(w_N)| + \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| \right) \\
 & \quad + O\left(\frac{(\psi(1) - 1)f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N) - 1)}\right) + O\left(\frac{(\psi(1) - 1) \log f^{-1}(K_N)}{N(\log f^{-1}(K_N) - 1)}\right) \\
 & \quad + O\left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) + O\left(\frac{f^{-1}(k_0) + m}{N}\right) \\
 & \leq \frac{2}{N} \sum_{k=k_0}^{K_N-1} B(k) \sup_{x \in [0,1]} |\Psi_k(x) - \psi(x)| + (\psi(1) + 1) \frac{B(K_N)}{N} \sup_{x \in [0,1]} |\Psi_{K_N}(x) - \psi(x)| \\
 & \quad + \left(\psi(1) \frac{B(K_N)}{N} + 1 \right) |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\
 & \quad + O\left(\frac{\psi(1)B(K_N)}{N(\log f^{-1}(K_N))^2}\right) + O\left(\frac{1}{N} \sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) \\
 & \quad + O\left(\frac{f^{-1}(k_0) + m}{N}\right) + O\left(\frac{\psi(1) \log f^{-1}(K_N)}{N(\log f^{-1}(K_N) - 1)}\right).
 \end{aligned}$$

This proves the theorem. $\square$

From Theorem 1, we deduce the following corollaries.

COROLLARY 1. *Suppose f is a strictly increasing differentiable function on $[p_{m+1}, \infty)$ for some integer $m \geq 0$, and satisfies*

$$\lim_{t \rightarrow \infty} t f'(t) = \rho > 0. \quad (23)$$

Let $0 < w \leq 1$ and $N_i = \pi(f^{-1}(i + w))$. Then

$$\lim_{i \rightarrow \infty} \{f(p_{N_i})\} = w$$

and

$$\lim_{i \rightarrow \infty} D_{N_i}^*(f(p_{m+n}), g_w) = 0,$$

where

$$g_w(x) = \frac{1}{e^{\frac{w}{\rho}}} \frac{e^{\frac{x}{\rho}} - 1}{e^{\frac{1}{\rho}} - 1} + \frac{\min(e^{\frac{x}{\rho}}, e^{\frac{w}{\rho}}) - 1}{e^{\frac{w}{\rho}}} \quad (0 \leq x \leq 1).$$

Proof. Let $0 < w \leq 1$, $K_{N_i} = \lfloor f(p_{N_i}) \rfloor$, $w_{N_i} = \{f(p_{N_i})\}$. By the prime number theorem, we have

$$\begin{aligned} & \pi(f^{-1}(i + w)) - \pi(f^{-1}(i)) \\ &= \frac{f^{-1}(i + w)}{\log f^{-1}(i + w)} - \frac{f^{-1}(i)}{\log f^{-1}(i)} + O\left(\frac{f^{-1}(i + w)}{\log^2 f^{-1}(i + w)}\right) \\ &= \frac{f^{-1}(i + w)}{\log f^{-1}(i + w)} \left\{ 1 - \frac{f^{-1}(i)}{f^{-1}(i + w)} \frac{\log f^{-1}(i + w)}{\log f^{-1}(i)} + O\left(\frac{1}{\log f^{-1}(i + w)}\right) \right\}. \end{aligned} \quad (24)$$

By the assumption, we obtain

$$\lim_{x \rightarrow \infty} f^{-1}(x) = \infty, \quad \text{and so} \quad \lim_{i \rightarrow \infty} \frac{f^{-1}(i + w)}{\log f^{-1}(i + w)} = \infty.$$

By Lemma 4, we have $\lim_{i \rightarrow \infty} \frac{f^{-1}(i + w)}{f^{-1}(i)} = e^{\frac{w}{\rho}} > 1$. Hence

$$\lim_{i \rightarrow \infty} \frac{\log f^{-1}(i + w)}{\log f^{-1}(i)} = \lim_{i \rightarrow \infty} \left(\frac{1}{\log f^{-1}(i)} \log \frac{f^{-1}(i + w)}{f^{-1}(i)} \right) + 1 = 1. \quad (25)$$

Therefore, (24) and (25) yield

$$\lim_{i \rightarrow \infty} \pi(f^{-1}(i + w)) - \pi(f^{-1}(i)) = \infty.$$

Hence, there exists i_0 such that $\pi(f^{-1}(i + w)) - \pi(f^{-1}(i)) \geq 1$ for $i \geq i_0$. Thus, for $i \geq i_0$

$$f^{-1}(i) \leq p_{N_i} < f^{-1}(i + w) \leq p_{N_i+1}, \quad \text{so} \quad i \leq f(p_{N_i}) < i + w \leq f(p_{N_i+1}),$$

and so

$$K_{N_i} = \lfloor f(p_{N_i}) \rfloor = i. \quad (26)$$

Successively, we obtain

$$\begin{aligned} w_{N_i} &= \{f(p_{N_i})\} = f(p_{N_i}) - i < w, \\ w > w_{N_i} &= w_{N_i} - w + w = f(p_{N_i}) - (i + w) + w \geq f(p_{N_i}) - f(p_{N_i+1}) + w, \\ 0 < w - w_{N_i} &\leq f(p_{N_i+1}) - f(p_{N_i}). \end{aligned} \quad (27)$$

It is shown that

$$p_{n+1} - p_n = O(p_n^\theta) \quad (28)$$

for some $0 < \theta < 1$, namely, $p_{n+1} - p_n = O(p_n^{0.525})$ due to R. C. Baker, G. Harman and J. Pintz [1]. Hence, by the assumption (23)

$$f(p_{N_i+1}) - f(p_{N_i}) = (p_{N_i+1} - p_{N_i})f'(\xi) = O(p_{N_i}^{\theta-1}), \quad p_{N_i} \leq \xi \leq p_{N_i+1}.$$

Therefore, by (27) we have

$$w_{N_i} \rightarrow w \quad (i \rightarrow \infty). \quad (29)$$

Applying Theorem 1 with $\psi(x) = e^{\frac{x}{\rho}}$, we have

$$\begin{aligned} &D_{N_i}^*(f(p_{m+n}), g_w) \\ &\leq \frac{2}{N_i} \sum_{k=k_0}^{i-1} B(k) \sup_{x \in [0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}} \right| + \left(e^{\frac{1}{\rho}} + 1 \right) \frac{B(i)}{N_i} \sup_{x \in [0,1]} \left| \frac{B(i+x)}{B(i)} - e^{\frac{x}{\rho}} \right| \\ &\quad + \left(e^{\frac{1}{\rho}} \frac{B(i)}{N_i} + 1 \right) \left| e^{\frac{w}{\rho}} - e^{\frac{w_{N_i}}{\rho}} \right| + \frac{B(k_0)}{N_i} + O \left(\frac{1}{N_i} \sum_{k=k_0+1}^{i+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3} \right) \\ &\quad + O \left(\frac{e^{\frac{1}{\rho}} B(i)}{N_i (\log f^{-1}(i))^2} \right) + \frac{O(f^{-1}(k_0 + m))}{N_i} + O \left(\frac{e^{\frac{1}{\rho}} \log f^{-1}(i)}{N_i (\log f^{-1}(i) - 1)} \right). \end{aligned} \quad (30)$$

We note that

$$\begin{aligned} \frac{1}{N_i} \sum_{k=k_0}^{i-1} B(k) \left| \frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}} \right| &= \frac{1}{N_i} \sum_{k=k_0}^{i-1} (B(k+1) - B(k)) \times \\ &\quad \frac{\sup_{x \in [0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}} \right|}{\frac{B(k+1)}{B(k)} - 1}. \end{aligned} \quad (31)$$

By Lemma 5 we have

$$\lim_{k \rightarrow \infty} \sup_{x \in [0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}} \right| = 0.$$

Combining this with $\lim_{k \rightarrow \infty} \frac{B(k+1)}{B(k)} = \psi(1) = e^{\frac{1}{\rho}} > 1$, we get

$$\lim_{k \rightarrow \infty} \frac{\sup_{x \in [0,1]} \left| \frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}} \right|}{\frac{B(k+1)}{B(k)} - 1} = 0. \quad (32)$$

By Lemma 5 and (29) we have

$$\begin{aligned} \left| \frac{B(K_{N_i} + w_{N_i})}{B(K_{N_i})} - e^{\frac{w}{\rho}} \right| &\leq \left| \frac{B(K_{N_i} + w_{N_i})}{B(K_{N_i})} - e^{\frac{w_{N_i}}{\rho}} \right| + \left| e^{\frac{w_{N_i}}{\rho}} - e^{\frac{w}{\rho}} \right| \\ &\leq \sup_{0 \leq x \leq 1} \left| \frac{B(K_{N_i} + x)}{B(K_{N_i})} - e^{\frac{x}{\rho}} \right| + \left| e^{\frac{w_{N_i}}{\rho}} - e^{\frac{w}{\rho}} \right| \rightarrow 0 \quad (i \rightarrow \infty), \end{aligned}$$

so we have

$$\lim_{i \rightarrow \infty} \frac{B(K_{N_i} + w_{N_i})}{B(K_{N_i})} = e^{\frac{w}{\rho}}. \quad (33)$$

Therefore, by (18) and (19) we have

$$\lim_{i \rightarrow \infty} \frac{B(i)}{N_i} = \frac{1}{e^{\frac{w}{\rho}}}. \quad (34)$$

By (31), (32) and (34), we have

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{i-1} B(k) \sup_{x \in [0,1]} |\Psi_k(x) - e^{\frac{x}{\rho}}| = 0.$$

Lastly, we show that $\frac{1}{N_i} \sum_{k=k_0+1}^{i+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3} \rightarrow 0$. By Cauchy-Stolz lemma, it suffices to prove that

$$\lim_{i \rightarrow \infty} \frac{(\log f^{-1}(i+1))^3 (N_i - N_{i-1})}{f^{-1}(i+1)} = \infty. \quad (35)$$

By (17) we have

$$N_i - N_{i-1} = B(i + w_{N_i}) - B(i - 1 + w_{N_{i-1}}) + O\left(\frac{f^{-1}(i + w_{N_i})}{(\log f^{-1}(i + w_{N_i}))^3}\right). \quad (36)$$

Since

$$\begin{aligned} &f^{-1}(i + w_{N_i})(\log f^{-1}(i - 1 + w_{N_{i-1}}) - 1) \\ &\quad - f^{-1}(i - 1 + w_{N_{i-1}})(\log f^{-1}(i + w_{N_i}) - 1) \\ &= f^{-1}(i + w_{N_i}) \log \frac{f^{-1}(i - 1 + w_{N_{i-1}})}{f^{-1}(i + w_{N_i})} \\ &\quad + \{f^{-1}(i + w_{N_i}) - f^{-1}(i - 1 + w_{N_{i-1}})\} \{\log f^{-1}(i + w_{N_i}) - 1\} \\ &\geq f^{-1}(i) \log \frac{f^{-1}(i - 1 + w_{N_{i-1}})}{f^{-1}(i + w_{N_i})}, \end{aligned}$$

we obtain

$$B(i + w_{N_i}) - B(i - 1 + w_{N_{i-1}}) \geq \frac{f^{-1}(i) \log \frac{f^{-1}(i - 1 + w_{N_{i-1}})}{f^{-1}(i + w_{N_i})}}{(\log f^{-1}(i) - 1)(\log f^{-1}(i + 1) - 1)}.$$

Therefore,

$$\begin{aligned}
 & \frac{(\log f^{-1}(i))^3 (B(i + w_{N_i}) - B(i - 1 + w_{N_{i-1}}))}{f^{-1}(i)} \\
 & \geq \frac{(\log f^{-1}(i))^3 \log \frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}}{(\log f^{-1}(i) - 1)(\log f^{-1}(i + 1) - 1)} \\
 & \geq (\log f^{-1}(i)) \frac{1}{1 - \frac{1}{\log f^{-1}(i)}} \frac{1}{\frac{\log f^{-1}(i+1)}{\log f^{-1}(i)} - \frac{1}{\log f^{-1}(i)}} \log \frac{f^{-1}(i - 1 + w_{N_{i-1}})}{f^{-1}(i + w_{N_i})}.
 \end{aligned} \tag{37}$$

By Lemma 4 and (29), we have

$$\lim_{i \rightarrow \infty} \frac{f^{-1}(i + w_{N_i})}{f^{-1}(i)} = \psi(w), \tag{38}$$

so

$$\frac{f^{-1}(i - 1 + w_{N_{i-1}})}{f^{-1}(i + w_{N_i})} = \frac{\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i-1)}}{\frac{f^{-1}(i+w_{N_i})}{f^{-1}(i)} \cdot \frac{f^{-1}(i)}{f^{-1}(i-1)}} \rightarrow \frac{\psi(w)}{\psi(w)\psi(1)} = \frac{1}{\psi(1)} \quad (i \rightarrow \infty). \tag{39}$$

From (25), (37), and (39), it follows that

$$\lim_{i \rightarrow \infty} \frac{(\log f^{-1}(i))^3 (B(i + w_{N_i}) - B(i - 1 + w_{N_{i-1}}))}{f^{-1}(i)} = \infty. \tag{40}$$

By (25) and (38) we have

$$\lim_{i \rightarrow \infty} \frac{\log f^{-1}(i)}{\log f^{-1}(i + w_{N_i})} = 1. \tag{41}$$

Hence, by (36), (40), and (41) we have

$$\begin{aligned}
 & \frac{(\log f^{-1}(i))^3 (N_i - N_{i-1})}{f^{-1}(i)} = \\
 & \frac{(\log f^{-1}(i))^3 (B(i + w_{N_i}) - B(i - 1 + w_{N_{i-1}}))}{f^{-1}(i)} + \\
 & O \left(\frac{f^{-1}(i + w_{N_i})}{f^{-1}(i)} \cdot \left(\frac{\log f^{-1}(i)}{\log f^{-1}(i + w_{N_i})} \right)^3 \right) \rightarrow \infty \\
 & \text{as } i \rightarrow \infty.
 \end{aligned} \tag{42}$$

From (42),

$\lim_{i \rightarrow \infty} f^{-1}(i)/f^{-1}(i+1) = 1/e^{1/\rho}$, and $\lim_{i \rightarrow \infty} \log f^{-1}(i+1)/\log f^{-1}(i) = 1$, we arrive at (35). Consequently, the right hand side of (30) tends to 0 as $i \rightarrow \infty$. Thus $\lim_{i \rightarrow \infty} D_{N_i}^*(f(p_{m+n}) \bmod 1, g_w) = 0$. $\square$

COROLLARY 2. *Let $r > 0$, $0 < w \leq 1$, let $N_i = \pi(b^{\frac{i+w}{r}})$, $i = 1, 2, \dots$, and let*

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \leq x \leq 1).$$

Then we have

$$\begin{aligned} D_{N_i}^*(\log_b p_n^r, g_w) &\leq b^{\frac{1}{r}}(\log b) \left(2 + (b^{\frac{1}{r}} + 1) \frac{\log b}{r} \right) \frac{\log p_{N_i}}{(\log p_{N_i} - (1 + \frac{1}{r}) \log b)^2} \\ &\quad + C_1 \left(\frac{b^{\frac{2}{r}} r}{\log b} + b^{\frac{1}{r}} \log b \right) \frac{\log p_{N_i}}{(\log p_{N_i} - (1 + \frac{1}{r}) \log b)^3} \\ &\quad + C_2 \left(b^2 + \frac{b^2}{2 \log b - 1} \right) b^{\frac{2}{r}} r \frac{\log p_{N_i}}{p_{N_i}^{1-\theta}} \\ &\quad + C_3 b^{\frac{2}{r}} r \frac{(\log p_{N_i})^2 + \log p_{N_i} \log \log p_{N_i}}{p_{N_i}(r \log p_{N_i} - r - \log b)}, \end{aligned} \quad (43)$$

where $N_i \geq 6$ for $i = 1, 2, \dots$ and C_j is a constant for $j = 1, 2, 3$.

Proof. Applying Theorem 1 and Corollary 1 with

- $f(t) = \log_b t^r$ for $t \geq 1$,
- $m = 0$,
- $k_0 = \lceil 2r \rceil$,
- $f^{-1}(t) = b^{\frac{t}{r}}$ for $t \geq 0$,
- $\rho = \lim_{t \rightarrow \infty} t f'(t) = \frac{r}{\log b}$,
- $\psi(x) = b^{\frac{x}{r}}$,
- $B(x) = \frac{r b^{\frac{x}{r}}}{x \log b - r}$,
- $N_i = \pi(b^{\frac{i+w}{r}})$,
- $K_{N_i} = \lfloor r \log_b p_{N_i} \rfloor = i$ by (26), $w_{N_i} = \{r \log_b p_{N_i}\}$,

we have

$$\begin{aligned}
 D_{N_i}^*(\log_b p_n^r, g_w) &\leq \frac{2r}{N_i} \sum_{2r \leq k \leq i-1} \frac{b^{\frac{k}{r}}}{k-r} \sup_{x \in [0,1]} \left| b^{\frac{x}{r}} \left(\frac{k-r}{k+x-r} - 1 \right) \right| \\
 &+ \frac{(b^{\frac{1}{r}} + 1)rb^{\frac{i}{r}}}{(i-r)N_i} \sup_{x \in [0,1]} \left| b^{\frac{x}{r}} \left(\frac{i-r}{i+x-r} - 1 \right) \right| + \left(\frac{rb^{\frac{i+1}{r}}}{N_i} + 1 \right) \left| b^{\frac{w}{r}} - b^{\frac{wN_i}{r}} \right| \\
 &+ \frac{b^{2+\frac{1}{r}}}{(2 \log b - 1)N_i} + O\left(\frac{r^3}{(\log b)^3 N_i} \sum_{2r+1 \leq k \leq i+1} \frac{b^{\frac{k}{r}}}{k^3} \right) + O\left(\frac{r^3 b^{\frac{i+1}{r}}}{(\log b)^2 i^2 (i-r)N_i} \right) \\
 &+ O\left(\frac{b^{2+\frac{1}{r}}}{N_i} \right) + O\left(\frac{b^{\frac{1}{r}}(\log b)i}{N_i(i \log b - r)} \right). \tag{44}
 \end{aligned}$$

Since

$$\sup_{x \in [0,1]} \left| b^{\frac{x}{r}} \left(\frac{k-r}{k+x-r} - 1 \right) \right| < \frac{b^{\frac{1}{r}}}{k-r} \tag{45}$$

and that $\frac{b^{\frac{x}{r}}}{(x-r)^2}$ is increasing for $x \geq 2r$, we have

$$\sum_{2r \leq k \leq i-1} \frac{b^{\frac{k}{r}}}{k-r} \sup_{x \in [0,1]} \left| b^{\frac{x}{r}} \left(\frac{k-r}{k+x-r} - 1 \right) \right| \leq b^{\frac{1}{r}} \int_{2r}^i \frac{b^{\frac{x}{r}}}{(x-r)^2} dx.$$

Using the integration by parts, we have

$$\begin{aligned}
 \int_{2r}^i \frac{b^{\frac{x}{r}}}{(x-r)^2} dx &\leq \frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2} + 2 \frac{r}{\log b} \int_{2r}^i \frac{b^{\frac{x}{r}}}{(x-r)^3} dx \\
 &\leq \frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2} \\
 &\quad + 2 \frac{r}{\log b} \left\{ \frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^3} + 3 \frac{r}{\log b} \int_{2r}^i \frac{b^{\frac{x}{r}}}{(x-r)^4} dx \right\} \\
 &\leq \frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2} + \frac{8r^2 b^{\frac{i}{r}}}{(\log b)^2 (i-r)^3},
 \end{aligned}$$

because $\frac{e^{\frac{x}{r}}}{(x-r)^4}$ is increasing for $x \geq 2r$. Hence

$$\begin{aligned}
 \sum_{2r \leq k < i} \frac{b^{\frac{k}{r}}}{k-r} \sup_{x \in [0,1]} \left| b^{\frac{x}{r}} \left(\frac{k-r}{k+x-r} - 1 \right) \right| \\
 \leq \frac{rb^{\frac{i+1}{r}}}{(\log b)^2 (i-r)^2} \left(\log b + \frac{8r}{i-r} \right). \tag{46}
 \end{aligned}$$

Similarly, we have

$$\sum_{2r+1 \leq k \leq i+1} \frac{b^{\frac{k}{r}}}{k^3} \leq \frac{4rb^{\frac{i+2}{r}}}{(\log b)(i+2)^3}. \quad (47)$$

By (27), we have for $w \in [0, 1)$,

$$\begin{aligned} \left| b^{\frac{w}{r}} - b^{\frac{w_{N_i}}{r}} \right| &\leq \frac{b^{\frac{1}{r}} \log b}{r} (f(p_{N_i+1}) - f(p_{N_i})) \\ &\leq b^{\frac{1}{r}} \left(\frac{p_{N_i+1} - p_{N_i}}{p_{N_i}} \right). \end{aligned} \quad (48)$$

By (28) and (48), we obtain

$$\left| b^{\frac{u}{r}} - b^{\frac{w_{N_i}}{r}} \right| = O\left(\frac{b^{\frac{1}{r}}}{p_{N_i}^{1-\theta}} \right) = O\left(\frac{b^{\frac{1}{r}}}{N_i^{1-\theta} (\log N_i)^{1-\theta}} \right), \quad (49)$$

where $\theta = 0.525$.

By (44), (45), (46), (47), and (49), we obtain

$$\begin{aligned} D_{N_i}^*(\log_b p_n^r, g_w) &\leq \left(\frac{2}{\log b} b^{\frac{1}{r}} r^2 + (b^{\frac{2}{r}} + b^{\frac{1}{r}}) r \right) \frac{b^{\frac{i}{r}}}{(i-r)^2 N_i} \\ &\quad + \frac{16b^{\frac{1}{r}} r^3}{(\log b)^2} \frac{b^{\frac{i}{r}}}{(i-r)^3 N_i} \\ &\quad + \text{const } b^{\frac{2}{r}} r \frac{b^{\frac{i}{r}}}{N_i^{2-\theta} (\log N_i)^{1-\theta}} + \text{const } b^{\frac{1}{r}} \frac{1}{N_i^{1-\theta} (\log N_i)^{1-\theta}} \\ &\quad + \frac{b^{2+\frac{1}{r}}}{2 \log b - 1} \frac{1}{N_i} \\ &\quad + \text{const } \frac{b^{\frac{2}{r}} r^4}{(\log b)^4} \frac{b^{\frac{i}{r}}}{i^3 N_i} + \text{const } \frac{b^{\frac{1}{r}} r^3}{(\log b)^2} \frac{b^{\frac{i}{r}}}{i^2 (i-r) N_i} \\ &\quad + \text{const } b^{2+\frac{1}{r}} \frac{1}{N_i} + \text{const } (\log b) b^{\frac{1}{r}} \frac{i}{N_i (i \log b - r)}, \end{aligned} \quad (50)$$

where each “const” means a positive constant. Since

$$r \log_b p_{N_i} = i + w_{N_i} \leq i + 1,$$

$$i \geq \frac{r}{\log b} \log p_{N_i} - 1. \quad (51)$$

Since

$$i < i + w_{N_i} = \frac{r}{\log b} \log p_{N_i} \quad (52)$$

and that

$$p_n < n(\log n + \log \log n) \quad \text{for } n \geq 6$$

by J. B. Rosser and L. Schoenfeld [10], we have

$$i < \frac{r}{\log b} (\log N_i + \log(\log N_i + \log \log N_i)), \quad (53)$$

and so

$$b^{\frac{i}{r}} < N_i(\log N_i + \log \log N_i) \quad (54)$$

for $N_i \geq 6$. By (51), (53), (54), and

$$n \log n < p_n \quad \text{for } n \geq 1,$$

by J. B. Rosser and L. Schoenfeld [10], we have

$$\begin{aligned} \frac{b^{\frac{i}{r}}}{(i-r)^2 N_i} &< \frac{\log(N_i \log N_i)}{(\frac{r}{\log b} \log p_{N_i} - r - 1)^2} < \frac{\log p_{N_i}}{(\frac{r}{\log b} \log p_{N_i} - r - 1)^2}, \\ \frac{b^{\frac{i}{r}}}{(i-r)^3 N_i} &< \frac{\log(N_i \log N_i)}{(\frac{r}{\log b} \log p_{N_i} - r - 1)^3} < \frac{\log p_{N_i}}{(\frac{r}{\log b} \log p_{N_i} - r - 1)^3}, \\ \frac{b^{\frac{i}{r}}}{N_i^{2-\theta} (\log N_i)^{1-\theta}} &< \frac{\log(N_i \log N_i)}{N_i^{1-\theta} (\log N_i)^{1-\theta}}, \\ \frac{b^{\frac{i}{r}}}{i^3 N_i} &< \frac{\log(N_i \log N_i)}{(\frac{r}{\log b} \log p_{N_i} - 1)^3} < \frac{\log p_{N_i}}{(\frac{r}{\log b} \log p_{N_i} - 1)^3}, \\ \frac{b^{\frac{i}{r}}}{i^2 (i-r) N_i} &< \frac{\log(N_i \log N_i)}{(\frac{r}{\log b} \log p_{N_i} - 1)^2 (\frac{r}{\log b} \log p_{N_i} - r - 1)} \\ &< \frac{\log p_{N_i}}{(\frac{r}{\log b} \log p_{N_i} - 1)^2 (\frac{r}{\log b} \log p_{N_i} - r - 1)}, \\ \frac{i}{N_i (i-r)} &< \frac{r}{\log b} \frac{\log(N_i \log(N_i \log N_i))}{N_i (\frac{r}{\log b} \log p_{N_i} - r - 1)} \\ &< \frac{r}{\log b} \frac{\log p_{N_i}}{N_i (\frac{r}{\log b} \log p_{N_i} - r - 1)}, \\ \frac{1}{N_i} &< \frac{1}{\pi(b^{\frac{i}{r}})} \sim \frac{\log b}{r} \frac{i}{b^{\frac{i}{r}}} < \frac{\log b}{r} \frac{\frac{r}{\log b} \log p_{N_i}}{b^{(\log_b p_{N_i} - \frac{1}{r})}} \\ &= b^{\frac{1}{r}} \frac{\log p_{N_i}}{p_{N_i}}. \end{aligned}$$

Overall, by (50), we have

$$\begin{aligned}
D_{N_i}^*(\log_b p_n^r, g_w) &\leq \left(\frac{2b^{\frac{1}{r}} r^2}{\log b} + (b^{\frac{2}{r}} + b^{\frac{1}{r}}) r \right) \frac{\log p_{N_i}}{(\frac{r}{\log b} \log p_{N_i} - r - 1)^2} \\
&\quad + \frac{16b^{\frac{1}{r}} r^3}{(\log b)^2 (\frac{r}{\log b} \log p_{N_i} - r - 1)^3} + \text{const } b^{\frac{2}{r}} r \frac{\log p_{N_i}}{p_{N_i}^{1-\theta}} \\
&\quad + \text{const } b^{\frac{1}{r}} \frac{1}{p_{N_i}^{1-\theta}} + \text{const } \frac{b^{2+\frac{2}{r}}}{2 \log b - 1} \frac{\log p_{N_i}}{p_{N_i}} \\
&\quad + \text{const } \frac{b^{\frac{2}{r}} r^4}{(\log b)^4 (\frac{r}{\log b} \log p_{N_i} - 1)^3} \\
&\quad + \text{const } \frac{b^{\frac{1}{r}} r^3}{(\log b)^2 (\frac{r}{\log b} \log p_{N_i} - 1)^2 (\frac{r}{\log b} \log p_{N_i} - r - 1)} \\
&\quad + \text{const } b^{2+\frac{2}{r}} \frac{\log p_{N_i}}{p_{N_i}} \\
&\quad + \text{const } b^{\frac{2}{r}} r \frac{(\log p_{N_i})^2 + \log p_{N_i} \log \log p_{N_i}}{p_{N_i} (r \log p_{N_i} - r - \log b)} \\
&\leq b^{\frac{1}{r}} (\log b) \left(2 + (b^{\frac{1}{r}} + 1) \frac{\log b}{r} \right) \frac{\log p_{N_i}}{(\log p_{N_i} - (1 + \frac{1}{r}) \log b)^2} \\
&\quad + \text{const } \left(\frac{b^{\frac{2}{r}} r}{\log b} + b^{\frac{1}{r}} \log b \right) \frac{\log p_{N_i}}{(\log p_{N_i} - (1 + \frac{1}{r}) \log b)^3} \\
&\quad + \text{const } \left(b^2 + \frac{b^2}{2 \log b - 1} \right) b^{\frac{2}{r}} r \frac{\log p_{N_i}}{p_{N_i}^{1-\theta}} \\
&\quad + \text{const } b^{\frac{2}{r}} r \frac{(\log p_{N_i})^2 + \log p_{N_i} \log \log p_{N_i}}{p_{N_i} (r \log p_{N_i} - r - \log b)}
\end{aligned}$$

for $N_i \geq 6$. □

From Corollary 2 it follows immediately:

COROLLARY 3. *Let $r > 0$, $0 < w \leq 1$, let $N_i = \pi(b^{\frac{i+w}{r}})$, $i = 1, 2, \dots$, and let*

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \leq x \leq 1).$$

Then we have

$$\lim_{i \rightarrow \infty} (\log p_{N_i}) D_{N_i}^*(\log_b p_n^r, g_w) \leq b^{\frac{1}{r}} (\log b) \left(2 + (b^{\frac{1}{r}} + 1) \frac{\log b}{r} \right).$$

COROLLARY 4. *Let r be a positive integer. Then*

$$\lim_{r \rightarrow \infty} r D_{\pi(b^r)}^*(\log_b p_n^r, g_1) \leq 2,$$

where

$$g_1(x) = \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} \quad (0 \leq x \leq 1).$$

Proof. Let $r \geq 3$. Put $w = 1$ and $i = r^2 - 1$ in Corollary 3. Then $N_i = \pi(b^r)$. By (52), we have

$$\log p_{N_i} > \left(r - \frac{1}{r}\right) \log b,$$

and so

$$\frac{r}{\log p_{N_i}} \leq \frac{r}{(r - \frac{1}{r}) \log b} \rightarrow \frac{1}{\log b}$$

as $r \rightarrow \infty$. From this and Corollary 3, the desired inequality follows. $\square$

REFERENCES

- [1] BAKER, R. C.—HARMAN, G.—PINTZ, J.: *The difference between consecutive primes, II*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 532–562.
- [2] BINGHAM, N. H.—GOLDIE, C. M.—TEUGELS, J. L.: *Regular Variation*. In: *Encyclopedia of Mathematics and its Applications Vol. 27*. Cambridge University Press, Cambridge, 1987.
- [3] DIACONIS, P.: *The distribution of leading digits and uniform distribution mod 1*, Ann. Probability **5** (1977), no. 1, 72–81.
- [4] ELIAHOU, S.—MASSÉ, B.—SCHNEIDER, D.: *On the mantissa distribution of powers of natural and prime numbers*, Acta Math. Hungar. **139** (1–2) (2013), 49–63.
- [5] INGHAM, A. E.: *The Distribution of Prime Numbers*, Cambridge Univ. Press, New York, 1932.
- [6] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*. In: *Pure and Applied Mathematics*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974; Reprint: Dover Publications, Inc. Mineola, New York, 2006.
- [7] OHKUBO, Y.: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238.
- [8] OHKUBO, Y.—STRAUCHI, O.: *Distribution of leading digits of numbers*, Unif. Distrib. Theory **11** (2016), no. 1, 23–45.
- [9] PAVLOV, A. I.: *On distribution of fractions parts and Benford's law* Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 4, 760–774. (In Russian)
- [10] ROSSER, J. B.—SCHOENFELD, L.: *Approximate formulas for some functions of prime numbers* Illinois J. Math. **6** (1962), 64–94.

- [11] SENETA, E.: *Regularly Varying Functions*. In: *Lecture Notes in Math. Vol. 508*. Springer-Verlag, Berlin, 1976.
- [12] STRAUCH, O.—PORUBSKÝ, Š.: *Distribution of Sequences: A Sampler*. In: *Series of the Slovak Academy of Sciences*. Peter Lang, Frankfurt am Main, 2005.
- [13] WINTNER, A.: *On the cyclical distribution of logarithms of the prime numbers*, Quart. J. Math. **6** (1935), no.1, 65–68.

Received November 6, 2017

Accepted May 16, 2018

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