AN OPERATIONAL CALCULUS MODEL FOR THE n^{TH} — ORDER FORWARD DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of a non-classical Bittner operational calculus, in which the derivative is understood as a forward difference $\Delta_n\{x(k)\} := \{x(k+n) - x(k)\}$. Next, considering the operation $\Delta_{n,b}\{x(k)\} := \{x(k+n) - b x(k)\}$, the presented model has been generalized.

Key words:

operational calculus, derivative, integrals, limit conditions, forward difference.

THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [1–4]

$$CO(L^0, L^1, S, T_q, s_q, Q)^1 \quad (1)$$

is understood as a system, in which L^0 and L^1 are linear spaces (over a field F of scalars), such that $L^1 \subset L^0$. Moreover, a linear operation $S : L^1 \longrightarrow L^0$ (which is described as $S \in \mathcal{L}(L^1, L^0)$), called a *derivative*, is a surjection. Q is a set of indices q for the operations $T_q \in \mathcal{L}(L^0, L^1)$ and $s_q \in \mathcal{L}(L^1, L^1)$, called *integrals* and *limit conditions*, respectively. These operations have to fulfil the properties $ST_q f = f, f \in L^0$ and $s_q x = x - T_q Sx, x \in L^1$. The kernel of S , i.e. $\text{Ker } S$, is called a set of *constants* for the derivative S . The limit conditions $s_q, q \in Q$ are projections of L^1 on the subspace $\text{Ker } S$.

* Polish Naval Academy, Faculty of Mechanical and Electrical Engineering, Śmidowicza 69 Str., 81-127 Gdynia, Poland; e-mail: h.wysocki@amw.gdynia.pl

¹ The abbreviation *CO* is derived from the French *calcul opératoire* (operational calculus).

When speaking of a *representation* or a *model* of an operational calculus, we have in mind a system (1), in which all the objects are defined. A classic example of an operational calculus (1) is a *discrete* model with the derivative as a forward difference $\Delta\{x(k)\} := \{x(k+1) - x(k)\}$.

THE FORWARD DIFFERENCE MODEL

Let $\mathbb{N}_0$ and $\mathbb{C}$ mean sets of non-negative integers and complexes, respectively. Moreover, let $L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k \in \mathbb{N}_0}$ with usual operations on sequences. In [1, 2, 4] Bittner considered a model with a derivative

$$Sx \equiv \Delta x := \{x(k+1) - x(k)\}, \quad (2)$$

to which there was corresponding an integral

$$T_0 x := \begin{cases} 0 & \text{for } k = 0 \\ \sum_{i=0}^{k-1} x(i) & \text{for } k > 0 \end{cases}, \quad k \in \mathbb{N}_0$$

and a limit condition

$$s_0 x := \{x(0)\},$$

where $x = \{x(k)\} \in L^0 = L^1$. Later there appeared, mentioned officially in [5], a model with the derivative (2), integrals

$$T_{k_0} x := \begin{cases} -\sum_{i=k}^{k_0-1} x(i) & \text{for } k < k_0 \\ 0 & \text{for } k = k_0 \\ \sum_{i=k_0}^{k-1} x(i) & \text{for } k > k_0 \end{cases}, \quad k \in \mathbb{N}_0 \quad (3)$$

and limit conditions

$$s_{k_0} x := \{x(k_0)\}, \quad (4)$$

where $k_0 \equiv q \in Q := \mathbb{N}_0^2$ (see also [11]). This model was generalized in [7] by Mieloszyk. He proved that to the so-called *forward difference on the basis* $b = \{b(k)\}$, i.e.

² Due to the definition of T_{k_0} , we assume that $\sum_{i=k_0}^{k_0-1} x(i) := 0$.

$$S_b x := \{x(k+1) - b(k)x(k)\},$$

where $b(k) \neq 0$ for each $k \in \mathbb{N}_0$ and $\{b(k)\}\{x(k)\} := \{b(k)x(k)\}$ means the usual multiplication of sequences b, x in the algebra L^0 , there correspond the integrals

$$T_{b,k_0} x = \{e(k)\} T_{k_0} \left\{ \frac{x(k)}{e(k+1)} \right\}$$

and limit conditions

$$s_{b,k_0} x = \left\{ \frac{e(k)}{e(k_0)} \right\} s_{k_0} \{x(k)\},$$

where $e(k) := \prod_{i=0}^{k-1} b(i)$, $e(0) := 1$.

Having in mind further considerations, let us notice that the integrals (3) can be presented in the concise form of

$$T_{k_0} x = \left\{ \sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i) \right\}. \quad (5)$$

A HIGHER ORDER FORWARD DIFFERENCE MODEL

A generalization of the operation $\Delta \equiv \Delta_1$ is the n^{th} — order forward difference

$$\Delta_n \{x(k)\} := \{x(k+n) - x(k)\}, \quad (6)$$

where n is a given natural number.

We will determine integrals T_{k_0} and limit conditions s_{k_0} corresponding to (6) understood as the derivative S . Firstly, let us notice that an arbitrary constant c for (6) is an n -periodic sequence, i.e. it satisfies the condition $c(k+n) = c(k)$ for each $k \in \mathbb{N}_0$. What is more, for any sequence $c \in \text{Ker } \Delta_n$ there exist numbers $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$ such that

$$c = \{a_0 \varepsilon_0^k + a_1 \varepsilon_1^k + \dots + a_{n-1} \varepsilon_{n-1}^k\}, \quad (7)$$

where

$$\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1} \quad (8)$$

are n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos \frac{2j\pi}{n} + i \sin \frac{2j\pi}{n}, \quad j \in \overline{0, n-1}^3,$$

while 'i' is the imaginary unit.

In what follows, we shall use the below sequence (8) properties:

$$\varepsilon_j^{k+n} = \varepsilon_j^k, \quad j \in \overline{0, n-1}, k \in \mathbb{N}_0,$$

$$\varepsilon_0^m + \varepsilon_1^m + \dots + \varepsilon_{n-1}^m = 0, \quad m \neq \ell n, \ell, m \in \mathbb{Z}^4, n \in \mathbb{N} \setminus \{1\}.$$

We shall prove the following:

Theorem. *The system (1), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{N}_0$ and*

$$Sx := \{x(k+n) - x(k)\}, \quad (9)$$

$$T_{k_0}x := \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=0}^{k-1} \varepsilon_j^{k-i} x(i) - \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} x(i) \right] \right\}, \quad (10)$$

$$s_{k_0}x := \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i} x(i) \right\} \quad (11)$$

forms a discrete Bittner operational calculus model⁵.

Proof. It is obvious that operations (9)–(11) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Then

$$\begin{aligned} S\{y(k)\} &= \{y(k+n) - y(k)\} \\ &= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=0}^{k+n-1} \varepsilon_j^{k+n-i} x(i) - \sum_{i=0}^{k_0-1} \varepsilon_j^{k+n-i} x(i) \right] \right. \\ &\quad \left. - \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=0}^{k-1} \varepsilon_j^{k-i} x(i) - \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} x(i) \right] \right\} = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k}^{k+n-1} \varepsilon_j^{k-i} x(i) \right\}. \end{aligned}$$

³ $\overline{0, n-1} := \{0, 1, \dots, n-1\}$.

⁴ $\mathbb{Z}$ denotes the set of integers.

⁵ We assume that $\sum_{i=0}^{n-1} x(i) := 0$.

It is not difficult to notice that for $n = 1$ we have $S\{y(k)\} = \{x(k)\}$, while for $n > 1$ we can write

$$S\{y(k)\} = \{x(k)\} + \frac{1}{n} \left\{ \sum_{i=k+1}^{k+n-1} [\varepsilon_0^{k-i} + \varepsilon_1^{k-i} + \cdots + \varepsilon_{n-1}^{k-i}] x(i) \right\} = \{x(k)\}. \quad (12)$$

Finally, it can be stated that the property $ST_{k_0}\{x(k)\} = \{x(k)\}$ is satisfied.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+n) - x(k)\}$. Then

$$\begin{aligned} T_{k_0}S\{x(k)\} &= T_{k_0}\{f(k)\} = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=0}^{k-1} \varepsilon_j^{k-i} f(i) - \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} f(i) \right] \right\} \\ &= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=0}^{k-1} \varepsilon_j^{k-i} [x(i+n) - x(i)] - \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} [x(i+n) - x(i)] \right] \right\} \\ &= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=n}^{k+n-1} \varepsilon_j^{k-i} x(i) - \sum_{i=0}^{k-1} \varepsilon_j^{k-i} x(i) - \sum_{i=n}^{k_0+n-1} \varepsilon_j^{k-i} x(i) + \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} x(i) \right] \right\} \\ &= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{i=n}^{k+n-1} \varepsilon_j^{k-i} x(i) + \sum_{i=0}^{n-1} \varepsilon_j^{k-i} x(i) - \sum_{i=0}^{k-1} \varepsilon_j^{k-i} x(i) \right. \right. \\ &\quad \left. \left. - \left(\sum_{i=n}^{k_0+n-1} \varepsilon_j^{k-i} x(i) + \sum_{i=0}^{n-1} \varepsilon_j^{k-i} x(i) - \sum_{i=0}^{k_0-1} \varepsilon_j^{k-i} x(i) \right) \right] \right\} \\ &= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k}^{k+n-1} \varepsilon_j^{k-i} x(i) \right\} - \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i} x(i) \right\}. \end{aligned}$$

By analogy to (12), we eventually get

$$T_{k_0}S\{x(k)\} = \{x(k)\} - \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i} x(i) \right\}.$$

So the property $T_{k_0}S\{x(k)\} = \{x(k)\} - s_{k_0}\{x(k)\}$ is also fulfilled. $\square$

Let us observe that (2), (4), (5) constitute a particular case of the above model for $n = 1$.

Example. The limit condition (11) allows to present an arbitrary n -periodic sequence $c = \{c(k)\}$ with a recurring cycle $(c_0, c_1, \dots, c_{n-1})$, i.e.

$$c = \{(c_0, c_1, \dots, c_{n-1})\} := \{c_0, c_1, \dots, c_{n-1}, c_0, c_1, \dots, c_{n-1}, \dots\}$$

in the form of (7). For, if $c \in \text{Ker } \Delta_n$, we have $s_{k_0}c = c$. Then for $k_0 = 0$ we get

$$c = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \varepsilon_j^{k-i} c_i \right\}. \quad (13)$$

A lot of interesting examples of periodic sequences are included in *The On-Line Encyclopedia of Integer Sequences* OEIS^{®6}. Some of them are related to the *Fibonacci sequence* $\{F(k)\}$, whose terms meet the conditions

$$F(k+1) = F(k) + F(k-1), \quad k \in \mathbb{N}$$

and $F(0) = 0, F(1) = 1$.

In 1960, Wall proved [9] that for each natural number $m > 1$, the sequence

$$\{F_m(k)\} := \{F(k) \pmod{m}\} \quad (14)$$

is $p(m)$ -periodic⁷.

Thus, if in (13) we take $n := p(m)$ and $c_i := F_m(i)$, then

$$F_m(k) = \frac{1}{p(m)} \sum_{j=0}^{p(m)-1} \sum_{i=1}^{p(m)-1} \varepsilon_j^{k-i} F_m(i), \quad k \in \mathbb{N}_0. \quad (15)$$

The below table contains sequences (14) chosen from OEIS[®] as well as trigonometric forms of their general terms obtained on the basis of (15) by using the *Mathematica*[®] program.

⁶ <https://oeis.org/>.

⁷ The number $p(m)$ is called the *Pisano period* of the sequence (14) [10]. The Pisano periods for $m \leq 10 \cdot 10^7$ can be calculated directly using the Marc Renault [8] web browser applet available at <http://webpace.ship.edu/msrenault/fibonacci/fibfactory.htm>.

OEIS®	m	$p(m)$	Cycle	$F_m(k)$
A011655	2	3	(0, 1, 1)	$\frac{4}{3} \sin^2 \frac{k\pi}{3}$
A082115	3	8	(0, 1, 1, 2, 0, 2, 2, 1)	$\frac{1}{8} \left(9 - 2\sqrt{2} \cos \frac{k\pi}{4} - 6 \cos \frac{k\pi}{2} \right.$ $\left. + 2\sqrt{2} \cos \frac{3k\pi}{4} - 3 \cos k\pi \right.$ $\left. - 2 \sin \frac{k\pi}{4} + 2 \sin \frac{3k\pi}{4} \right)$
A079343	4	6	(0, 1, 1, 2, 3, 1)	$\frac{2}{3} \sin \frac{k\pi}{6} \left(\sqrt{3} \cos \frac{k\pi}{2} \right.$ $\left. + 4 \sin \frac{k\pi}{6} + \sin \frac{k\pi}{2} \right)$
A079344	8	12	(0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1)	$\frac{2}{3} \sin \frac{k\pi}{6} \left(2\sqrt{3} \cos \frac{k\pi}{3} + \sqrt{3} \cos \frac{k\pi}{2} + 2\sqrt{3} \cos \frac{2k\pi}{3} \right.$ $\left. + 8 \sin \frac{k\pi}{6} + (5 - 4\sqrt{3} \cos \frac{k\pi}{6}) \sin \frac{k\pi}{2} \right)$

SOME GENERALIZATION

The operation

$$S_b\{x(k)\} := \{x(k+n) - bx(k)\}, \quad (16)$$

where $\{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $b \in \mathbb{C} \setminus \{0\}$, is a generalization of the derivative (9).

In order to construct an operational calculus model related to the derivative (16), we will use the idea of solving the equation $x(k+1) - b(k)x(k) = f(k)$ described in [6] as well as the following auxiliary theorems:

Lemma 1 (Th. 3 [4]). *An abstract differential equation*

$$Sx = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = x_{0,q}, \quad x_{0,q} \in \text{Ker } S$$

has exactly one solution

$$x = x_{0,q} + T_q f. \quad (17)$$

Lemma 2 (Th. 4 [4]). *With a given derivative $S \in \mathcal{L}(L^1, L^0)$, the projection $s_q \in \mathcal{L}(L^1, \text{Ker } S)$ determines the integral $T_q \in \mathcal{L}(L^0, L^1)$ from the condition*

$$x = T_q f \quad \text{if and only if} \quad Sx = f, s_q x = 0.$$

Moreover, s_q is a limit condition corresponding to the integral T_q .

Let us notice that one of the elements of the space $\text{Ker } S_b$ is the sequence

$$e(k) := b^{\frac{k}{n}}, \quad k \in \mathbb{N}_0.$$

So

$$e(k+n) = be(k), \quad k \in \mathbb{N}_0.$$

Let us consider the difference equation

$$S_b \{x(k)\} = \{f(k)\},$$

i.e.

$$x(k+n) - bx(k) = f(k), \quad k \in \mathbb{N}_0. \quad (18)$$

Hence we get

$$\frac{x(k+n)}{e(k+n)} - \frac{x(k)}{e(k)} = \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0,$$

that is

$$y(k+n) - y(k) = g(k), \quad k \in \mathbb{N}_0, \quad (19)$$

where

$$y(k) := \frac{x(k)}{e(k)}, \quad g(k) := \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0. \quad (20)$$

The equation (19) can be presented in the form of

$$S \{y(k)\} = \{g(k)\}, \quad (21)$$

where $S \equiv \Delta_n$ is the operation (9).

From Lemma 1 it follows that the sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (10) and (11), is the solution of the equation (21).

From (20) we get $x(k) = e(k)y(k)$, $k \in \mathbb{N}_0$. Finally,

$$\{x(k)\} = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\} + \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\} \quad (22)$$

constitutes the solution of the equation (18).

If we take

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

then the sequence $\{\widetilde{c}(k)\} \in \text{Ker } S$ is n -periodic, i.e.

$$\widetilde{c}(k+n) = \widetilde{c}(k), \quad k \in \mathbb{N}_0.$$

Let

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\}, \quad k_0 \in Q := \mathbb{N}_0, \{x(k)\} \in L^1. \quad (23)$$

Thus, for each $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned} S_b s_{b,k_0} x(k) &= e(k+n)\widetilde{c}(k+n) - b e(k)\widetilde{c}(k) \\ &= e(k+n)(\widetilde{c}(k+n) - \widetilde{c}(k)) = e(k+n) \cdot 0 = 0, \end{aligned}$$

which means that $s_{b,k_0} \in \mathcal{L}(L^1, \text{Ker } S_b)$. Moreover, for each $k \in \mathbb{N}_0$ holds the below

$$\begin{aligned} s_{b,k_0}^2 x(k) &= s_{b,k_0}[e(k)\widetilde{c}(k)] = e(k)s_{k_0}\left[\frac{e(k)\widetilde{c}(k)}{e(k)}\right] \\ &= e(k)s_{k_0}\widetilde{c}(k) = e(k)\widetilde{c}(k) = s_{b,k_0}x(k), \end{aligned}$$

since $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$. Eventually, s_{b,k_0} is a projection of L^1 onto $\text{Ker } S_b$ for each $k_0 \in \mathbb{N}_0$. From Lemma 2 it follows that the projection s_{b,k_0} determines the *integral* T_{b,k_0} from the formula (22). Namely,

$$T_{b,k_0}\{f(k)\} := \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \quad (24)$$

What is more, s_{b,k_0} is the *limit condition* corresponding to the integral (24). Hence we arrive at the

Corollary. *The system (16), (23), (24) constitutes a discrete model of the Bittner operational calculus*

$$CO(C(\mathbb{N}_0, \mathbb{C}), C(\mathbb{N}_0, \mathbb{C}), S_b, T_{b,k_0}, s_{b,k_0}, \mathbb{N}_0).$$

REFERENCES

- [1] Bittner R., *On certain axiomatics for the operational calculus*, 'Bull. Acad. Polon. Sci.', 1959, Cl. III, 7(1), pp. 1–9.
- [2] Bittner R., *Operational calculus in linear spaces*, 'Studia Math.', 1961, 20, pp. 1–18.
- [3] Bittner R., *Algebraic and analytic properties of solutions of abstract differential equations*, 'Rozprawy Matematyczne' ['Dissertationes Math.'], 41, PWN, Warszawa 1964.
- [4] Bittner R., *Rachunek operatorów w przestrzeniach liniowych*, PWN, Warszawa 1974 [*Operational Calculus in Linear Spaces* — available in Polish].
- [5] Bittner R., Mieloszyk E., *Properties of eigenvalues and eigenelements of some difference equations in a given operational calculus*, 'Zeszyty Naukowe Wydz. Mat. Fiz. Chem. Uniwersytetu Gdańskiego, Matematyka', 1981, 5, pp. 5–18.
- [6] Levy H., Lessman F., *Finite difference equations*, Pitman and Sons, London 1959.
- [7] Mieloszyk E., *Example of operational calculus*, 'Zeszyty Naukowe Politechniki Gdańskiej, Matematyka XIII', 1985, 383, pp. 151–157.
- [8] Renault M., [online], <http://webpace.ship.edu/msrenault/> [access 28.05.2017].
- [9] Wall D. D., *Fibonacci series modulo m*, 'Amer. Math. Monthly', 1960, 67(6), pp. 525–532.
- [10] Weisstein E. W., 'Pisano Period', from MathWorld — A Wolfram Web Resource, [online], <http://mathworld.wolfram.com/PisanoPeriod.html> [access 28.05.2017].
- [11] Wysocki H., *Taylor's formula for the forward difference via operational calculus*, 'Studia Sci. Math. Hungar.', 2010, 47(1), pp. 46–53.
- [12] Wysocki H., *The operational calculus model for the nth-order backward difference*, 'Zeszyty Naukowe Akademii Marynarki Wojennej' ['Scientific Journal of PNA'], 2015, 3(202), pp. 75–88.

MODEL RACHUNKU OPERATORÓW DLA RÓŻNICY PROGRESYWNEJ RZĘDU n

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna rozumiana jest jako różnica progresywna $\Delta_n\{x(k)\} := \{x(k+n) - x(k)\}$. Następnie dokonano uogólnienia opracowanego modelu, rozważając operację $\Delta_{n,b}\{x(k)\} := \{x(k+n) - b x(k)\}$.

Słowa kluczowe:

rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica progresywna.